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Relations between multizeta values in characteristic p

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ABSTRACT

We study relations between the multizeta values for function fields introduced by D. Thakur. The product $\zeta(a)\zeta(b)$ is a linear combination of multizeta values. For $q = 2$, a full conjectural description of how the product of two zeta values can be described as the sum of multizetas was given by Thakur. The recursion part of this recipe was generalized by the author. In this paper, the main conjecture formulated by the author, as well as some conjectures of Thakur are proved. Moreover, for general q, we prove closed formulas as well as a recursive recipe to express $\zeta(a)\zeta(b)$ as a sum of multizeta values.

1. Introduction

We refer to [Wal05], and the references in there, for a survey of many exciting recent developments related to the multizeta values introduced by Euler. We refer to [Gos96,Tha04] for discussion of Goss zeta functions in connection with the function field arithmetic.

Dinesh Thakur [Tha04, Section 5.10] introduced two types of multizeta values for function fields over finite fields of characteristic p, one complex valued (generalizing the Artin–Weil zeta function) and the other with values in Laurent series over finite fields (generalizing the Carlitz–Goss zeta function). In this paper, we only focus on the latter. For its properties, connections with Drinfeld modules and Anderson t-motives, we refer the reader to [AT09,Tha04,Tha09,Tha10].

Thakur proves the existence of "shuffle" relations for the multizeta values (for a general A with a rational place at infinity) [Tha10]. In particular, he shows that the product of multizeta values can also be expressed as a sum of some multizeta values, so that the $\mathbb{F}_p$-span of all multizeta values is an algebra. In the function field case, the identities are much more complicated than the classical shuffle identities. In fact there are two types of identities, one with $\mathbb{F}_p(t)$ coefficients and the other with $\mathbb{F}_p$

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coefficients. (Note that although for many purposes a good analog of $\mathbb{Q}$ is $\mathbb{F}_q(t)$, the prime field in characteristic p (0 respectively) is $\mathbb{F}_p$ ($\mathbb{Q}$ respectively).) We concentrate only on the latter type.

The results in [Tha10], although effective, are not explicit and bypass the explicit conjectures formulated in [Tha09,Lar09,Lar10]. In this paper, we use the ideas of the process in [Tha10] to prove the main conjecture formulated in [Lar09,Lar10] and to give a closed formula for the expression of the product of zeta values as the sum of multizeta values (Theorems 6.3 and 7.1). More precisely, for general q, we prove that the expression of $\zeta(a)\zeta(b)$ as a sum of multizeta values is obtained by a recursive process; we effectively determine the recursion length, the terms and the number of terms to be added at each step of the recursion (Theorem 5.13). Also, when $q=2$, we give formulas to compute the initial values (Theorem 6.1 and Corollary 7.2) and, using one of these formulas, some conjectures in [Tha09] are proved.

2. Frequently used notation

$\mathbb{Z}$	{integers}
$\mathbb{Z}_+$	{positive integers}
q	a power of a prime p, $q=p^s$
$\mathbb{F}_q$	a finite field of q elements
t	an independent variable
A	the polynomial ring $\mathbb{F}_q[t]$
A_+	monics in A
K	the function field $\mathbb{F}_q(t)$
K_∞	$=\mathbb{F}_q((1/t))=$ the completion of K at ∞
A_d	{elements of A of degree d}
A_{d+}	$A_d \cap A_+$
$[n]$	$=t^{q^n}-t$
‘even’	multiple of $q-1$
deg	function assigning to $a \in A$ its degree in t
$\ell(k)$	sum of the digits of the base q expansion of k
$\lfloor x \rfloor$	the largest integer not greater than x
$\lceil x \rceil$	the smallest integer not less than x.

3. Thakur multizeta values

For $s \in \mathbb{Z}_+$, the *Carlitz zeta values* [Gos96,Tha04] are defined as

$$\zeta_A(s) := \sum_{a \in A_+} \frac{1}{a^s} \in K_\infty.$$

For $s \in \mathbb{Z}$ and $d \geqslant 0$, write

$$S_d(s) := \sum_{a \in A_{d+}} \frac{1}{a^s} \in K.$$

Given integers $s_i \in \mathbb{Z}_+$ and $d \geqslant 0$ put

$$S_d(s_1, \ldots, s_r) = S_d(s_1) \sum_{d > d_2 > \cdots > d_r \geqslant 0} S_{d_2}(s_2) \cdots S_{d_r}(s_r) \in K.$$

For $s_i \in \mathbb{Z}_+$, Thakur defined the *multizeta values* [Tha04,Tha09] by:

$$\zeta(s_1,\ldots,s_r) := \sum_{d_1>\cdots>d_r\geqslant 0} S_{d_1}(s_1)\cdots S_{d_r}(s_r) = \sum \frac{1}{a_1^{s_1}\cdots a_r^{s_r}} \in K_\infty,$$

where the second sum is over all $a_i \in A_+$ of degree d_i such that $d_1 > \cdots > d_r \geqslant 0$. We say that this multizeta value has depth r and weight $\sum s_i$.

4. Relations between multizeta values

Recall that Euler's multizeta values ζ (only in this paragraph, the Greek letter ζ will be used to denote the classical multizeta values) are defined by $\zeta(s_1,\ldots,s_r) = \sum (n_1^{s_1}\cdots n_r^{s_r})^{-1}$, where the sum is over positive integers $n_1 > n_2 > \cdots > n_r$ and s_i are positive integers, with $s_1 > 1$ (this condition is required for convergence). Since $n_1 = n_2$, $n_1 > n_2$ or $n_2 > n_1$, we have the "sum shuffle relation"

$$\begin{aligned}\zeta(a)\zeta(b) &= \sum_{n_1=1}^{\infty}\frac{1}{n_1^a}\sum_{n_2=1}^{\infty}\frac{1}{n_2^b} = \sum_{n_1=n_2}\frac{1}{n_1^{a+b}} + \sum_{n_1>n_2}\frac{1}{n_1^a n_2^b} + \sum_{n_2>n_1}\frac{1}{n_2^b n_1^a}\\ &= \zeta(a+b) + \zeta(a,b) + \zeta(b,a).\end{aligned}$$

In the function field case, this sum shuffle relation fails because there are many polynomials of a given degree. In contrast to the classical sum shuffle, in the function field case the identities we get are much more involved.

For $s_1, s_2 \in \mathbb{Z}_+$ put

$$S_d(s_1,s_2) = \sum_{\substack{d=d_1>d_2\\ a_i\in A_+}} \frac{1}{a_1^{s_1}a_2^{s_2}}$$

where $d_i = \deg(a_i)$. For $a, b \in \mathbb{Z}_+$, we define

$$\Delta_d(a,b) = S_d(a)S_d(b) - S_d(a+b).$$

We write $\Delta(a,b)$ for $\Delta_1(a,b)$. The definition implies $\Delta_d(a,b) = \Delta_d(b,a)$.

The next two theorems (the second theorem in the reference has implications to higher genus function fields, but we state only a special case relevant to us) are due to Thakur [Tha10, Theorems 1, 2].

Theorem 4.1. *Given $a, b \in \mathbb{Z}_+$, there are $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}_+$, so that*

$$\Delta_d(a,b) = \sum f_i S_d(a_i, a+b-a_i) \tag{4.1.1}$$

holds for $d = 1$.

Theorem 4.2. *Fix A. If* (4.1.1) *holds for some $f_i \in \mathbb{F}_p^\times$ and distinct $a_i \in \mathbb{Z}_+$ for $d = 1$, then* (4.1.1) *holds for all $d \geqslant 0$. In this case, we have the shuffle relation*

$$\zeta(a)\zeta(b) - \zeta(a+b) - \zeta(a,b) - \zeta(b,a) = \sum f_i\zeta(a_i, a+b-a_i).$$

5. Main results

In this section we shall prove the recursion part of the main conjecture formulated by Lara [Lar10, Lar09], except the items concerning the initial values.

