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The representation type of Ariki–Koike algebras and cyclotomic q-Schur algebras

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Abstract

We give a necessary and sufficient condition on parameters for Ariki–Koike algebras (resp. cyclotomic q-Schur algebras) to be of finite representation type.

Keywords: Representation type; Cyclotomic q-Schur algebra; Ariki–Koike algebra; Hecke algebra

0. Introduction

Let F be an algebraically closed field, and $\mathcal{A}$ be a finite dimensional associative algebra over F. We say that $\mathcal{A}$ is of finite representation type (simply, finite type) if there are only a finite number of isomorphism classes of indecomposable $\mathcal{A}$-modules, and that $\mathcal{A}$ is of infinite representation type (infinite type) otherwise. Moreover, the infinite representation type has two classes, namely, tame type and wild type. $\mathcal{A}$ is of tame type if indecomposable modules in each dimension come in one parameter families with finitely many exceptions. $\mathcal{A}$ is of wild type if its module category is comparable with that of the free algebra in two variables. For precise definitions, see [10] or [5]. By Drozd's theorem, it is known that any finite dimensional algebra has finite type, tame type or wild type.

We consider the representation type of the Ariki–Koike algebra $\mathscr{H}_{n,r} = \mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ over F with parameters $q, Q_1, \dots, Q_r \in F$ and of the cyclotomic q-Schur algebra

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$\mathscr{S}_{n,r} = \mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ associated to $\mathscr{H}_{n,r}$. In the case where $r = 1$, $\mathscr{S}_{n,1}$ is the q-Schur algebra, and the representation type of $\mathscr{S}_{n,1}$ has been determined by Erdmann and Nakano [13]. On the other hand, the representation type of Hecke algebras of classical type has been determined by Uno [23], Erdmann and Nakano [12], Ariki and Mathas [4] and Ariki [3]. In this paper, we will give a necessary and sufficient condition (here, we denote this condition by (CF)) for $\mathscr{H}_{n,r}$ and $\mathscr{S}_{n,r}$ to be of finite type (Theorem 3.13).

Suppose that $\mathscr{S}_{n,r} = \mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ satisfies the condition (CF). In order to show the finiteness, we will see that any block of $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ $(r \geqslant 3)$ is Morita equivalent to a certain block of $\mathscr{S}_{n',2}(q, Q_i, Q_j)$ for some $i, j \in \{1, \dots, r\}$ (Corollary 3.20). Thus, the finiteness of $\mathscr{S}_{n,r}$ $(r \geqslant 3)$ is reduced to the case where $r = 2$. For the case where $r = 2$, the finiteness is shown in a similar way as in the case of Hecke algebras of type B [4, Theorem 6.2]. The finiteness for $\mathscr{H}_{n,r}$ follows from the finiteness for $\mathscr{S}_{n,r}$ (see Lemma 2.6).

On the other hand, suppose that $\mathscr{H}_{n,r}$ does not satisfy the condition (CF). In order to show the infiniteness, we will make use of some properties of the structures of $\mathscr{H}_{n,r}$ which follows from the structures of $\mathscr{S}_{n,r}$ obtained in [22,24]. By adding some facts to results in [22,24], we have the following picture (Theorem 2.10).

$$\begin{array}{ccccccc}
\bigoplus_{\eta=(n_1,\dots,n_g)} e_\eta \mathscr{S}_{n,r} e_\eta & \hookrightarrow & \mathscr{S}^{\mathbf{p}} & \hookrightarrow & \widetilde{\mathscr{S}}^{\mathbf{p}} & \hookrightarrow & \mathscr{S}_{n,r} \\
\big\downarrow & & & & & & \\
\overline{\mathscr{S}}^{\mathbf{p}} \cong \bigoplus_{\eta=(n_1,\dots,n_g)} \mathscr{S}_{n_1,r_1}(q, \mathbf{Q}_1) \otimes \mathscr{S}_{n_2,r_2}(q, \mathbf{Q}_2) \otimes \cdots \otimes \mathscr{S}_{n_g,r_g}(q, \mathbf{Q}_g). & & & & & &
\end{array}$$

$$\mathscr{S}^{\mathbf{p}} \twoheadrightarrow \overline{\mathscr{S}}^{\mathbf{p}}, \qquad \widetilde{\mathscr{S}}^{\mathbf{p}} \twoheadrightarrow \overline{\mathscr{S}}^{\mathbf{p}}$$

This implies the surjective homomorphism

$$\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r) \to \mathscr{H}_{n,k}(q, Q_{r-k+1}, \dots, Q_r)$$

for $k = 1, \dots, r-1$ (Proposition 2.12). By using this surjection, the infiniteness of $\mathscr{H}_{n,r}$ is reduced to some special cases, namely the case where $r = 2$ (in [4]), Proposition 3.21 and Proposition 3.22. Finally, the infiniteness for $\mathscr{S}_{n,r}$ follows from the infiniteness for $\mathscr{H}_{n,r}$ (Lemma 2.6).

1. The representation type of algebras

Throughout this paper, we suppose that F is an algebraically closed field, and any algebra $\mathcal{A}$ is a finite dimensional unital associative algebra over F. We say just an $\mathcal{A}$-module for a right $\mathcal{A}$-module. The following results for the representation type are well known (see [10]).

Lemma 1.1. *Let $\mathcal{A}$ be an F-algebra.*

- (i) *Let I be a two-sided ideal of $\mathcal{A}$. If $\mathcal{A}/I$ is of infinite (resp. wild) type then $\mathcal{A}$ is also of infinite (resp. wild) type.*
- (ii) *Let e be an idempotent of $\mathcal{A}$. If $e\mathcal{A}e$ is of infinite (resp. wild) type then $\mathcal{A}$ is also of infinite (resp. wild) type.*
- (iii) *If an idempotent $e \in \mathcal{A}$ is primitive then $\operatorname{End}_{\mathcal{A}}(e\mathcal{A}) \cong e\mathcal{A}e$ is local. Moreover, $\operatorname{End}_{\mathcal{A}}(e\mathcal{A})$ is of finite type if and only if $\operatorname{End}_{\mathcal{A}}(e\mathcal{A}) \cong F[x]/\langle x^m \rangle$ for some integer $m \geqslant 0$, where $F[x]$ is*

a polynomial ring over F with an indeterminate x, and $\langle x^m \rangle$ is the ideal generated by the polynomial x^m.

(iv) *Let $P_1, \dots, P_k$ be the complete set of non-isomorphic projective indecomposable $\mathcal{A}$-modules. Then $\mathcal{A}$ is Morita equivalent to $\mathrm{End}_{\mathcal{A}}(P_1 \oplus \cdots \oplus P_k)$. Thus if $\mathrm{End}_{\mathcal{A}}(P_i)$ is of infinite type (resp. wild) for some i, then $\mathcal{A}$ is of infinite (resp. wild) type.*

1.2. *Cellular algebras.* A cyclotomic q-Schur algebra is a cellular algebra in the sense of [15]. So, we give some fundamental properties of cellular algebras which we will be needed in later discussions. For more details for cellular algebras, see [15] or [20].

Let $\mathscr{A}$ be a cellular algebra over F with respect to a poset $(\Lambda^+, \geqslant)$ and an algebra anti-automorphism $*$ of $\mathscr{A}$. Then we can define a cell module W^λ for each $\lambda \in \Lambda^+$. Let $\operatorname{rad} W^\lambda$ be the radical of W^λ with respect to the canonical bilinear form on W^λ. Put $L^\lambda = W^\lambda / \operatorname{rad} W^\lambda$. Since $\operatorname{rad} W^\lambda$ is an $\mathscr{A}$-submodule of W^λ, L^λ is also an $\mathscr{A}$-module. Set $\Lambda_0^+ = \{\lambda \in \Lambda^+ \mid L^\lambda \neq 0\}$. Note that, for $\lambda \in \Lambda_0^+$, $\operatorname{rad} W^\lambda$ coincides with the Jacobson radical of W^λ. Then $\{L^\lambda \mid \lambda \in \Lambda_0^+\}$ is a complete set of non-isomorphic simple $\mathscr{A}$-modules. It is known that a cellular algebra $\mathcal{A}$ is a quasi-hereditary algebra if and only if $\Lambda_0^+ = \Lambda^+$.

For $\lambda \in \Lambda^+$ and $\mu \in \Lambda_0^+$, let $d_{\lambda\mu} = [W^\lambda : L^\mu]$ be the decomposition number, namely the multiplicity of L^μ in the composition series of W^λ. If $d_{\lambda\mu} \neq 0$ then $\lambda \geqslant \mu$. The decomposition matrix of $\mathscr{A}$ is a matrix $D = (d_{\lambda\mu})_{\lambda \in \Lambda^+, \mu \in \Lambda_0^+}$. For $\lambda \in \Lambda_0^+$, let P^λ be the projective cover of L^λ. The Cartan matrix of $\mathscr{A}$ is a matrix $C = (p_{\lambda\mu})_{\lambda, \mu \in \Lambda_0^+}$, where $p_{\lambda\mu} = \dim_F \mathrm{Hom}_{\mathscr{A}}(P^\lambda, P^\mu) = [P^\mu : L^\lambda]$. It is known that

$$C = {}^t D D. \tag{1.2.1}$$

Moreover, for $\lambda \in \Lambda_0^+$, P^λ has a cell module filtration in which each cell module W^μ occurs with multiplicity $d_{\mu\lambda}$. The following properties are well known (see [13, 2.5]).

Lemma 1.3. *Let $\mathscr{A}$ be a cellular algebra with respect to a poset $(\Lambda^+, \geqslant)$.*

(i) *L^λ $(\lambda \in \Lambda_0^+)$ is self-dual. Thus we have, for $\lambda, \mu \in \Lambda_0^+$,*

$$\mathrm{Ext}^i_{\mathscr{A}}(L^\lambda, L^\mu) \cong \mathrm{Ext}^i_{\mathscr{A}}(L^\mu, L^\lambda) \quad \textit{for any } i \geqslant 0.$$

In particular,

$$\begin{aligned} \mathrm{Ext}^1_{\mathscr{A}}(L^\lambda, L^\mu) &\cong \mathrm{Hom}_{\mathscr{A}}(\operatorname{rad} P^\lambda / \operatorname{rad}^2 P^\lambda, L^\mu) \\ &\cong \mathrm{Hom}_{\mathscr{A}}(\operatorname{rad} P^\mu / \operatorname{rad}^2 P^\mu, L^\lambda) \\ &\cong \mathrm{Ext}^1_{\mathscr{A}}(L^\mu, L^\lambda). \end{aligned}$$

(ii) *If $\mathscr{A}$ is a quasi-hereditary algebra, then for $\lambda, \mu \in \Lambda^+$ such that $\lambda \geqslant \mu$, we have*

$$\begin{aligned} \mathrm{Ext}^1_{\mathscr{A}}(L^\lambda, L^\mu) &\cong \mathrm{Hom}_{\mathscr{A}}(\operatorname{rad} P^\lambda / \operatorname{rad}^2 P^\lambda, L^\mu) \\ &\cong \mathrm{Hom}_{\mathscr{A}}(\operatorname{rad} W^\lambda / \operatorname{rad}^2 W^\lambda, L^\mu). \end{aligned}$$

Moreover, if $\mathrm{Ext}^1_{\mathscr{A}}(L^\lambda, L^\mu) \neq 0$ then $\lambda > \mu$ or $\mu > \lambda$.

1.4. *Tensor products and representation type.* For two cellular algebras $\mathscr{A}$ and $\mathscr{B}$, the tensor product $\mathscr{A} \otimes_F \mathscr{B}$ becomes a cellular algebra again in the natural way. For the representation type of $\mathscr{A} \otimes_F \mathscr{B}$, we have the following lemma.

Lemma 1.5. *Let $\mathscr{A}$, $\mathscr{B}$ be cellular algebras.*

(i) *If $\mathscr{B}$ is semisimple, then the representation type of $\mathscr{A} \otimes_F \mathscr{B}$ coincides with the representation type of $\mathscr{A}$.*
(ii) *If neither $\mathscr{A}$ nor $\mathscr{B}$ are semisimple, then $\mathscr{A} \otimes_F \mathscr{B}$ is of infinite type.*
(iii) *If neither $\mathscr{A}$ nor $\mathscr{B}$ are semisimple, and $\mathscr{A}$ contains a block which has at least three non-isomorphic simple modules as composition factors, then $\mathscr{A} \otimes_F \mathscr{B}$ is of wild type.*

Proof. (i) is clear. We show only (iii) since (ii) is proven in a similar way.

In order to apply [3, Lemma 17] to $\mathscr{A} \otimes_F \mathscr{B}$, we consider the Gabriel quiver of $\mathscr{A} \otimes_F \mathscr{B}$. By assumption and Lemma 1.3(i), the Gabriel quiver of $\mathscr{A}$ contains the quiver $\bullet \rightleftarrows \bullet \rightleftarrows \bullet$, and the Gabriel quiver of $\mathscr{B}$ contains the quiver $\bullet \rightleftarrows \bullet$ as a subquiver. Thus, by [17, Lemma 1.3], the Gabriel quiver of $\mathscr{A} \otimes_F \mathscr{B}$ contains the quiver

$$\begin{array}{ccccc} \bullet & \rightleftarrows & \bullet & \rightleftarrows & \bullet \\ \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\ \bullet & \rightleftarrows & \bullet & \rightleftarrows & \bullet \end{array}$$

as subquiver. Thus $\mathscr{A} \otimes_F \mathscr{B}$ is of wild type by [3, Lemma 17]. □

1.6. Now we study a particular algebra defined by a quiver and relations. Let m be a positive integer, and let Q be a quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{m-2}}{\overset{\alpha_{m-2}}{\rightleftarrows}} m-1 \underset{\beta_{m-1}}{\overset{\alpha_{m-1}}{\rightleftarrows}} m$$

and $\mathcal{I}$ be the two-sided ideal of the path algebra FQ generated by the relations

$$\alpha_{m-1}\beta_{m-1} = 0, \qquad \alpha_{i+1}\alpha_i = 0,$$
$$\beta_i\beta_{i+1} = 0, \qquad \alpha_i\beta_i = \beta_{i+1}\alpha_{i+1} \quad \text{for } 1 \leqslant i \leqslant m-2,$$

where we denote by $\alpha_{i+1}\alpha_i$ the path $i \xrightarrow{\alpha_i} i+1 \xrightarrow{\alpha_{i+1}} i+2$, etc. We define $\mathcal{A}_m = FQ/\mathcal{I}$. Under the natural surjection $FQ \to \mathcal{A}_m$, we denote the image of paths in FQ by the same symbol. By definition, $\mathcal{A}_m$ has an F-basis

$$e_1, \quad \begin{array}{cccccccc} e_2, & \beta_1, & e_3, & \beta_2, & & e_m, & \beta_{m-1}, \\ \alpha_1, & \alpha_1\beta_1, & \alpha_2, & \alpha_2\beta_2, & \cdots & \alpha_{m-1}, & \alpha_{m-1}\beta_{m-1}, \end{array}$$

where e_i is the path of length 0 on the vertex i. It is known that $\mathcal{A}_m$ is of finite type by [11, 3.1]. The following proposition was inspired by [11, Proposition 3.2], and will be used to prove the finiteness of cyclotomic q-Schur algebras.

Proposition 1.7. *Let $\mathscr{A}$ be a cellular algebra with respect to the poset $(\Lambda^+, \geqslant)$. If $\mathscr{A}$ is a quasi-hereditary algebra with decomposition matrix*

$$D = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 1 \end{pmatrix} \quad (\textit{all omitted entries are zero}),$$

and if any projective indecomposable $\mathscr{A}$-module has the simple socle, then $\mathscr{A}$ is Morita equivalent to $\mathcal{A}_m$ with $m = |\Lambda^+|$. In particular, $\mathscr{A}$ is of finite type.

Proof. Let $\Lambda^+ = \{\lambda_1, \ldots, \lambda_m\}$ such that $i < j$ if $\lambda_i < \lambda_j$. We denote the simple $\mathscr{A}$-module by L^{λ_i}, and its projective cover by P^{λ_i} for $\lambda_i \in \Lambda^+$. By (1.2.1), the Cartan matrix of $\mathscr{A}$ is

$$C = {}^t D D = \begin{pmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 1 \end{pmatrix} \quad (\text{all omitted entries are zero}).$$

Combined with the second isomorphism in Lemma 1.3(ii), we have

$$P^{\lambda_1} = \begin{matrix} L^{\lambda_1} \\ L^{\lambda_2} \\ L^{\lambda_1} \end{matrix}, \qquad P^{\lambda_i} = \begin{matrix} L^{\lambda_i} \\ L^{\lambda_{i-1}} \oplus L^{\lambda_{i+1}} \\ L^{\lambda_i} \end{matrix} \quad \text{for } i = 2, \ldots, m-1, \qquad P^{\lambda_m} = \begin{matrix} L^{\lambda_m} \\ L^{\lambda_{m-1}} \end{matrix}, \tag{1.7.1}$$

where the ith row in the right-hand side of equation means the ith radical layer of P^{λ_i}. We also have

$$\dim_F \operatorname{Hom}_{\mathscr{A}}\left(P^{\lambda_i}, P^{\lambda_j}\right) = \begin{cases} 1 & \text{if } |i-j| = 1 \text{ or } i = j = m, \\ 2 & \text{if } i = j = 1, \ldots, m-1, \\ 0 & \text{if } |i-j| > 1. \end{cases} \tag{1.7.2}$$

Let $\mathscr{A}' = \operatorname{End}_{\mathscr{A}}(\bigoplus_{i=1}^m P^{\lambda_i})$. Then $\mathscr{A}$ is Morita equivalent to $\mathscr{A}'$. Let $e'_i \in \mathscr{A}'$ be the identity map on P^{λ_i} and be the 0-map on P^{λ_j} $(i \neq j)$ for $i = 1, \ldots, m$.