Let us start with a definition.

Definition 5.1. Let $a \in \mathbb{Z}_+$.

(1) We set

$$r_a = (q-1)p^m,$$

where m is the smallest integer such that $a \leqslant p^m$.

(2) Put

$$\phi(j) := r_a - a - j(q-1).$$

(3) We define

$$j_{a,\max} = \left\lfloor \frac{r_a - a}{q-1} \right\rfloor.$$

(4) Let q be prime. For $0 \leqslant j \leqslant j_{a,\max}$, let $c_{a,j} \in \mathbb{F}_p$ be defined by:

$$c_{a,j} = \begin{cases} 1 & \text{if } j = 0, \\ \lceil \frac{j(q-1)}{j_{a,\max}} \rceil^{-1} \binom{r_a - a}{j(q-1)} & 0 < j \leqslant j_{a,\max}. \end{cases}$$

(5) For each j, $0 \leqslant j \leqslant p-1$, let μ_j be the number of j's in the p expansion of $a-1$. Set

$$t_a = \prod_{j=0}^{p-2} (p-j)^{\mu_j}.$$

Remark 5.2. For q prime and $0 < j \leqslant j_{a,\max}$, $\lceil j(q-1)/j_{a,\max} \rceil$ is not zero in $\mathbb{F}_p$ because $0 < j(q-1)/j_{a,\max} \leqslant q-1 < p$. Therefore, $c_{a,j}$ in Definition 5.1 (4) makes sense. For q non-prime, $c_{a,j}$ is not always defined (e.g., $q=4$, $a=5$, $j=3$).

Let $a, b \in \mathbb{Z}_+$. Let $j \in 0, 1, \dots, p^m - a$. Since p^m and $q-1$ are coprime, p^m is a unit in $\mathbb{Z}/(q-1)\mathbb{Z}$; so, the equation $j \equiv -ip^m \bmod (q-1)$ has always a solution. There exists exactly one solution in the range $0 \leqslant i < q-1$.

Definition 5.3. Let i_j be the unique integer $0 \leqslant i_j < q-1$ such that $j + i_j p^m \equiv 0 \bmod (q-1)$. Let l_j be the non-negative integer defined by

$$l_j = \frac{j + i_j p^m}{q-1}.$$

In general, the correspondence $j \mapsto i_j$ between $\{0, 1, \dots, p^m - a\}$ and $0, 1, \dots, q-2$ is neither injective (e.g., $q=3$, $a=4$) nor surjective (e.g., $q=4$, $a=3$).

Proposition 5.4. *The map*

$$\{0, 1, \ldots, p^m - a\} \to \{l(q-1) \mid 0 \leqslant l \leqslant j_{a,\max}\},$$
$$j \mapsto j + i_j p^m$$

is injective.

Proof. Since $0 \leqslant j \leqslant p^m - a$ and $0 \leqslant i_j \leqslant q-2$, it follows that $0 \leqslant j + i_j p^m \leqslant p^m - a + p^m(q-2) = r_a - a$. This shows that the map is well defined. If $a = p^m$, then the map is clearly injective. Assume $a < p^m$. If $j_1 + i_{j_1} p^m = j_2 + i_{j_2} p^m$, then $p^m \mid j_1 - j_2$. Since $0 \leqslant j_1, j_2 \leqslant p^m - a < p^m$, we conclude that $j_1 = j_2$. $\square$

When $q = 2$, the map of the above proposition is a bijection, but in general the map is not surjective. For instance, consider $q = 3$ and $a = 2$.

Proposition 5.5. *The following statements hold in* $\mathbb{F}_p$:

a) *If* $0 \leqslant i \leqslant q-1$, *then*

$$\binom{q-1}{i} = (-1)^i. \tag{5.5.1}$$

b) *If* $0 \leqslant i \leqslant p-2$, *then*

$$\binom{p-2}{i} = (-1)^i(i+1).$$

c) *The number of* j*'s,* $0 \leqslant j \leqslant p^m - a$, *such that* $\binom{p^m-a}{j}$ *is not zero modulo* p *is* t_a.

Proof. a) First consider the case $0 \leqslant i \leqslant p-1$. Then, $(p-1)!$, $i!$, and $(p-1-i)!$ are non-zero modulo p; therefore, in $\mathbb{F}_p$, we have

$$\binom{p-1}{i} = \frac{(p-1)(p-2)\cdots(p-i)}{i!} = \frac{(-1)(-2)\cdots(-i)}{i!} = (-1)^i.$$

Thus, (5.5.1) holds in this special case.

For general i, let $i = \sum_{l=0}^{s-1} i_l p^l$ be the base p expansion of i. The base p expansion of $q-1$ is $\sum_{l=0}^{s-1}(p-1)p^l$. Note that $(-1)^{a_l} = (-1)^{a_l p^l}$. By the Lucas theorem, we conclude

$$\binom{q-1}{i} = \prod_{l=0}^{s-1}\binom{p-1}{i_l} = \prod_{l=0}^{s-1}(-1)^{i_l p^l} = (-1)^i.$$

Thus, (5.5.1) holds in the general case.

b) For the second part, we have

$$\binom{p-2}{i} = \frac{(p-2)\cdots(p-2-(i-1))}{i!} = \frac{(-2)\cdots(-(i+1))}{i!} = (-1)^i(i+1).$$

c) Let t'_a be the number of j's in $0, 1, \dots, p^m - a$ such that $\binom{p^m-a}{j} \not\equiv 0 \bmod p$. Let us prove that $t'_a = t_a$. Since $a \leqslant p^m$, then $a - 1 \leqslant p^m - 1 = \sum_{l=0}^{m-1} (p-1)p^l$. Let $a - 1 = \sum_{l=0}^{m-1} a_l p^l$ and $j = \sum_{l=0}^{m-1} b_l p^l$ be the base p expansions of $a-1$ and j, respectively. Since $p^m - 1 - (a-1) = p^m - a$, the base p expansion of $p^m - a$ is $\sum_{l=0}^{m-1} (p-1-a_l)p^l$. By the Lucas theorem,

$$\binom{p^m - a}{j} \equiv \prod_{l=0}^{m-1} \binom{p-1-a_l}{b_l} \mod p.$$

Since $\binom{\alpha}{\beta}$ vanishes modulo p if $\beta > \alpha$, by choosing b_l in the set $\{0, 1, \dots, p-1-a_l\}$, we guarantee that $\binom{p^m-a}{j}$ does not vanish modulo p. Therefore,

$$t'_a = \prod_{l=0}^{m-1} (p - a_l) = \prod_{k=0}^{p-2} (p-k)^{\mu_k} = t_a. \qquad \square$$

Proposition 5.6. *Let $q = 2$. Then, $j_{a,\max} = 0$ if and only if $a = p^m$. If $q > 2$, then $j_{a,\max} = 0$ if and only if $a = 1$.*

Proof. This follows by a straight calculation from the definitions. $\square$

Definition 5.7. For each j, $0 \leqslant j \leqslant p^m - a$, let $f_{a,j} \in \mathbb{F}_p$ be defined by

$$f_{a,j} = \binom{p^m - a}{j} (-1)^j.$$

Proposition 5.8. *The number of j's such that $f_{a,j} \neq 0$ is t_a.*

Proof. It follows immediately from Proposition 5.5(c). $\square$

When q is prime, we have another description for $f_{a,j}$ which is consistent with the part 3 of the main conjecture in [Lar10] (see Remark 5.19(1)).

Proposition 5.9. *If q is prime (so that $q = p$), then for $j \in \{1, 2, \dots, p^m - a\}$*

$$f_{a,j} = \left\lceil \frac{j + i_j p^m}{j_{a,\max}} \right\rceil^{-1} \binom{r_a - a}{j + i_j p^m} = c_{a,l_j},$$

where c_{a,l_j} is as defined in Definitions 5.1 and 5.3.