For $i = 2, \ldots, m$, let M^{λ_i} and N^{λ_i} be the $\mathscr{A}$-modules such that

$$M^{\lambda_i} = \begin{matrix} L^{\lambda_{i-1}} \\ L^{\lambda_i} \end{matrix}, \qquad N^{\lambda_i} = \begin{matrix} L^{\lambda_i} \\ L^{\lambda_{i-1}} \end{matrix}.$$

Note that any projective indecomposable $\mathcal{A}$-module has the simple socle. Then, by (1.7.1), one can take the natural surjective homomorphism $\varphi_i : P^{\lambda_i} \to M^{\lambda_{i+1}}$ for $i = 1, \ldots, m-2$, and take the injective homomorphism $\psi_i : M^{\lambda_i} \to P^{\lambda_i}$ for $i = 2, \ldots, m-1$. Put $\alpha'_i = \psi_{i+1} \circ \varphi_i \in \operatorname{Hom}_{\mathcal{A}}(P^{\lambda_i}, P^{\lambda_{i+1}})$ for $i = 1, \ldots, m-2$. We also define $\alpha'_{m-1} \in \operatorname{Hom}_{\mathcal{A}}(P^{\lambda_{m-1}}, P^{\lambda_m})$ by the composition of the natural surjection $P^{\lambda_{m-1}} \to L^{\lambda_{m-1}}$ and the injection $L^{\lambda_{m-1}} \to P^{\lambda_m}$. Similarly, one can define $\beta'_i \in \operatorname{Hom}_{\mathcal{A}}(P^{\lambda_{i+1}}, P^{\lambda_i})$ for $i = 1, \ldots, m-1$ such that $\operatorname{Im} \beta'_i = N^{\lambda_{i+1}}$.

We regard α'_i (resp. β'_i) as an element of $\mathscr{A}'$ by $\alpha'_i(P^{\lambda_j}) = 0$ (resp. $\beta'_i(P^{\lambda_j}) = 0$) for any $j \neq i$ (resp. $j \neq i+1$). Then we have $\operatorname{Im} \beta'_i \alpha'_i = L^{\lambda_i}$ for $i = 1, \dots, m-1$, $\operatorname{Im} \alpha'_i \beta'_i = L^{\lambda_{i+1}}$ for $i = 1, \dots, m-2$ and $\operatorname{Im} \alpha'_{m-1} \beta'_{m-1} = 0$. Since $\dim_F \operatorname{Hom}_{\mathscr{A}}(P^{\lambda_i}, L^{\lambda_i}) = 1$, we have $\alpha'_i \beta'_i = \beta'_{i+1} \alpha'_{i+1}$ for $i = 1, \dots, m-2$ by multiplying α'_i by a scalar if necessary. Moreover, we have $\alpha'_{i+1} \alpha'_i = 0$ and $\beta'_i \beta'_{i+1} = 0$ since $\operatorname{Hom}_{\mathscr{A}}(P^{\lambda_i}, P^{\lambda_j}) = 0$ for $|i-j| > 1$. Now we can define a surjective homomorphism of algebras from $\mathcal{A}_m$ to $\mathscr{A}'$ by $X \mapsto X'$ ($X \in \{e_i, e_m, \alpha_i, \beta_i \mid i = 1, \dots, m-1\}$), and we see that this gives an isomorphism by comparing the dimensions. □

The following lemma will be used to prove the infiniteness of Ariki–Koike algebras.

Lemma 1.8. *Let $\mathcal{A}$ be a cellular algebra with respect to the poset $\Lambda^+ = \{\lambda_0, \lambda_1, \lambda_2\}$. If $\mathcal{A}$ has the following decomposition matrix (thus, $\mathcal{A}$ is a quasi-hereditary)*

	L^{λ_0}	L^{λ_1}	L^{λ_2}
W^{λ_0}	1	0	0
W^{λ_1}	a	1	0
W^{λ_2}	b	0	1

$(a, b > 0)$,

then $\mathcal{A}$ is of infinite type.

Proof. From the decomposition matrix, we see that W^{λ_1} (resp. W^{λ_2}) has the unique simple top L^{λ_1} (resp. L^{λ_2}), and that any other composition factor of W^{λ_1} (resp. W^{λ_2}) is isomorphic to L^{λ_0}. By the general theory of cellular algebras, P^{λ_0} has the filtration $P^{\lambda_0} = P_0 \supsetneq P_1 \supsetneq P_2 \supsetneq 0$ such that $P_0/P_1 \cong W^{\lambda_0} \cong L^{\lambda_0}$, $P_1/P_2 \cong (W^{\lambda_1})^{\oplus a}$ and $P_2 \cong (W^{\lambda_2})^{\oplus b}$. This filtration implies that L^{λ_1} appears exactly a times in the second radical layer of P^{λ_0}. On the other hand, by Lemma 1.3, we see that L^{λ_2} appears at least once in the second radical layer of P^{λ_0}. Thus, L^{λ_0} appears at least twice in the third radical layer of P^{λ_0}. This implies that $\operatorname{End}_{\mathcal{A}}(P^{\lambda_0}/\operatorname{rad}^4 P^{\lambda_0})$ is not isomorphic to $F[x]/\langle x^m \rangle$ for any $m \geqslant 0$. Combining with Lemma 1.1, we have that $\mathcal{A}/\operatorname{rad}^4 \mathcal{A}$ is of infinite type, thus $\mathcal{A}$ is also of infinite type. □

2. Ariki–Koike algebras and cyclotomic q-Schur algebras

In this section, we introduce Ariki–Koike algebras and cyclotomic q-Schur algebras. We also give some properties of them.

2.1. A composition $\mu = (\mu_1, \mu_2, \dots)$ is a finite sequence of non-negative integers, and $|\mu| = \sum_i \mu_i$ is called the size of μ. If $\mu_l \neq 0$ and $\mu_k = 0$ for any $k > l$, then $l = \ell(\mu)$ is called the length of μ. If the composition λ is a weakly decreasing sequence, λ is called a partition. An r-tuple $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ of compositions is called an r-composition, and the size $|\mu|$ of μ is defined by $|\mu| = \sum_{i=1}^r |\mu^{(i)}|$. In particular, if all $\mu^{(i)}$ are partitions, μ is called an r-partition. For $n, r \in \mathbb{Z}_{>0}$ and an r-tuple $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r_{>0}$, we denote by $\Lambda_{n,r}(\mathbf{m})$ the set of r-compositions $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ such that $|\mu| = n$ and that $\ell(\mu^{(k)}) \leqslant m_k$ for $k = 1, \dots, r$. We define $\Lambda^+_{n,r}(\mathbf{m})$ as the subset of Λ consisting of r-partitions. Throughout this paper, we assume the following condition for $\Lambda_{n,r}(\mathbf{m})$.

(CL) $m_i \geqslant n$ for any $i = 1, \dots, r$.

Under this condition, $\Lambda^+_{n,r}(\mathbf{m})$ coincides with the set of r-partitions of size n. In particular, $\Lambda^+_{n,r}(\mathbf{m})$ is independent of a choice of $\mathbf{m}$ satisfying (CL). Thus, we write it simply as $\Lambda^+_{n,r}$ instead of $\Lambda^+_{n,r}(\mathbf{m})$. Similarly, we may write $\Lambda_{n,r}(\mathbf{m})$ simply as $\Lambda_{n,r}$ if there is no fear of confusion.

For $\mu \in \Lambda_{n,r}(\mathbf{m})$, the diagram of μ is the set

$$[\mu] = \{(i, j, k) \in \mathbb{N} \times \mathbb{N} \times \{1, 2, \dots, r\} \mid 1 \leqslant j \leqslant \mu_i^{(k)}\}.$$

We call an element of $[\mu]$ a node.

We define a partial order, the so-called "dominance order", on $\Lambda_{n,r}(\mathbf{m})$ by $\mu \trianglerighteq \nu$ if and only if

$$\sum_{i=1}^{l-1} |\mu^{(i)}| + \sum_{j=1}^{k} \mu_j^{(l)} \geqslant \sum_{i=1}^{l-1} |\nu^{(i)}| + \sum_{j=1}^{k} \nu_j^{(l)}$$

for any $1 \leqslant l \leqslant r$, $1 \leqslant k \leqslant m_l$. If $\mu \trianglerighteq \nu$ and $\mu \neq \nu$, we write it as $\mu \triangleright \nu$.

2.2. *Ariki–Koike algebras.* Let F be an algebraically closed field, and take $q \neq 0, Q_1, \dots, Q_r \in F$. The Ariki–Koike algebra $\mathscr{H}_{n,r} = \mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ is an associative algebra over F with generators $T_0, T_1, \dots, T_{n-1}$ and relations

$$\begin{gathered}
(T_0 - Q_1)(T_0 - Q_2) \cdots (T_0 - Q_r) = 0, \\
T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, \\
(T_i + 1)(T_i - q) = 0 \quad \text{for } i = 1, \dots, n-1, \\
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for } i = 1, \dots, n-2, \\
T_i T_j = T_j T_i \quad \text{if } |i - j| \geqslant 2.
\end{gathered}$$

By [6], it is known that $\mathscr{H}_{n,r}$ is a cellular algebra with a cellular basis

$$\{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda) \text{ for some } \lambda \in \Lambda^+_{n,r}\},$$

where $\operatorname{Std}(\lambda)$ is the set of standard tableaux of shape λ (see [6] for definitions).

We denote by S^λ the cell module of $\mathscr{H}_{n,r}$ corresponding to $\lambda \in \Lambda^+_{n,r}$, which is called the Specht module. Put $D^\lambda = S^\lambda / \operatorname{rad} S^\lambda$, where $\operatorname{rad} S^\lambda$ is the radical of S^λ with respect to the canonical bilinear form on S^λ. When $q \neq 1$ and $Q_i \neq 0$ for $1 \leqslant i \leqslant r$, it is known that $D^\lambda \neq 0$ if and only if λ is a Kleshchev multipartition by [2]. Thus $\{D^\lambda \mid \lambda \in \Lambda^+_{n,r}\colon \text{Kleshchev multipartition}\}$ gives a complete set of non-isomorphic simple $\mathscr{H}_{n,r}$-modules (see [2] or [21, §3.4] for the definition of Kleshchev multipartitions and more details).

2.3. *Cyclotomic q-Schur algebras.* The cyclotomic q-Schur algebra $\mathscr{S}_{n,r} = \mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ associated to $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ is defined by

$$\mathscr{S}_{n,r} = \mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m})) = \operatorname{End}_{\mathscr{H}_{n,r}}\Big(\bigoplus_{\mu \in \Lambda_{n,r}(\mathbf{m})} M^\mu\Big),$$

where M^μ is a certain $\mathscr{H}_{n,r}$-module introduced in [6] with respect to $\mu \in \Lambda_{n,r}(\mathbf{m})$. By [6], $\mathscr{S}_{n,r}$ is a cellular algebra with respect to the poset $(\Lambda^+_{n,r}, \trianglerighteq)$. More precisely, $\mathscr{S}_{n,r}$ has a cellular basis

$$\mathcal{C} = \{\varphi_{ST} \mid S, T \in \mathcal{T}_0(\lambda) \text{ for some } \lambda \in \Lambda^+_{n,r}\},$$

where $\mathcal{T}_0(\lambda) = \bigcup_{\mu \in \Lambda_{n,r}} \mathcal{T}_0(\lambda, \mu)$, and $\mathcal{T}_0(\lambda, \mu)$ is the set of semistandard tableaux of shape λ with type μ (see [6] for definitions). By definition, for $S \in \mathcal{T}_0(\lambda, \mu)$ and $T \in \mathcal{T}_0(\lambda, \nu)$, we have that $\varphi_{ST} \in \mathrm{Hom}_{\mathscr{H}_{n,r}}(M^\nu, M^\mu)$ and $\varphi_{ST}|_{M^\kappa} = 0$ $(\kappa \neq \nu)$. For $\mu \in \Lambda_{n,r}$, let $\varphi_\mu \in \mathscr{S}_{n,r}$ be the identity map on M^μ and zero-map on M^κ $(\kappa \neq \mu)$. Then we have

$$1_{\mathscr{S}_{n,r}} = \sum_{\mu \in \Lambda_{n,r}} \varphi_\mu,$$

and $\{\varphi_\mu \mid \mu \in \Lambda_{n,r}\}$ is a set of pairwise orthogonal idempotents. Thus, for $S \in \mathcal{T}_0(\lambda, \mu)$, $T \in \mathcal{T}_0(\lambda, \nu)$ and $\kappa \in \Lambda_{n,r}$, we have

$$\varphi_\kappa \varphi_{ST} = \delta_{\kappa\mu}\varphi_{ST}, \qquad \varphi_{ST}\varphi_\kappa = \delta_{\kappa\nu}\varphi_{ST}, \tag{2.3.1}$$

where $\delta_{\kappa\mu} = 1$ if $\kappa = \mu$, and $\delta_{\kappa\mu} = 0$ if $\kappa \neq \mu$.

Let W^λ be the cell module for $\mathscr{S}_{n,r}$ corresponding to $\lambda \in \Lambda^+_{n,r}$, which is called the Weyl module, and $\mathrm{rad}\, W^\lambda$ be the radical of W^λ with respect to the canonical bilinear form on W^λ. Put $L^\lambda = W^\lambda / \mathrm{rad}\, W^\lambda$. It is known that $L^\lambda \neq 0$ for any $\lambda \in \Lambda^+_{n,r}$, namely the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ is a quasi-hereditary algebra. Thus $\{L^\lambda \mid \lambda \in \Lambda^+_{n,r}\}$ is a complete set of non-isomorphic simple $\mathscr{S}_{n,r}$-modules.

Remarks 2.4.

(i) By a general theory, for a cellular algebra $\mathscr{A}$ over any field F, F is a splitting field for $\mathscr{A}$. Thus, we may assume that F is an algebraically closed field without loss of generality.

(ii) A cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ depends on a choice of $\Lambda_{n,r}(\mathbf{m})$. But, it is known that $\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}))$ is Morita equivalent to $\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}'))$ with same parameters if both of $\mathbf{m}$ and $\mathbf{m}'$ satisfy the condition (CL).

(iii) By definitions, we have the following properties.

(a) For any $0 \neq c \in F$, we have an isomorphism $\mathscr{H}_{n,r}(q, Q_1, \ldots, Q_r) \cong \mathscr{H}_{n,r}(q, cQ_1, \ldots, cQ_r)$. We also have an isomorphism $\mathscr{S}_{n,r}(q, Q_1, \ldots, Q_r) \cong \mathscr{S}_{n,r}(q, cQ_1, \ldots, cQ_r)$.

(b) For any permutation σ of r letters, we have an isomorphism $\mathscr{H}_{n,r}(q, Q_1, \ldots, Q_r) \cong \mathscr{H}_{n,r}(q, Q_{\sigma(1)}, \ldots, Q_{\sigma(r)})$. However, $\mathscr{S}_{n,r}(q, Q_1, \ldots, Q_r)$ is not isomorphic to $\mathscr{S}_{n,r}(q, Q_{\sigma(1)}, \ldots, Q_{\sigma(r)})$ in general.

2.5. Put $\omega = (-, \ldots, -, (1^n))$, where "$-$" means the empty partition, then $\omega \in \Lambda^+_{n,r}$ by condition (CL) (see 2.1). By the definition, φ_ω is the identity map on M^ω and 0-map on M^κ $(\kappa \neq \omega)$. In particular, φ_ω is an idempotent of $\mathscr{S}_{n,r}$. It is well known that the subalgebra $\varphi_\omega \mathscr{S}_{n,r} \varphi_\omega$ of $\mathscr{S}_{n,r}(q, Q_1, \ldots, Q_r)$ is isomorphic to $\mathscr{H}_{n,r}(q, Q_1, \ldots, Q_r)$ as algebras. Thus, by Lemma 1.1(ii), we have the following lemma.