Proof. We claim that (A) $i_j + 1 = \lceil \frac{j + i_j p^m}{j_{a,\max}} \rceil$, which is equivalent to showing

$$i_j j_{a,\max} \leqslant i_j p^m < j + i_j p^m \leqslant (i_j + 1) j_{a,\max}.$$

It is enough to prove the rightmost inequality. To do so, we first compute i_j for $j > 0$, as follows. Write $j = l(q-1) + r$, $0 \leqslant r < q - 1$. If $r > 0$, then

$$j + (q-1-r)p^m = \big(l + p^m\big)(q-1) + r\big(1 - p^m\big) \equiv 0 \mod (q-1),$$

and $0 < q-1-r < q-1$. If $r=0$, then $j \equiv 0 \bmod (q-1)$. By definition, it follows that $i_j = q-1-r$ if $r>0$ and $i_j = 0$, otherwise. Note $0 \leqslant i_j \leqslant q-2$, so that i_j+1 is non-zero in $\mathbb{F}_p$. Write $a = l'(q-1)+a'$, $0 \leqslant a' < q-1$. Then $j_{a,\max} = p^m - l' - y$, where $y=0$ if $a'=0$ and $y=1$ otherwise. Let us consider first the case $j \not\equiv 1 \bmod (q-1)$. Thus $i_j \neq q-2$. Let $j_0 \in \{2,\dots,p^m-a\} \cap \{2,\dots,q-1\}$ such that $i_j = i_{j_0}$. Then $j_0 = q-u$ for some u, $1 \leqslant u \leqslant q-2$ and $i_{j_0} = u-1$, $0 \leqslant i_{j_0} < q-2$. Then

$$\begin{aligned} j + i_j p^m &\leqslant j_0 + i_{j_0} p^m + (q-1)\left\lfloor \frac{p^m - a + i_j p^m - (j_0 + i_{j_0} p^m)}{q-1} \right\rfloor \\ &= q - u + (u-1)p^m + (q-1)\left\lfloor \frac{p^m - a - j_0}{q-1} \right\rfloor \\ &= q - u + (u-1)p^m + p^m - 1 - l'(q-1) - z(q-1), \end{aligned}$$

where $z=1$ if $0 < (a'+j_0-1)/(q-1) \leqslant 1$ and $z=2$ if $1 < (a'+j_0-1)/(q-1) < 1 + (q-2)/(q-1)$. Since $u \leqslant q-1$, then $l'u \leqslant l'(q-1) \leqslant l'(q-1) + (z-1)(q-1)$. Therefore

$$j + i_j p^m = up^m - u - l'(q-1) - (z-1)(q-1) \leqslant up^m - l'u - yu = (i_j+1) j_{a,\max}.$$

On the other hand, if $j \equiv 1 \bmod (q-1)$, then $i_j + 1 = q-1$. Since $j \leqslant p^m - a$, then $j + i_j p^m \leqslant r_a - a$. Now,

$$\frac{j + i_j p^m}{i_j + 1} = \frac{j + i_j p^m}{q-1} \leqslant \frac{r_a - a}{q-1}.$$

The claim now follows as the left side is an integer.

Let $j = b_0 + b_1 p + \cdots + b_{m-1} p^{m-1}$ and $a - 1 = a_0 + a_1 p + \cdots + a_{m-1} p^{m-1}$ be the base p expansions of j and $a-1$, respectively. The base p expansion of $p^m(p-1) - 1$ is $\sum_{l=0}^{m-1} (p-1)p^l + (p-2)p^m$. Consequently, the base p expansion of $r_a - a = r_a - 1 - (a-1)$ is $\sum_{l=0}^{m-1} (p-1-a_l)p^l + (p-2)p^m$. Finally, the base p expansion of $j + i_j p^m$ is $\sum_{l=0}^{m-1} b_l p^l + i_j p^m$.

Note $j + i_j \equiv j + i_j p^m \equiv 0 \bmod (q-1)$, so, (B) $(-1)^{i_j} = (-1)^j$. By the Lucas theorem, b) of Proposition 5.5, and (A), (B) respectively, we have

$$\begin{aligned} \binom{r_a - a}{j + i_j p^m} &= \binom{p-1-a_0}{b_0} \cdots \binom{p-1-a_{m-1}}{b_{m-1}} \binom{p-2}{i_j} \\ &= \binom{p^m - a}{j} (-1)^{i_j} (i_j + 1) \\ &= f_{a,j} \left\lceil \frac{j + i_j p^m}{j_{a,\max}} \right\rceil. \qquad \square \end{aligned}$$

Proposition 5.10. *Let q be arbitrary and $a \in \mathbb{Z}_+$. For each $n \in A_{1+}$, let g_n be defined by*

$$g_n = -\frac{[1]^{p^m - a}}{n^{p^m - a}} [1]^a S_1(a).$$

Then,

$$g_n = 1 + \sum_{j=1}^{p^m - a} f_{a,j} n^{r_a - (j + i_j p^m)}$$

where $f_{a,j}$ is as in Definition 5.7 and i_j is the unique integer, $0 \leqslant i_j < q-1$, such that $j + i_j p^m \equiv 0 \bmod (q-1)$.

Proof. Note

$$-\frac{[1]^{p^m-a}}{n^{p^m-a}} = -\left(n^{q-1}-1\right)^{p^m-a} = \sum_{j_1=0}^{p^m-a} \binom{p^m-a}{j_1} (-1)^{p^m-a-j_1+1} n^{j_1(q-1)}.$$

On the other hand, by specializing [Tha09, 3.3] to $d=1$ we see that

$$S_1(k+1) = \frac{(-1)^{k+1}}{[1]^{k+1}} \left(1 + \sum_{k_1=1}^{\lfloor k/q \rfloor} \binom{k-k_1(q-1)}{k_1} (-1)^{k_1} [1]^{k_1(q-1)}\right).$$

Hence,

$$[1]^a S_1(a) = (-1)^a + \sum_{j_2=1}^{n_a} \binom{a-1-j_2(q-1)}{j_2} (-1)^{a+j_2} n^{j_2(q-1)} \left(n^{q-1}-1\right)^{j_2(q-1)},$$

where $n_a = \lfloor (a-1)/q \rfloor$. Since

$$n^{j_1(q-1)} n^{j_2(q-1)} \left(n^{q-1}-1\right)^{j_2(q-1)} = \sum_{j_3=0}^{j_2(q-1)} \binom{j_2(q-1)}{j_3} (-1)^{j_2(q-1)-j_3} n^{(j_1+j_2+j_3)(q-1)},$$

we see that $g_n = -(n^{q-1}-1)^{p^m-a}[1]^a S_1(a) \in \mathbb{F}_p[n]$. Note that all powers of n involved are 'even'. If $n = t-\theta$ for some $\theta \in \mathbb{F}_q$, then $g_n(t+\theta) = g_t$. If $g_n = \sum_j c_{n,j} n^{j(q-1)}$, it follows that $c_{n,j} = c_{t,j}$ for all j. Now it suffices to compute $c_{t,j}$. To do so let us put $h_\theta = (t-\theta)^{p^m-a}((t-\theta)^{q-1}-1)^{p^m}$, $\theta \in \mathbb{F}_q^*$. Then,

$$\begin{aligned} h_\theta &= \sum_{j=0}^{p^m-a} \binom{p^m-a}{j} t^{p^m-a-j} \theta^j (-1)^j \left(\sum_{i=0}^{q-1} \binom{q-1}{i}^{p^m} t^{p^m(q-1-i)} \theta^{ip^m} (-1)^i - 1\right) \\ &= \sum_{j=0}^{p^m-a} f_{a,j} t^{p^m-a-j} \left(\sum_{i=0}^{q-1} t^{p^m(q-1-i)} \theta^{j+ip^m} - \theta^j\right). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{\theta \in \mathbb{F}_q^*} h_\theta &= \sum_{j=0}^{p^m-a} f_{a,j} t^{p^m-a-j} \left(\sum_{i=0}^{q-1} t^{p^m(q-1-i)} \sum_{\theta \in \mathbb{F}_q^*} \theta^{j+ip^m} - \sum_{\theta \in \mathbb{F}_q^*} \theta^j\right) \\ &= \sum_{j=0}^{p^m-a} f_{a,j} t^{p^m-a-j} \left(\sum_{i=0}^{q-2} t^{p^m(q-1-i)} \sum_{\theta \in \mathbb{F}_q^*} \theta^{j+ip^m}\right) \\ &= \sum_{j=0}^{p^m-a} f_{a,j} t^{p^m-a-j} \left(t^{p^m(q-1-i_j)}(-1)\right) \\ &= -\sum_{j=0}^{p^m-a} f_{a,j} t^{r_a-a-(j+i_j p^m)+p^m}. \end{aligned}$$