Lemma 2.6. *If $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ is of infinite (resp. wild) type then $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ is of infinite (resp. wild) type.*

2.7. *Schur functor.* Since $\mathscr{H}_{n,r}$ is isomorphic to the subalgebra $\varphi_\omega \mathscr{S}_{n,r} \varphi_\omega$ of $\mathscr{S}_{n,r}$, we can define a functor $F = \mathrm{Hom}_{\mathscr{S}_{n,r}}(\varphi_\omega \mathscr{S}_{n,r}, -)$ from the category of finite dimensional $\mathscr{S}_{n,r}$-modules to the category of finite dimensional $\mathscr{H}_{n,r}$-modules.

The following lemma is known (see e.g. [21, Theorem 5.1], [8, Appendix]).

Lemma 2.8.

(i) *$F(W^\lambda) \cong S^\lambda$ as $\mathscr{H}_{n,r}$-modules for $\lambda \in \Lambda^+_{n,r}$.*
(ii) *$F(L^\lambda) \cong D^\lambda$ as $\mathscr{H}_{n,r}$-modules for $\lambda \in \Lambda^+_{n,r}$.*
(iii) *For $\lambda \in \Lambda^+_{n,r}$, let P^λ be the projective cover of L^λ. Then we have that $\{F(P^\lambda) \mid \lambda \in \Lambda^+_{n,r}$ such that $F(L^\lambda) \neq 0\}$ gives a complete set of non-isomorphic projective indecomposable $\mathscr{H}_{n,r}$-modules.*

2.9. For later discussions, we describe some structural properties of $\mathscr{S}_{n,r}$, the essential part of which has been obtained in [22,24]. Here, we modify such results for our purpose. Take and fix $\mathbf{p} = (r_1, \dots, r_g) \in \mathbb{Z}^g_{>0}$ such that $r_1 + \dots + r_g = r$. Put $p_i = \sum_{j=1}^{i-1} r_j$ with $p_1 = 0$. For $\mu \in \Lambda_{n,r}$, put $\mu^{[k]} = (\mu^{(p_k+1)}, \mu^{(p_k+2)}, \dots, \mu^{(p_k+r_k)})$ for $k = 1, \dots, g$. Thus, $\mu^{[k]}$ is an r_k-composition. We define a map $\alpha_\mathbf{p} : \Lambda_{n,r} \to \mathbb{Z}^g_{\geqslant 0}$ by $\mu \mapsto (|\mu^{[1]}|, |\mu^{[2]}|, \dots, |\mu^{[g]}|)$. Thus, we have

$$\operatorname{Im} \alpha_\mathbf{p} = \Delta_{n,g} := \left\{(n_1, \dots, n_g) \in \mathbb{Z}^g_{\geqslant 0} \;\middle|\; n_1 + \dots + n_g = n\right\}.$$

Recall that, for $\lambda \in \Lambda^+$, $\mathcal{T}_0(\lambda) = \bigcup_{\mu \in \Lambda_{n,r}} \mathcal{T}_0(\lambda, \mu)$ is the set of semistandard tableaux of shape λ. We define two subsets $\mathcal{T}^+_\mathbf{p}(\lambda)$ and $\mathcal{T}^-_\mathbf{p}(\lambda)$ of $\mathcal{T}_0(\lambda)$ by

$$\mathcal{T}^+_\mathbf{p}(\lambda) = \bigcup_{\substack{\mu \in \Lambda_{n,r} \\ \alpha_\mathbf{p}(\lambda) = \alpha_\mathbf{p}(\mu)}} \mathcal{T}_0(\lambda, \mu), \qquad \mathcal{T}^-_\mathbf{p} = \mathcal{T}_0(\lambda) \setminus \mathcal{T}^+_\mathbf{p}(\lambda).$$

Moreover, for each $\eta \in \Delta_{n,g}$, we define the subset $\mathcal{T}^\eta_\mathbf{p}(\lambda)$ of $\mathcal{T}_0(\lambda)$ by

$$\mathcal{T}^\eta_\mathbf{p}(\lambda) = \bigcup_{\substack{\mu \in \Lambda_{n,r} \\ \alpha_\mathbf{p}(\mu) = \eta}} \mathcal{T}_0(\lambda, \mu).$$

By definition, we have

$$\begin{cases} \mathcal{T}^\eta_\mathbf{p}(\lambda) = \mathcal{T}^+_\mathbf{p}(\lambda) & \text{if } \alpha_\mathbf{p}(\lambda) = \eta, \\ \mathcal{T}^\eta_\mathbf{p}(\lambda) \subseteq \mathcal{T}^-_\mathbf{p}(\lambda) & \text{if } \alpha_\mathbf{p}(\lambda) \neq \eta. \end{cases} \tag{2.9.1}$$

For $\eta \in \Delta_{n,g}$, put

$$e_\eta = \sum_{\substack{\mu \in \Lambda_{n,r} \\ \alpha_{\mathbf{p}}(\mu)=\eta}} \varphi_\mu.$$

Since $\{\varphi_\mu \mid \mu \in \Lambda_{n,r}\}$ is a set of pairwise orthogonal idempotents, e_η is also an idempotent. Thus, $\mathscr{S}^\eta = e_\eta \mathscr{S}_{n,r} e_\eta$ is a subalgebra of $\mathscr{S}_{n,r}$. The following theorem is a modification of the results in [22,24].

Theorem 2.10. *For each $\eta \in \Delta_{n,g}$, we have the following.*

(i) *$\mathscr{S}^\eta$ is a subalgebra of $\mathscr{S}_{n,r}$. Moreover, $\mathscr{S}^\eta$ is a cellular algebra with a cellular basis*

$$\mathcal{C}^\eta = \{\varphi_{ST} \mid S, T \in \mathcal{T}_{\mathbf{p}}^\eta(\lambda) \text{ for some } \lambda \in \Lambda_{n,r}^+\}.$$

(ii) *Let $\widehat{\mathscr{S}^\eta}$ be the F-subspace of $\mathscr{S}^\eta$ spanned by $\{\varphi_{ST} \mid S, T \in \mathcal{T}_{\mathbf{p}}^\eta(\lambda)$ for $\lambda \in \Lambda_{n,r}^+$ such that $\alpha_{\mathbf{p}}(\lambda) \neq \eta\}$. Then $\widehat{\mathscr{S}^\eta}$ is a two-sided ideal of $\mathscr{S}^\eta$. Thus one can define a quotient algebra $\overline{\mathscr{S}^\eta} = \mathscr{S}^\eta / \widehat{\mathscr{S}^\eta}$.*

(iii) *$\overline{\mathscr{S}^\eta}$ is a cellular algebra with a cellular basis*

$$\bar{\mathcal{C}}^\eta = \{\bar{\varphi}_{ST} \mid S, T \in \mathcal{T}_{\mathbf{p}}^+(\lambda) \text{ for } \lambda \in \Lambda_{n,r}^+ \text{ such that } \alpha_{\mathbf{p}}(\lambda) = \eta\}.$$

(iv) *There exists an isomorphism of algebras*

$$\overline{\mathscr{S}^\eta} \cong \mathscr{S}_{n_1,r_1}(q, \mathbf{Q}_1) \otimes \mathscr{S}_{n_2,r_2}(q, \mathbf{Q}_2) \otimes \cdots \otimes \mathscr{S}_{n_g,r_g}(q, \mathbf{Q}_g),$$

where $\eta = (n_1, \ldots, n_g)$ and $\mathbf{Q}_k = \{Q_{p_k+1}, \ldots, Q_{p_k+r_k}\}$ for $k = 1, \ldots, g$.

Proof. It is already shown that $\mathscr{S}^\eta$ is a subalgebra of $\mathscr{S}_{n,r}$. By (2.3.1), we see that $\mathcal{C}^\eta$ is a basis of $\mathscr{S}^\eta$. Thus, $\mathscr{S}^\eta$ inherits the cellular structure from $\mathscr{S}_{n,r}$. This proves (i). By noting (2.9.1), (ii) follows from [24, Lemma 2.11]. Now (iii) is clear. (iv) follows from the proof of [22, Theorem 4.15]. □

Remark 2.11. The statements in Theorem 2.10 except (iv) also hold for certain types cellular algebras under the setting in [24].

Theorem 2.10 implies the following proposition for Ariki–Koike algebras.