Here, we used that $\sum_{\theta\in\mathbb{F}_q^*}\theta^l$ is 0 if $q-1$ does not divide l, and -1 if $l\geqslant 0$ is divisible by $q-1$. Now,

$$[1]^{p^m}S_1(a)=\sum_{n\in A_{1+}}\frac{[1]^{p^m}}{n^a}=\sum_{n\in A_{1+}}n^{p^m-a}\big(n^{q-1}-1\big)^{p^m}=t^{p^m-a}\big(t^{r_a}-1\big)+\sum_{\theta\in\mathbb{F}_q^*}h_\theta.$$

It follows that

$$g_t=-t^{r_a}+1+\sum_{j=0}^{p^m-a}f_{a,j}t^{r_a-(j+i_jp^m)}=1+\sum_{j=1}^{p^m-a}f_{a,j}t^{r_a-(j+i_jp^m)}.\qquad\square$$

Example 5.11. Let $q=9$ and $a=17$. Then, $m=3$ and $r_a=27(8)=216$. Since $a-1=1+2p+p^2$, $t_a=3^02^2=4$. The j's for which $f_{a,j}\neq 0$ are 0, 1, 9, and 10. Using that $0\mapsto 0$, $1\mapsto 5$, $9\mapsto 5$, and $10\mapsto 2$, we have

$$g_t=1+f_{a,1}t^{216-136}+f_{a,9}t^{216-144}+f_{a,10}t^{216-64}=1+2t^{80}+2t^{72}+t^{152}.$$

Definition 5.12. Let q be arbitrary. For $a,b\in\mathbb{Z}_+$, let $S(a,b)$ denote the pairs (f_i,a_i) such (4.1.1) holds for $d=1$.

The following theorem proves parts 1 and 7 of the main Conjecture 1.5 in [Lar10], and is more precise in that it gives a complete recipe for general q.

Theorem 5.13. *Let $a,b\in\mathbb{Z}_+$. Let r_a, i_j, $f_{a,j}$ be as in Definitions* 5.1, 5.3, 5.7. *Then*

$$\Delta(a,b+r_a)-\Delta(a,b)=\sum_{j=0}^{p^m-a}f_{a,j}S_1\big(a+b+\big(j+i_jp^m\big)\big).$$

In particular, the sets $S(a,b)$ can be found recursively with recursion length r_a, by

$$S(a,b+r_a)=S(a,b)\cup T(a,b+r_a),$$

where

$$T(a,b+r_a)=\big\{\big(f_{a,j},a+b+\big(j+i_jp^m\big)\big)\;\big|\;0\leqslant j\leqslant p^m-a,\ f_{a,j}\neq 0\big\},$$

or equivalently,

$$T(a,b+r_a)=\big\{\big(f_{a,j},b+r_a-\phi(l_j)\big)\;\big|\;0\leqslant j\leqslant p^m-a,\ f_{a,j}\neq 0\big\},$$

where $\phi(l_j)=r_a-a-l_j(q-1)$.

The set $T(a,b+r_a)$ is a set of size t_a.

Proof. By definition of $\Delta(a,b+r_a)$, it follows that

$$\Delta(a,b+r_a)=\sum_{\substack{n_1\neq n_2\\ n_1,n_2\in A_{1+}}}\frac{1}{n_1^an_2^{b+r_a}}=\sum_{n_2\in A_{1+}}\frac{1}{n_2^{b+r_a}}\left(S_1(a)-\frac{1}{n_2^a}\right).$$

By Proposition 5.10, for any $n \in A_{1+}$, we have $g_n = 1 + \sum_{j=1}^{p^m-a} f_{a,j} n^{r_a-(j+i_j p^m)}$. Let $\Sigma = \sum_{j=1}^{p^m-a} f_{a,j} \times S_1(b + r_a - \phi(l_j))$. Then,

$$\Sigma = \sum_{n_2 \in A_{1+}} \sum_{j=1}^{p^m-a} \frac{f_{a,j}}{n_2^{b+r_a-(r_a-a+(j+i_j p^m))}} = \sum_{n_2 \in A_{1+}} \frac{1}{n_2^{b+r_a}} \frac{1}{n_2^a} \sum_{j=1}^{p^m-a} f_{a,j} n_2^{r_a-(j+i_j p^m)}.$$

Using this leads to

$$\begin{aligned}
\Delta(a, b + r_a) - \Sigma &= \sum_{n_2 \in A_{1+}} \frac{S_1(a)}{n_2^{b+r_a}} \left(1 - \frac{1}{S_1(a) n_2^a} \left(1 + \sum_{j=1}^{p^m-a} f_{a,j} n_2^{r_a-(j+i_j p^m)}\right)\right) \\
&= \sum_{n_2 \in A_{1+}} \frac{S_1(a)}{n_2^{b+r_a}} \left(1 - \frac{1}{S_1(a) n_2^a} g_{n_2}\right) \\
&= \sum_{n_2 \in A_{1+}} \frac{S_1(a)}{n_2^{b+r_a}} \left(1 + \frac{[1]}{n_2}\right)^{p^m} \\
&= \sum_{n_2 \in A_{1+}} \frac{S_1(a)}{n_2^{b+r_a}} \left(n_2^{q-1}\right)^{p^m} \\
&= S_1(a) S_1(b).
\end{aligned}$$

Therefore, $\Delta(a, b + r_a) - S_1(a) S_1(b) = \Sigma$. From this, we get

$$\begin{aligned}
\Delta(a, b + r_a) - \Delta(a, b) &= \Sigma + S_1(a + b) \\
&= \sum_{j=1}^{p^m-a} f_{a,j} S_1\big(b + r_a - \phi(l_j)\big) + S_1\big(b + r_a - \phi(0)\big) \\
&= \sum_{j=0}^{p^m-a} f_{a,j} S_1\big(b + r_a - \phi(l_j)\big).
\end{aligned}$$

This shows that $T(a, b + r_a)$ is exactly as claimed. By Proposition 5.8 it follows that the size of $T(a, b + r_a)$ is precisely t_a. $\square$

Remark 5.14.

(1) The set T_a of pairs $(f_{a,j}, \phi(l_j))$ with $f_{a,j} \neq 0$ clearly is independent of b as predicted in Conjecture 1.5(1c) in [Lar10], which did not have precise description for it as above.
(2) The number of terms t_a to be added at each step of the recursion depends on q only through p.

Examples 5.15 *(Special large indices).* (See [Tha09, p. 2338].) Let $q = 2$.

(1) Let $a = 2^n - 1$. Then, $m = n$ and $a - 1 = 2 + \cdots + 2^{n-1}$. There is only one zero in the base 2 expansion of $a - 1$. Thus, $t_a = (2 - 0)^1 = 2$. At each step of recursion, two terms must be added. Furthermore, $T_a = \{(1, r_a - a), (1, r_a - a - 1)\}$ because in this case $l_j = j$ for $j = 0, 1$.

(2) For $a=2^n+1$, 2^n top terms are added. Then, $m=n+1$. The base 2 expansion of $a-1$ has n zeros and so $t_a=2^n$. As before, for any j, $0\leqslant j\leqslant 2^n-1$, $i_j=0$, and $l_j=j$. Thus, $T_a=\{(1, r_a-a-j)\mid 0\leqslant j\leqslant 2^n-1\}$.

Example 5.16. (See Example in [Tha09, p. 2337].) Let $q=2$ and $a=19$. Then, $m=5$, $t_{19}=8$, and $r_{19}=32$. In this case, $i_j=0$, and $l_j=j$ for all j. The j's for which $f_{19,j}\neq 0$ are 0, 1, 4, 5, 8, 9, 12, and 13. The polynomial g_t is $g_t=1+t^{31}+t^{28}+t^{27}+t^{24}+t^{23}+t^{20}+t^{19}$. Therefore,

$$\begin{aligned}\Delta(19,b+32)-\Delta(19,b)&=\sum_{j=0}^{13} f_{19,j}S_1(19+b+j)\\&=S_1(b+19)+S_1(b+20)+S_1(b+23)+S_1(b+24)\\&\quad+S_1(b+27)+S_1(b+28)+S_1(b+31)+S_1(b+32).\end{aligned}$$

The following corollary proves parts 2, 5, and 6 of the main conjecture in [Lar10].

Corollary 5.17. *Let notations be the same as before.*

a) $(1,\phi(0))=(1,r_a-a)\in T_a$.
b) $T_a=\{(1,\phi(0))\}$ *if and only if* $a=p^m$.
c) *If* $a'=p^{m'}a$, *then*

$$T_{a'}=\bigl\{\bigl(f_{a,j},p^{m'}\phi(l_j)\bigr)\bigm| 0\leqslant j\leqslant p^m-a,\ f_{a,j}\neq 0\bigr\}.$$

d) *If* q *is prime,*

$$T_a=\bigl\{\bigl(c_{a,l_j},\phi(l_j)\bigr)\bigm| 0\leqslant j\leqslant p^m-a,\ f_{a,j}\neq 0\bigr\}.$$

Proof. To prove a), just note that $f_{a,0}=1$.

b) When $a=p^m$, then $g_n=1$ and, thus, $T_a=\{(1,r_a-a)\}$. Conversely, since $t_a=1$, by definition of t_a, it follows that $a-1=\sum_{i=0}^{m-1}(p-1)p^i=p^m-1$.

c) Let $a'=p^{m'}a$ for some integer $m'\in\mathbb{Z}_+$. Then, $m+m'$ is the smallest integer such that $a'\leqslant p^{m+m'}$. We have

$$\begin{aligned}-\frac{[1]^{p^{m+m'}}}{n^{p^{m+m'}-a'}}S_1(a')&=\left(-\frac{[1]^{p^m}}{n^{p^m-a}}S_1(a)\right)^{p^{m'}}\\&=(g_n)^{p^{m'}}\\&=\left(1+\sum_{j=1}^{p^m-a}f_{a,j}n^{r_a-(j+i_jp^m)}\right)^{p^{m'}}\\&=1+\sum_{j=1}f_{a,j}^{p^{m'}}n^{p^{m'}(r_a-(j+i_jp^m))}\\&=1+\sum_{j=1}f_{a,j}n^{p^{m'}(r_a-(j+i_jp^m))}.\end{aligned}$$

d) If q is prime, by Proposition 5.9 $f_{a,j}=c_{a,l_j}$ follows. $\square$

Proposition 5.18. *If $\binom{r_a-a}{l(q-1)} \neq 0$ in $\mathbb{F}_p$ for some l, $0 \leqslant l \leqslant j_{a,\max}$, then there exists j, $0 \leqslant j \leqslant p^m - a$, such that $l(q-1) = j + i_j p^m$ and $\binom{p^m-a}{j} \neq 0$.*

Proof. Writing the base p expansion of $a-1$ as $\sum a_k p^k$ as before, we have

$$r_a - a = p^m - a + p^m(q-2) = \sum_{k=0}^{m-1}(p-1-a_k)p^k + (p-2)p^m + \sum_{k=m+1}^{m+s-1}(p-1)p^k.$$

Let $l(q-1) = \sum_{k=0}^{m+s-1} b_k p^k$ be the base p expansion of $l(q-1)$. Since $\binom{r_a-a}{l(q-1)} \neq 0$, by the Lucas theorem, we have $\binom{p-1-a_k}{b_k} \neq 0$ for $k = 0, 1, \ldots, m-1$ and $\binom{p-2}{b_m} \neq 0$. Therefore, $b_k \leqslant p-1-a_k$ for $k = 0, 1, \ldots, m-1$ and $b_m \leqslant p-2$. Let $j = \sum_{k=0}^{m-1} b_k p^k$ and $i = \sum_{k=0}^{s-1} b_{m+k} p^k$. Thus, $0 \leqslant j \leqslant p^m - a$ and $0 \leqslant i \leqslant q-2$. Since $j + ip^m = l(q-1) \equiv 0 \mod (q-1)$, we have $i = i_j$. It follows that $\binom{p^m-a}{j} \neq 0$. $\square$

Remark 5.19.

(1) This proposition proves part 3 of the main conjecture in [Lar10]: If there is no carry over base p in the sum of $l(q-1)$ and $\phi(l)$, then $\binom{r_a-a}{l(q-1)} \neq 0$ and, so, by Proposition 5.18 and Theorem 5.13, $(f_{a,j}, \phi(l_j))$ belongs to T_a.

(2) By Proposition 5.9, if q is prime, $f_{a,j} \neq 0$ if and only if $\binom{r_a-a}{l_j(q-1)} \neq 0$. If q is not prime, we could have $\binom{p^m-a}{j} \neq 0$ and $\binom{r_a-a}{l_j(q-1)} = 0$ (e.g., $q=4$, $a=5$, $j=1$).

Example 5.20. (See [Tha09, Theorems 3 and 7].) We give another proof of Theorems 3 and 7 in [Tha09] using Theorem 5.13. Let $q = 2$.

a) Let $a = 1$. Then $m = 0$, $r_1 = 1$, $g_t = 1$, and $t_1 = 1$. By Theorem 5.13, we have $\Delta(1, b+1) - \Delta(1, b) = S_1(1+b)$. Then, $\Delta(1,2) = \Delta(1,1) + S_1(2) = S_1(2)$. By repeating this process, we get $\Delta(1,3) = \Delta(1,2) + S_1(3) = S_1(2) + S_1(3)$. It follows that $\Delta(1,b) = \sum_{i=2}^{b} S_1(i)$. Therefore, we get

$$\zeta(1)\zeta(b) = \zeta(1+b) + \sum_{i=1}^{b-1}\zeta(i, b+1-i).$$

b) Now let $a = 2$. Then $r_2 = 2$, $g_t = 1$, and $t_2 = 1$. Since $\Delta(2,1) = S_1(2)$ and $\Delta(2,2) = 0$, proceeding as in part a), it follows that

$$\Delta(2,b) = \begin{cases} \sum_{i=1}^{(b-1)/2} S_1(2i+1) + S_1(2) & \text{if } b \text{ is odd,} \\ \sum_{i=2}^{b/2} S_1(2i) & \text{if } b \text{ is even.} \end{cases}$$

6. Closed formulas

Let $a, b \in \mathbb{Z}_+$. By Theorem 4.1, there are $f_i \in \mathbb{F}_p$ and $a_i \in \mathbb{Z}_+$ so that $\Delta(a,b) = \sum_{i=0}^{\delta} f_i S_1(a_i)$. From this, we get the partial fraction decomposition of $\Delta(a,b)$:

$$\Delta(a,b) = \sum_{\mu \in \mathbb{F}_q} \frac{h_\mu(t)}{(t-\mu)^n}, \tag{6.0.1}$$

where $h_\mu(t) = f_0 + f_1(t-\mu)^{a_0-a_1} + \cdots + f_{a_\delta}(t-\mu)^{a_0-a_\delta}$ (we assume that $n = a_0 > a_1 > \cdots > a_\delta$). Uniqueness of the partial fraction decomposition guarantees uniqueness of f_i's.