Proposition 2.12. *For $k = 1, 2, \ldots, r-1$, there exists a surjective homomorphism*

$$\mathscr{H}_{n,r}(q, Q_1, \ldots, Q_r) \to \mathscr{H}_{n,k}(q, Q_{r-k+1}, \ldots, Q_{r-1}, Q_r).$$

Proof. Put $\mathbf{p} = (r-k, k)$ and $\eta = (0, n)$. Then, by Theorem 2.10, we have a surjective homomorphism $\mathscr{S}^\eta \to \overline{\mathscr{S}^\eta}$, where $\overline{\mathscr{S}^\eta}$ is isomorphic to $1 \otimes \mathscr{S}_{n,k}(q, Q_{r-k+1}, \ldots, Q_r)$. Since $\alpha_{\mathbf{p}}(\omega) = \eta$, we have $\varphi_\omega e_\eta = e_\eta \varphi_\omega = \varphi_\omega$. This implies that $\varphi_\omega \mathscr{S}^\eta \varphi_\omega \cong \mathscr{H}_{n,r}(q, Q_1, \ldots, Q_r)$. On the other hand, one see easily that

$$\overline{\varphi_\omega}\, \overline{\mathscr{S}^\eta}\, \overline{\varphi_\omega} \cong \mathscr{H}_{n,k}(q, Q_{r-k+1}, \ldots, Q_r)$$

through the isomorphism $\overline{\mathscr{S}^\eta} \cong 1 \otimes \mathscr{S}_{n,k}(q, Q_{r-k+1}, \dots, Q_r)$, where $\overline{\varphi_\omega}$ is the image of φ_ω under the surjection $\mathscr{S}^\eta \to \overline{\mathscr{S}^\eta}$. Hence the surjection $\mathscr{S}^\eta \to \overline{\mathscr{S}^\eta}$ implies the proposition. □

The following corollary plays an important role in later discussions.

Corollary 2.13. *Take a subset $\{i_1, \dots, i_k\} \subset \{1, \dots, r\}$. If $\mathscr{H}_{n,k}(q, Q_{i_1}, \dots, Q_{i_k})$ is of infinite (resp. wild) type, then $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ is also of infinite (resp. wild) type.*

Proof. By Remarks 2.4(iii), we may suppose that $\{i_1, i_2, \dots, i_k\} = \{r-k+1, \dots, r-1, r\}$. Hence, the corollary follows from Proposition 2.12 together with Lemma 1.1(i). □

3. The representation type of Ariki–Koike algebras and cyclotomic q-Schur algebras

In this section, we study the representation type of Ariki–Koike algebras and cyclotomic q-Schur algebras. First, we recall a necessary and sufficient condition for Ariki–Koike algebras (resp. cyclotomic q-Schur algebras) to be semisimple. By Ariki [1], the condition for Ariki–Koike algebras to be semisimple was obtained, and through the double centralizer property [21, Theorem 5.3], we have the following theorem.

Theorem 3.1. *$\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ (resp. $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$) is semisimple if and only if*

$$\prod_{i=1}^{n}\left(1+q+\cdots+q^{i-1}\right) \prod_{1\leqslant i<j\leqslant r} \prod_{-n<a<-n} \left(q^a Q_i - Q_j\right) \neq 0.$$

3.2. In order to compute the decomposition numbers of $\mathscr{H}_{n,r}$ or $\mathscr{S}_{n,r}$ in some cases, we will use the Jantzen sum formula obtained by James and Mathas [16]. Here, we review on the Jantzen sum formula briefly (see [16] for more details).

Let R be a discrete valuation ring with the unique maximal ideal $\wp$, K be the quotient field of R, and F be the residue field $R/\wp$. Let $\nu_\wp : R^\times \to \mathbb{N}$ be the $\wp$-adic valuation map, and we extend $\nu_\wp$ to a map $K^\times \to \mathbb{Z}$ in the natural way.

Take $\widehat{q}, \widehat{Q}_1, \dots, \widehat{Q}_r \in R$, where $\widehat{q}$ is invertible in R. Let $q, Q_1, \dots, Q_r \in F$ be the image of $\widehat{q}, \widehat{Q}_1, \dots, \widehat{Q}_r \in R$ under the natural surjection $R \to F$. Then, (K, R, F) turns out to be a modular system. We denote by $\mathscr{H}_{n,r}^R$ (resp. $\mathscr{S}_{n,r}^R$) the Ariki–Koike algebra (resp. cyclotomic q-Schur algebra) over R with the parameters $\widehat{q}, \widehat{Q}_1, \dots, \widehat{Q}_r$. Put $\mathscr{H}_{n,r}^K = K \otimes_R \mathscr{H}_{n,r}^R$ (resp. $\mathscr{S}_{n,r}^K = K \otimes_R \mathscr{S}_{n,r}^R$) and $\mathscr{H}_{n,r}^F = F \otimes_R \mathscr{H}_{n,r}^R$ (resp. $\mathscr{S}_{n,r}^F = F \otimes_R \mathscr{S}_{n,r}^R$), where we regard F as the R-module through the natural surjection $R \to F$. By using the modular system, for the Weyl module W^λ ($\lambda \in \Lambda_{n,r}^+$) of $\mathscr{S}_{n,r}^F$, we can define the Jantzen filtration

$$W^\lambda = W^\lambda(0) \supseteq W^\lambda(1) \supseteq W^\lambda(2) \supseteq \cdots,$$

where $W^\lambda(1) = \operatorname{rad} W^\lambda$. Similarly, we have the Jantzen filtration

$$S^\lambda = S^\lambda(0) \supseteq S^\lambda(1) \supseteq S^\lambda(2) \supseteq \cdots$$

for the Specht module S^λ of $\mathscr{H}_{n,r}^F$.

For $x=(i,j,k)\in[\lambda]$ ($\lambda\in\Lambda^+_{n,r}$), put

$$\mathrm{res}_R(x)=\widehat{q}^{\,j-i}\,\widehat{Q}_k.$$

For $\lambda,\mu\in\Lambda^+_{n,r}$, we define the integer $J_{\lambda\mu}$ called a **Jantzen coefficient** by

$$J_{\lambda\mu}=\begin{cases}\sum_{x\in[\lambda]}\sum_{\substack{y\in[\mu]\\ [\mu]\setminus r_y=[\lambda]\setminus r_x}}(-1)^{\ell\ell(r_x)+\ell\ell(r_y)}\nu_{\wp}(\mathrm{res}_R(f_x)-\mathrm{res}_R(f_y)), & \text{if } \lambda\rhd\mu,\\ 0, & \text{otherwise,}\end{cases} \tag{3.2.1}$$

where r_x is the rim hook of x, $\ell\ell(r_x)$ is the leg length of r_x and f_x is the foot node of r_x (for definitions, see [16]). Here, we remark that f_x is a node in $\lambda^{(k)}$ if x is a node in $\lambda^{(k)}$. Then we have the following theorem.

Theorem 3.3. *(See [16].) Let (K,R,F) be a modular system. Suppose that $\mathscr{S}^K_{n,r}$ is semisimple. Then, for $\lambda\in\Lambda^+_{n,r}$, we have*

$$\sum_{i>0}\bigl[W^\lambda(i)\bigr]=\sum_{\substack{\mu\in\Lambda^+_{n,r}\\ \lambda\rhd\mu}}J_{\lambda\mu}\bigl[W^\mu\bigr]\quad\Bigl(\textit{resp. }\sum_{i>0}\bigl[S^\lambda(i)\bigr]=\sum_{\substack{\mu\in\Lambda^+_{n,r}\\ \lambda\rhd\mu}}J_{\lambda\mu}\bigl[S^\mu\bigr]\Bigr)$$

in the Grothendieck group of $\mathscr{S}^F_{n,r}$ (resp. $\mathscr{H}^F_{n,r}$).

3.4. Let $e\in\{1,2,\dots,\infty\}$ be the multiplicative order of q in F, namely q is a primitive e-th root of unity or $e=\infty$ if q is not a root of unity. In the case where $r=1$, $\mathscr{H}_{n,1}(q,Q_1)$ is the Iwahori–Hecke algebra $\mathscr{H}_n(q)$ of type A, and $\mathscr{S}_{n,1}(q,Q_1)$ is the q-Schur algebra $\mathscr{S}_n(q)$ associated to $\mathscr{H}_n(q)$. (Note that $\mathscr{H}_{n,1}(q,Q_1)$ (resp. $\mathscr{S}_{n,1}(q,Q_1)$) is independent from the parameter Q_1.) A condition for $\mathscr{H}_n(q)$ to be of finite representation type was obtained by Uno [23]. On the other hand, the representation type of $\mathscr{S}_n(q)$ has been determined by Erdmann and Nakano [13] as follows.