Conversely, since the denominator of $\Delta(a,b)$ is a power of [1], its partial fraction decomposition is of the form (6.0.1), where $h_\mu(t) \in \mathbb{F}_q[t]$ is relatively prime to $t-\mu$; also, $\deg h_\mu < n$. Now, $\Delta(a,b)$ is invariant with respect to the automorphisms $t \to t+\theta$, $\theta \in \mathbb{F}_q$ of A. By the uniqueness of the partial fraction decomposition, we have $h_0(t) = h_\theta(t+\theta)$ for any $\theta \in \mathbb{F}_q$. Let $h_0(t) = f_0 + f_1 t + \cdots + f_{n-1}t^{n-1}$. Then,

$$\Delta(a,b) = \sum_{\mu \in \mathbb{F}_q} \frac{h_0(t-\mu)}{(t-\mu)^n} = \sum_{i=0}^{n-1} f_i \sum_{\mu \in \mathbb{F}_q} \frac{1}{(t-\mu)^i} = \sum_{i=0}^{n-1} f_i S_1(n-i).$$

By the uniqueness of f_i's and by Theorem 4.1, it follows that $f_i \in \mathbb{F}_p$.

We proceed as follows to find $h_0(t)$. Since $[1]^n/t^n$ is a unit modulo t^n, it follows from (6.0.1) that modulo t^n, we have $[1]^n \Delta(a,b) = \frac{[1]^n}{t^n} h_0(t)$, so that $h_0(t) \bmod t^n = ([1]^n \Delta(a,b) \bmod t^n)(\frac{[1]^n}{t^n} \bmod t^n)^{-1}$. To finish, we pick the unique representative of $h_0(t) \bmod t^n$ of degree less than n.

We first explain the case $q=2$, when we can simplify a lot to get a nice expression, before giving the general case. When $a=b$, we know that $\Delta(a,b)=0$ [Tha09a, Theorem 8]. Since $S(a,b)=S(b,a)$, we can assume, without loss of generality, that $a>b$.

Theorem 6.1. *Let $q=2$. If $a > b \geqslant 1$, then*

$$\Delta(a,b) = \sum_{k=0}^{a-1} f_k S_1(a-k) \quad \textit{where } f_k = \sum_{\substack{i+j=k \\ i \leqslant 2^m - a \\ j \leqslant a-b-1}} \binom{2^m - a}{i}\binom{a-b}{j}.$$

In particular,

$$S(a,b) = \bigl\{ (f_k, a-k) \bigm| f_k \neq 0,\ 0 \leqslant k \leqslant a-1 \bigr\}.$$

Proof. Since $a>b$,

$$\Delta(a,b) = \frac{1}{t^a(t+1)^b} + \frac{1}{(t+1)^a t^b} = \frac{(t+1)^{a-b} + t^{a-b}}{t^a(t+1)^a} = \frac{S_1(b-a)}{[1]^a}.$$

The degree of the numerator of $\Delta(a,b)$ is less than $a-b$ and, thus, less than the degree of the denominator. Next, we apply the method explained above to write $\Delta(a,b)$ as a sum $\frac{h_0(t)}{t^a} + \frac{h_0(t-1)}{(t-1)^a}$. Since $(t+1)^{2^m-a}(t+1)^a \equiv 1 \bmod t^a$ we have that $(t+1)^{2^m-a} \bmod t^a$ is the inverse of $(t+1)^a \bmod t^a$. Then, $h_0(t) \bmod t^a = (t+1)^{2^m-a} S_1(b-a) \bmod t^a$. The only representative of degree less than a of $(t+1)^{2^m-a} \bmod t^a$ is $(t+1)^{2^m-a}$ because $a > 2^{m-1}$. The only representative of degree less than a of $S_1(b-a) \bmod t^a$ is $S_1(b-a)$. Now,

$$(t+1)^{2^m-a} = \sum_{i=0}^{2^m-a} \binom{2^m-a}{i} t^i,$$

$$S_1(b-a) = t^{a-b} + (t+1)^{a-b} = \sum_{j=0}^{a-b-1} \binom{a-b}{j} t^j.$$

The degree of $(t+1)^{2^m-a} S_1(b-a)$ is at most 2^m-b-1. By taking as $h_0(t)$ the remainder of $(t+1)^{2^m-a} S_1(b-a)$ when divided by t^a the theorem follows. $\square$

Remark 6.2.

(1) If $b \geqslant 2^m - a$, then $2^m - b - 1 < a$ and, therefore, it is not necessary to divide by t^a.
(2) If $k = 0$, then $i = j = 0$ and $f_0 = \binom{2^m-a}{0}\binom{a-b}{0} = 1$.

Now we make the recipe in Theorem 4.1 explicit by giving a closed formula for f_i there.

Theorem 6.3. *Let q be arbitrary. Let $a, b \in \mathbb{Z}_+$ such that $a \geqslant b \geqslant 1$; let m the smallest integer such that $a \leqslant p^m$. Then*

$$\Delta(a,b) = \sum_{i=0}^{a-1} f_i S_1(a-i), \tag{6.3.1}$$

where $f(t) := f_0 + \cdots + f_{a-1}t^{a-1} \in \mathbb{F}_p[t]$ is given by

$$-\left(t^{q-1}-1\right)^{p^m-a} \sum_{\theta\in\mathbb{F}_q} \sum_{\mu\in\mathbb{F}_q^*} \left(\sum_{j=1}^{q-1} \mu^{q-1-j}(t+\theta)^{j-1} \right)^a (t+\theta-\mu)^{a-b} \mod t^a.$$

Equivalently, $f(t)$ is given by

$$\sum_{i_3=0}^{p^m-a} \sum_{k} \sum_{i_1=0}^{a-b} \sum_{i_2=0}^{\sigma(k)+i_1-1} \binom{p^m-a}{i_3} \binom{a}{k_1,\ldots,k_{q-1}} \binom{a-b}{i_1} \times \binom{\sigma(k)+i_1}{i_2} (-1)^{b+i_1+i_3} t^{i_2+i_3(q-1)}, \tag{6.3.2}$$

where the second sum extends over all $(q-1)$-tuples $k = (k_1,\ldots,k_{q-1})$ of non-negative integers such that $k_1 + \cdots + k_{q-1} = a$; $\sigma(k) := \sum_{j=2}^{q-1}(j-1)k_j$; $\tau(k) := \sum_{j=1}^{q-2}(q-1-j)k_j$; and i_1, i_2, i_3 are subjected to $\sigma(k)+i_1-i_2$, $\tau(k)+a-b-i_1$ being both 'even', and $i_2 + i_3(q-1) < a$.

Proof. The proof method is the same as the one of Theorem 6.1, but with more combinatorial complications as we deal with any q. We now sketch the steps, omitting the routine calculations. First, the inverse modulo t^a of $[1]^a/t^a$ is $-(t^{q-1}-1)^{p^m-a}$ responsible for the first binomial coefficient. Let $\theta, \mu \in \mathbb{F}_q$, with $\mu \neq 0$. Then raising

$$\frac{[1]}{(t+\theta)(t+\theta-\mu)} = \frac{(t+\theta-\mu)^{q-1}-1}{t+\theta} = \sum_{j=1}^{q-1} \binom{q-1}{j} (t+\theta)^{j-1}(-\mu)^{q-1-j} = \sum_{j=1}^{q-1} \mu^{q-1-j}(t+\theta)^{j-1},$$

to the a-th power by the multinomial theorem brings in $\tau(k), \sigma(k)$, the multinomial coefficient; whereas the multiplication by $(t+\theta-\mu)^{a-b}$ the next binomial coefficient, and also $(t+\theta)^{\sigma(k)+i_1}$ bringing in the last binomial coefficient. Finally, we sum over θ and μ and use the fact that $\sum_{\xi\in\mathbb{F}_q^*} \xi^\ell$ is -1 or 0 as ℓ is 'even' or not, accounting for the conditions. $\square$

Corollary 6.4. *The i's in Eq.* (6.3.1) *are such that $b + i$ is 'even'.*

Proof. Each i is of the form $i_2 + i_3(q-1)$. Since both $\sum_{j=1}^{q-1}(j-1)k_j + i_1 - i_2$ and $\sum_{j=1}^{q-1}(q-1-j)k_j + a - b - i_1$ are 'even', and $\sum_{j=1}^{q-1}(j-1)k_j + \sum_{j=1}^{q-1}(q-1-j)k_j + a = a(q-1)$, it follows that $a(q-1)-(b+i_2)$ is 'even'. Then, $b+i_2$ is 'even', too. Therefore, $b+i = b+i_2+i_3(q-1)$ is 'even'. □

Remark 6.5.

(1) Corollary 6.4 proves the parity conjecture of Thakur [Tha09, 5.3]. We have already proved the parity conjecture by a different method [Lar].
(2) When $q=2$, in Eq. (6.3.2) instead of fours sums and four multinomial coefficients, we have two sums and two binomial coefficients and we reduce to formula for $q=2$.
(3) By the Lucas theorem the multinomial coefficient or the binomial coefficients in (6.3.2) are zero if there is carry over base p in the corresponding sum. So only terms where there is no carry over base p (in the corresponding sums) need to be considered. Similarly, the other conditions and vanishings of binomial coefficients reduce the number of terms in the sum a lot.

Example 6.6. (See [Tha09, Both indices large, p. 2338].) We use Theorem 6.1 to prove two conjectures due to Thakur. Let $q=2$.

a) Let $a = 2^n+1$ and $b = 2^n-1$. Then,

$$\Delta_d(a,b) = \sum_{k=2}^{2^n+1} S_d\big(k, 2^n+1-k\big).$$

b) Let $a = 2^n+1$ and $b = 2^{n-1}$. Then,

$$\Delta_d(a,b) = S_d(a,b) + \sum_{i=k}^{2^{n-1}+1} S_d\big(k, 3\cdot 2^{n-1}+1-k\big).$$

Proof. It is enough to prove the case $d=1$ as it is established in Theorem 4.1.

a) In this case, $m = n+1$, $p^m - a = 2^n-1$, and $a-b-1 = 1$. Then, $f_k = 1$ for $k = 1,\dots,2^n-1$ and $f_{a-1} = 0$ for $k = a-1 = 2^n$. By Theorem 6.1,

$$\Delta\big(2^n+1, 2^n-1\big) = \sum_{k=0}^{2^n-1} S_1\big(2^n+1-k\big).$$

b) Now, $m = n+1$, $p^m - a = 2^n-1$, and $a-b = 2^{n-1}+1$. Looking at the base 2 expansions of j and $2^{n-1}+1$ and using the Lucas theorem, we see that the values of j, $0 \leqslant j \leqslant 2^{n-1}$, such that $\binom{2^{n-1}+1}{j} \neq 0$ are $j = 0, 1, 2^{n-1}$. On the other hand, $\binom{2^n-1}{i} \neq 0$ for $i = 0, 1, \dots, 2^n-1$. We already know that $f_0 = 1$. For $1 \leqslant k \leqslant 2^n-1$, we have

$$f_k = \binom{2^n-1}{k}\binom{2^{n-1}+1}{0} + \binom{2^n-1}{k-1}\binom{2^{n-1}+1}{1} + \binom{2^n-1}{k-2^{n-1}}\binom{2^{n-1}+1}{2^{n-1}}.$$

Therefore, $f_k = 0$ for $1 \leqslant k < 2^{n-1}$ and $f_k = 1$ for $2^{n-1} \leqslant k \leqslant 2^n-1$. For $k = a-1 = 2^n$,

$$f_k = \binom{2^n-1}{k-1}\binom{2^{n-1}+1}{1} + \binom{2^n-1}{k-2^{n-1}}\binom{2^{n-1}+1}{2^{n-1}} = 0.$$

It follows that $S(a,b)=\{(f_0,a-0)\}\cup\{(f_k,a-k)\mid 2^{n-1}\leqslant k\leqslant 2^n-1\}$. Then

$$\Delta(a,b)=S_1\big(2^n+1\big)+\sum_{k=2^{n-1}}^{2^n-1}S_1\big(2^n+1-k\big). \qquad \square$$

The following corollary gives special cases of Theorem 6.1.

Corollary 6.7. *Let $q=2$.*

a) *If $a=2^m$ and $b<2^m$, then*

$$\Delta(2^m,b)=\sum_{k=0}^{2^m-b-1}\binom{2^m-b}{k}S_1(a-k).$$

$\binom{2^m-b}{k}\neq 0$ *if there is no carry over base 2 in the sum of k and $b-1$.*

b) *If $b=a-1$, then*

$$\Delta(a,a-1)=\sum_{k=0}^{2^m-a}\binom{2^m-a}{k}S_1(a-k).$$

The number of non-zero coefficients is t_a.

Proof. a) If $a=2^m$, then $i=0$ and $\binom{2^m-a}{0}=1$. Thus, $f_k=\binom{2^m-b}{k}$ for $0\leqslant k\leqslant 2^m-b-1$. Let $b-1=b_0+b_1 2+\cdots+b_{m-1}2^{m-1}$ and $k=k_0+k_1 2+\cdots+k_{m-1}2^{m-1}$ be the base p expansions of $b-1$ and k, respectively. Then the base 2 expansion of 2^m-b is $\sum(1-b_l)p^l$. If $k\leqslant 2^m-b-1$ and there is no carry over base 2 in the sum of k and $b-1$, then $f_k\neq 0$ because of the Lucas theorem.

b) If $b=a-1$, then $a-b-1=0$ and, so, $j=0$. Therefore, $f_k=\binom{2^m-a}{k}$ for $0\leqslant k\leqslant 2^m-a$. In this special case, the number of $f_k\neq 0$ is t_a because of Proposition 5.5(c). $\square$

Example 6.8. We use the corollary to prove the following conjectures for $q=2$ [Tha09, Section 4.1.3]:

$$\Delta_d\big(2^n,2^n-1\big)=S_d\big(2^n,2^n-1\big) \tag{6.8.1}$$

$$\Delta_d\big(2^n+1,2^n\big)=\sum_{i=2}^{2^n+1}S_d\big(i,2^{n+1}+1-i\big). \tag{6.8.2}$$

When $l<n-1$,

$$\Delta_d\big(2^n-2^l,2^n-2^l-1\big)=S_d\big(2^n-2^l,2^n-2^l-1\big)+S_d\big(2^n-2^{l+1},2^n-1\big). \tag{6.8.3}$$

In all cases, it is enough to prove for $d=1$. Conjecture (6.8.1) follows immediately either from a) or b) of Corollary 6.7, taking $a=2^n$ and $b=a-1$. For Conjecture (6.8.2), take $a=2^n+1$ and $b=2^n$. Then $m=n+1$. By Proposition 5.9(b), we have $\binom{2^n-1}{k}=1$ for $k=0,1,\ldots,2^n-1$; so,

$$\Delta\big(2^n+1,2^n\big)=\sum_{k=0}^{2^n-1}S_1\big(2^n+1-k\big)=\sum_{k=2}^{2^n+1}S_1(k).$$

Finally, for Conjecture (6.8.3), let $a = 2^n - 2^l$ and $b = a - 1$. Then, $m = n$, $p^m - a = 2^l$ and, thus,

$$\begin{aligned}\Delta(a,b) &= \sum_{k=0}^{2^l} \binom{2^l}{k} S_1(a-k) \\ &= S_1\bigl(2^n - 2^l\bigr) + S_1\bigl(2^n - 2^l - 2^l\bigr) + \sum_{k=1}^{2^l-1} \binom{2^l}{k} S_1(a-k).\end{aligned}$$

The 2-adic valuation of $\binom{2^l}{k}$ is $\ell(k) + \ell(2^l - k) - \ell(2^l)$, where $\ell(k)$ is the sum of the digits of k base $q = 2$. Since $k \neq 0$ and $2^l - k \neq 0$, we have that $\ell(k) + \ell(2^l - k) - \ell(2^l) \geqslant 1 + 1 - 1 = 1$ and $\binom{2^l}{k} = 0$ for $k = 1, \dots, 2^l - 1$ and the result follows.