Theorem 3.5. *(See [23,13].) Suppose that $q\neq 1$ and $r=1$, then we have the following.*

(i) *$\mathscr{H}_n(q)$ (resp. $\mathscr{S}_n(q)$) is semisimple if and only if $n<e$.*
(ii) *$\mathscr{H}_n(q)$ (resp. $\mathscr{S}_n(q)$) is of finite type if and only if $n<2e$.*
(iii) *$\mathscr{H}_n(q)$ (resp. $\mathscr{S}_n(q)$) is of wild type if and only if $n\geqslant 2e$.*

Remarks 3.6.

(i) In [12], the representation type for an each block of $\mathscr{H}_n(q)$ is determined.
(ii) In this paper, we are only concerned with cyclotomic q-Schur algebras satisfying the condition (CL) in 2.1. In [13], the representation type of q-Schur algebras with $m_1<n$ is also determined. It occurs that $\mathscr{S}_n(q)$ is of tame type for some cases with $m_1<n$. But, under the condition (CL), no $\mathscr{S}_n(q)$ has tame type.

(iii) In the case where $q = 1$, q-Schur algebras are nothing but classical Schur algebras. The representation type of (classical) Schur algebras has been determined by Doty, Erdmann, Martin and Nakano [9].

3.7. In order to describe the representation type of Ariki–Koike algebras and cyclotomic q-Schur algebras in the case where $r \geqslant 2$, we need the following theorem which has been proved by Dipper and Mathas [7].

Theorem 3.8. *(See [7].) Suppose that $I = I_1 \cup I_2 \cup \cdots \cup I_\kappa$ (disjoint union) is a partitioning of the index set $I = \{1, \dots, r\}$ of parameters $\mathbf{Q} = (Q_1, \dots, Q_r)$ such that*

$$\prod_{1 \leqslant \alpha < \beta \leqslant \kappa} \prod_{\substack{Q_i \in \mathbf{Q}_\alpha \\ Q_j \in \mathbf{Q}_\beta}} \prod_{-n < a < n} \left(q^a Q_i - Q_j\right) \neq 0,$$

where we put $\mathbf{Q}_\alpha = (Q_{\alpha_1}, \dots, Q_{\alpha_j})$ for $I_\alpha = \{\alpha_1, \dots, \alpha_j\}$. Then $\mathscr{H}_{n,r}(q, \mathbf{Q})$ (resp. $\mathscr{S}_{n,r}(q, \mathbf{Q})$) is Morita equivalent to the algebra

$$\bigoplus_{\substack{n_1, \dots, n_\kappa \geqslant 0 \\ n_1 + \cdots + n_\kappa = n}} \mathscr{H}_{n_1, r_1}(q, \mathbf{Q}_1) \otimes \mathscr{H}_{n_2, r_2}(q, \mathbf{Q}_2) \otimes \cdots \otimes \mathscr{H}_{n_\kappa, r_\kappa}(q, \mathbf{Q}_\kappa),$$

$$\left(\textit{resp.} \bigoplus_{\substack{n_1, \dots, n_\kappa \geqslant 0 \\ n_1 + \cdots + n_\kappa = n}} \mathscr{S}_{n_1, r_1}(q, \mathbf{Q}_1) \otimes \mathscr{S}_{n_2, r_2}(q, \mathbf{Q}_2) \otimes \cdots \otimes \mathscr{S}_{n_\kappa, r_\kappa}(q, \mathbf{Q}_\kappa)\right),$$

where $r_i = |\mathbf{Q}_i|$ $(i = 1, \dots, \kappa)$.

3.9. By Theorem 3.8 combined with Lemma 1.5 (together with multiplying $Q_1, \dots, Q_r$ by a scalar $c \in F$ simultaneously (see Remarks 2.4(iii))), we may assume that $Q_i = q^{f_i}$ $(i = 1, \dots, r)$ or $Q_1 = \cdots = Q_r = 0$. Then, in order to determine the representation type of cyclotomic q-Schur algebras, it is enough to consider the following cases.

Case 1. $q \neq 1$ and $Q_i = q^{f_i}$ $(f_i \in \mathbb{Z})$ for $i = 1, \dots, r$.
Case 2. $q = 1$ and $Q_1 = \cdots = Q_r = 1$.
Case 3. $q = 1$ and $Q_1 = \cdots = Q_r = 0$.
Case 4. $q \neq 1$ and $Q_1 = \cdots = Q_r = 0$.

In this paper, we are only concerned with Case 1 and Case 2. First, we consider Case 2.

Theorem 3.10. *Suppose that $q = 1$, $r \geqslant 2$ and $Q_1 = \cdots = Q_r = 1$. Then $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ (resp. $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$) is of finite type if and only if $n = 1$.*

Proof. It is clear that $\mathscr{H}_{n,r}$ (resp. $\mathscr{S}_{n,r}$) is of finite type if $n = 1$. Suppose that $n \geqslant 2$. Then, by [3, Proposition 41], $\mathscr{H}_{n,2}(q, Q_1, Q_2)$ $(q = Q_1 = Q_2 = 1)$ is of infinite type. (Note that the parameters are given by $Q_1 = Q_2 = -1$ in [3]. By multiplying Q_1 and Q_2 by -1, we obtain the above claim.) Thus $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ is of infinite type by Corollary 2.13, and $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ is of infinite type by Lemma 2.6. □

Remark 3.11. We can also prove that $\mathscr{H}_{n,r}$ (resp. $\mathscr{S}_{n,r}$) is of wild type if $r \geqslant 2$, $n \geqslant 3$ and $q = Q_1 = \cdots = Q_r = 1$ by [3, Proposition 41] in a similar way as in the above proof. But, we don't know whether $\mathscr{H}_{n,r}$ (resp. $\mathscr{S}_{n,r}$) is of tame type or of wild type if $n = 2$, $r \geqslant 2$ and $q = Q_1 = \cdots = Q_r = 1$.

3.12. From now on, we concentrate on Case 1 with $r \geqslant 2$. Hence we assume the following condition.

(CP) q is a primitive e-th root of unity. ($e = \infty$ if q is not a root of unity.) $Q_i = q^{f_i}$ ($0 \leqslant f_i \leqslant e-1$) for $i = 1, \dots, r$.

Note that, when $e = \infty$, we can take $f_i \in \mathbb{Z}_{\geqslant 0}$ without loss of generality by Remarks 2.4(iii), and we regard as $c < \infty$ for any integer c. Let

$$0 \leqslant f'_1 \leqslant f'_2 \leqslant \cdots \leqslant f'_r \leqslant e-1$$

be the increasing sequence of integers such that $f'_i = f_{\sigma(i)}$ ($i = 1, \dots, r$) for some permutation σ of r letters. Set $f'_{r+i} = e + f'_i$ and $g'_i = f'_{i+1} - f'_i$ ($i = 1, \dots, r$). We define the integers $f^{+1}(Q_1, \dots, Q_r)$, $f^{+2}(Q_1, \dots, Q_r)$ and $g(Q_1, \dots, Q_r)$ for parameters $Q_1, \dots, Q_r$ by

$$\begin{aligned} f^{+1}(Q_1, \dots, Q_r) &= \min\{ f'_{i+1} - f'_i \mid i = 1, \dots, r \}, \\ f^{+2}(Q_1, \dots, Q_r) &= \min\{ f'_{i+2} - f'_i \mid i = 1, \dots, r \}, \\ g(Q_1, \dots, Q_r) &= \min\{ g'_i + g'_j \mid 1 \leqslant i \neq j \leqslant r \}. \end{aligned}$$

The rest of this section is devoted to the proof of the following theorem.

Theorem 3.13. *Under the condition* (CP)*, we have the following.*

(i) *Assume that $r = 2$. Then $\mathscr{H}_{n,2}(q, Q_1, Q_2)$ (resp. $\mathscr{S}_{n,2}(q, Q_1, Q_2)$) is of finite type if and only if*

$$n < \min\{ e, 2f^{+1}(Q_1, Q_2) + 4 \}.$$

(ii) *Assume that $r \geqslant 3$. Then $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ (resp. $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$) is of finite type if and only if*

$$n < \min\{ 2f^{+1}(Q_1, \dots, Q_r) + 4, f^{+2}(Q_1, \dots, Q_r) + 1, g(Q_1, \dots, Q_r) + 2 \}.$$

3.14. In order to prove Theorem 3.13, we prepare some known results on blocks of $\mathscr{H}_{n,r}$ and $\mathscr{S}_{n,r}$. By a general theory of cellular algebras [15], for each $\lambda \in \Lambda^+_{n,r}$, all of the composition factors of the Specht module S^λ of $\mathscr{H}_{n,r}$ belong to the same block of $\mathscr{H}_{n,r}$. This result allows us to classify the blocks of $\mathscr{H}_{n,r}$ by using the Specht modules. Similar facts also hold for $\mathscr{S}_{n,r}$. By Lyle and Mathas [18], this classification has been described combinatorially as follows. Here, we only give the result under the condition (CP) though it is described in a general setting in [18].