Example 6.9. (See [Lar10, Conjecture 2.8.1].) Now we prove that

$$\Delta_d\bigl(q^n, q^n - 1\bigr) = -S_d\bigl(q^n\bigr), \tag{6.9.1}$$

for any q, which is a generalization of (6.8.1). In [Lar], we have already proved (6.9.1). Here, we prove it again, as a corollary of Theorem 6.3. By Theorem 4.2, it is enough to prove the case $d = 1$. Let $a = q^n$, $b = a - 1$, and $N = 1 + (q-2)q^n$. Then, $p^m - a = 0$, and the polynomial $f(t)$ of Theorem 6.3 becomes

$$\sum_{\theta \in \mathbb{F}_q} (t+\theta)^N = -1 - \sum_{0<l<\frac{N}{q-1}} \binom{N}{l(q-1)} t^{N-l(q-1)} \theta^l.$$

Taking a as $q^n - 1$ in Proposition 5.18, it follows that $\binom{N}{l(q-1)} = 0$ for $0 < l < N/(q-1)$. Also, it is easy to see that the result follows from the comparison of the coefficients of x^j in the identity $(1+x)^N = (1+x)(1+x^{q^n})^{q-2}$, which is valid over $\mathbb{F}_q$ (the author thanks the referee for pointing this out).

Example 6.10. We continue with Example 5.16. Let us evaluate $S(19, 20)$. Let $a = 20$ and $b = 19$. Then $m = 5$ and $a - b - 1 = 0$. Therefore, $f_k = \binom{12}{k}$ for $k = 0, 1, \dots, 12$. The k's for which $f_k \neq 0$ are 0, 4, 8 and 12. Then $S(19, 20) = S(20, 19) = \{(1, 20), (1, 16), (1, 12), (1, 8)\}$. The size of $S(19, 20)$ is $4 = t_{20}$.

7. Symmetric closed formulas

In this section we give a symmetric closed formula for the f_i in Theorem 4.1.

Theorem 7.1. *Let q be arbitrary. Let $a, b \in \mathbb{Z}_+$; let m be the smallest integer such that $a + b \leqslant p^m$. Then*

$$\Delta(a,b) = \sum_{i=0}^{b-1} f_i S_1(b-i) + \sum_{j=0}^{a-1} g_j S_1(a-j),$$

where $f(t) = H_{a,b}(t) = f_0 + f_1 t + \cdots + f_{b-1} t^{b-1}$, $g(t) = H_{b,a}(t) = g_0 + g_1 t + \cdots + g_{a-1} t^{a-1}$, $H_{a,b}(t)$ is given by

$$H_{a,b}(t) = \frac{1}{t^a} \left(-\bigl(t^{q-1} - 1\bigr)^{p^m - a} \sum_{\theta \in \mathbb{F}_q^*} \bigl((t+\theta)^{q-1} - 1\bigr)^a \bmod t^{a+b} \right),$$

and $H_{b,a}$ is obtained interchanging a and b. Equivalently,

$$H_{a,b}(t)=\sum_{j=0}^{p^m-a}\sum_{k}\binom{p^m-a}{j}\binom{a}{k_1,\dots,k_{q-1}}(-1)^{a+j+1}t^{j(q-1)+\sigma(k)-a}, \tag{7.1.1}$$

where the second sum runs over all $(q-1)$-tuples $k=(k_1,\dots,k_{q-1})$ of non-negative integers such that $k_1+\dots+k_{q-1}=a$ and $\sigma(k)$ is 'even', where $\sigma(k):=\sum_{j=1}^{q-1}jk_{j-1}$ and j and $\sigma(k)$ are subject to $j(q-1)+\sigma(k)<a+b$.

Proof. Following the same steps as in the proof of Theorem 6.1, it is easy to see that $\Delta(a,b)=\sum_{i=0}^{a+b-1}p_iS_1(a+b-i)$, where $P(t)=p_0+p_1t+\dots+p_{a+b-1}t^{a+b-1}\in\mathbb{F}_p[t]$ is given by

$$P(t)=-\frac{[1]^{p^m-a-b}}{t^{p^m-a-b}}[1]^{a+b}\Delta(a,b)\mod t^{a+b}.$$

Let us write $P(t)=P_a(t)+P_b(t)+P_{a,b}(t)$, where $P_a(t)=-\frac{[1]^{p^m-a}}{t^{p^m-a}}\sum_{\theta\in\mathbb{F}_q^*}\frac{[1]^a}{(t+\theta)^a}$ and $P_{a,b}(t)=-\frac{[1]^{p^m-a-b}}{t^{p^m-a-b}}\sum_{\theta,\mu\in\mathbb{F}_q^*,\theta\neq\mu}\frac{[1]^a}{(t+\theta)^a}\frac{[1]^b}{(t+\mu)^b}$. Since $v_t(P_{a,b}(t))\geqslant a+b$, where $v_t(\cdot)$ is the valuation at t, we have $P(t)\equiv P_a(t)+P_b(t)$ modulo t^{a+b}. Note that $H_{a,b}(t)=(1/t^a)(P_a(t)\mod t^{a+b})$ (observe that $v_t(P_a(t))\geqslant a$) and that $H_{b,a}(t)$ is obtained by interchanging a and b. Suppose for example that $\min\{a,b\}=a$. Then $P_a(t)+P_b(t)$ modulo t^{a+b} equals

$$t^af(t)+t^bg(t)=\sum_{i=a}^{b-1}f_{i-a}t^i+\sum_{i=b}^{a+b-1}(f_{i-a}+g_{i-b})t^i.$$

Therefore,

$$\Delta(a,b)=\sum_{i=0}^{b-1}f_iS_1(b-i)+\sum_{i=0}^{a-1}g_iS_1(a-i).$$

Now we compute $P_a(t)$. The factor $-[1]^{p^m-a}/t^{p^m-a}=-(t^{q-1}-1)^{p^m-a}$ brings in the binomial coefficient. Let $\theta\in\mathbb{F}_q^*$. Then, raising $[1]/(t+\theta)=\sum_{i=1}^{q-1}(-t)^i\theta^{q-1-i}$ to the a-th power, by the multinomial theorem, brings in $\sigma(k)$ and the multinomial coefficient. Summing over $\theta\in\mathbb{F}_q^*$ and using the fact that $\sum_{\xi\in\mathbb{F}_q^*}\xi^\ell$ is -1 or 0, depending on ℓ being 'even' or not, we get

$$P_a(t)=\sum_{j=0}^{p^m-a}\sum_{\substack{k_1+\dots+k_{q-1}=a\\ q-1\mid\sigma(k)}}\binom{p^m-a}{j}\binom{a}{k_1,\dots,k_{q-1}}(-1)^{a+j+1}t^{j(q-1)+\sigma(k)}.$$

Since $H_{a,b}(t)$ is $1/t^a$ times the remainder of $P_a(t)$ when divided by t^{a+b}, we see that $H_{a,b}(t)$ is exactly as claimed. $\square$

When $q=2$, we get the following compact formula.

Corollary 7.2. *Let $q = 2$. Let $a, b \in \mathbb{Z}_+$ and let m be the smallest integer such that $a + b \leqslant 2^m$. Then*

$$\Delta(a, b) = \sum_{j=0}^{b-1} \binom{2^m - a}{j} S_1(b - j) + \sum_{i=0}^{a-1} \binom{2^m - b}{i} S_1(a - i).$$

Proof. For $q = 2$, Eq. (7.1.1) becomes $H_{a,b}(t) = \sum_{j=0}^{b-1} \binom{p^m - a}{j} t^j$. $\square$

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