For $\lambda \in \Lambda^+_{n,r}$, we define the *residue* of the node $x = (i, j, k) \in [\lambda]$ by

$$\operatorname{res}(x) = j - i + f_k \pmod{e}.$$

For $\lambda, \mu \in \Lambda^+_{n,r}$, we say that λ and μ are *residue equivalent*, and write $\lambda \sim_C \mu$ if $\sharp\{x \in [\lambda] \mid \operatorname{res}(x) = a\} = \sharp\{y \in [\mu] \mid \operatorname{res}(y) = a\}$ for all $a \in \mathbb{Z}/e\mathbb{Z}$.

Theorem 3.15. *(See [18, Theorem 2.11].) For $\lambda, \mu \in \Lambda^+_{n,r}$, the following conditions are equivalent.*

(i) *S^λ and S^μ belong to the same block of $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$.*
(ii) *W^λ and W^μ belong to the same block of $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$.*
(iii) *$\lambda \sim_C \mu$.*

For a block $\mathcal{B}$ of $\mathscr{S}_{n,r}$, we write $\lambda \in \mathcal{B}$ if W^λ belongs to the block $\mathcal{B}$, and we say that $\mathcal{B}$ has a residue $(\operatorname{res}(x_1), \dots, \operatorname{res}(x_m))$, where $\{x_i \mid i = 1, \dots, m\} = [\lambda]$ for some $\lambda \in \mathcal{B}$. By Theorem 3.15, the residue $(\operatorname{res}(x_1), \dots, \operatorname{res}(x_m))$ of $\mathcal{B}$ is well-defined up to a permutation of components. Set $R(\mathcal{B}) = \{\operatorname{res}(x) \mid x \in [\lambda] \text{ for some } \lambda \in \mathcal{B}\}$. It is similar for $\mathscr{H}_{n,r}$, and we use the same notations.

Recall that F is the Schur functor defined in 2.7. As a corollary of Theorem 3.15, we have the following.

Corollary 3.16. *For $\lambda \in \Lambda^+_{n,r}$, let P^λ be the projective cover of L^λ. Then we have the following.*

(i) *$F(P^\lambda) \neq 0$ for any $\lambda \in \Lambda^+_{n,r}$.*
(ii) *If $F(L^\lambda) \neq 0$ and $F(L^\mu) \neq 0$, then the following two conditions are equivalent.*
  (a) *P^λ and P^μ belong to the same block of $\mathscr{S}_{n,r}$.*
  (b) *$F(P^\lambda)$ and $F(P^\mu)$ belong to the same block of $\mathscr{H}_{n,r}$.*

Proof. By the general theory of quasi-hereditary algebras, there exists a submodule K^λ of P^λ such that $P^\lambda/K^\lambda \cong W^\lambda$ for each $\lambda \in \Lambda^+_{n,r}$. Since F is an exact functor, we have that $F(P^\lambda)/F(K^\lambda) \cong F(W^\lambda) \cong S^\lambda$ by Lemma 2.8(i). This implies (i), and (ii) follows from Theorem 3.15. □

3.17. First, we prove the "if" part of Theorem 3.13. The following lemma plays an important role in the proof of the "if" part of Theorem 3.13.

Lemma 3.18. *Under the condition* (CP)*, suppose that $r \geqslant 3$ and*

$$n < \min\{f^{+2}(Q_1, \dots, Q_r) + 1, g(Q_1, \dots, Q_r) + 2\}. \tag{3.18.1}$$

Then, for each block $\mathcal{B}$ of $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$, there exist $i, j \in \{1, \dots, r\}$ such that $\lambda^{(k)} = \mu^{(k)}$ for any $\lambda, \mu \in \mathcal{B}$, and for $k \neq i, j$.

Proof. Let σ be a permutation of r letters. We define the bijective map $\sigma : \Lambda^+_{n,r} \to \Lambda^+_{n,r}$ by $\sigma((\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})) = (\lambda^{(\sigma^{-1}(1))}, \lambda^{(\sigma^{-1}(2))}, \dots, \lambda^{(\sigma^{-1}(r))})$. For $\lambda \in \Lambda^+_{n,r}$, we also define the bijection $\sigma : [\lambda] \to [\sigma(\lambda)]$ by $\sigma(x) = (i, j, \sigma(k))$ for $x = (i, j, k) \in [\lambda]$. Then, for $\lambda \in \Lambda^+_{n,r}$ and $x = (i, j, k) \in [\lambda]$, one sees easily that $\mathrm{res}(x)$ with respect to the parameters $q, Q_1, \dots, Q_r$ coincides with $\mathrm{res}(\sigma(x))$ with respect to the parameters $q, Q_{\sigma(1)}, \dots, Q_{\sigma(r)}$. Combining with Theorem 3.15, we have that λ, μ belong to the same block of $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ if and only if λ, μ belong to the same block of $\mathscr{S}_{n,r}(q, Q_{\sigma(1)}, \dots, Q_{\sigma(r)})$. Hence, it is enough to show the following cases:

$$0 \leqslant f_1 \leqslant f_2 \leqslant \cdots \leqslant f_r \leqslant e-1. \tag{3.18.2}$$

Hence, in this proof, we suppose that the condition (3.18.2) holds. Note that $f'_i = f_i$ $(i = 1, \dots, r)$ under the condition (3.18.2).

Take $\lambda, \mu \in \mathcal{B}$ such that $\lambda \neq \mu$. For a node $x = (a, b, i) \in [\lambda] \setminus [\mu]$, there exists a node $y = (c, d, j) \in [\mu] \setminus [\lambda]$ such that $\mathrm{res}(x) = \mathrm{res}(y)$ by Theorem 3.15. Since the condition $n < f^{+2}(Q_1, \dots, Q_r) + 1$ implies that $n < e$ by a direct calculation, we have $i \neq j$. We may assume that $i < j$ by interchanging λ and μ if necessary. Since $\mathrm{res}(x) = \mathrm{res}(y)$, one of the following two cases occurs:

$$\begin{cases} R(\mathcal{B}) \supseteq \{f_i, f_i + 1, \dots, f_j\}, \\ R(\mathcal{B}) \supseteq \{f_j, f_j + 1, \dots, e-1, 0, 1, \dots, f_i\}. \end{cases} \tag{3.18.3}$$

In fact, suppose that $a \geqslant b$, then $f_i, f_i + 1, \dots, \mathrm{res}(x)$ occur among the residues of λ. If $c \geqslant d$, then $f_j, f_j + 1, \dots, \mathrm{res}(x)$ occur in μ, and if $c < d$, then $\mathrm{res}(x), \mathrm{res}(x) + 1, \dots, f_j$ occur in μ. In either case, we see that $f_i, \dots, f_j$ occur in $R(\mathcal{B})$. Next suppose $a < b$, then $\mathrm{res}(x), \mathrm{res}(x) + 1, \dots, f_i$, occur in λ. Hence if $c \geqslant d$, again $f_i, f_i + 1, \dots, f_j$ occur in $R(\mathcal{B})$, and if $c < d$, we see that $\{f_j, \dots, e-1, 0, 1, \dots, f_i\} \in R(\mathcal{B})$.

If $|i - j| > 1$ and $\{i, j\} \neq \{1, r\}$, any case of (3.18.3) implies that

$$n \geqslant f^{+2}(Q_1, \dots, Q_r) + 1. \tag{3.18.4}$$

In fact, the first case implies that $f_j - f_i + 1 \leqslant n$. In the second case, we have $e - f_j + f_i \leqslant n - 1$, which implies that $n \geqslant f_{j+2} - f_j + 1$ since $\{i, j\} \neq \{1, r\}$. But (3.18.4) contradicts the condition (3.18.1). Thus we have

$$|i - j| = 1 \quad \text{or} \quad \{i, j\} = \{1, r\}.$$

Next we show that such a pair $\{i, j\}$ is determined uniquely for a given $\lambda, \mu \in \mathcal{B}$. Let $\lambda, \mu \in \mathcal{B}$, and take $x_k = (a_k, b_k, i_k) \in [\lambda] \setminus [\mu]$, $y_k = (c_k, d_k, j_k) \in [\mu] \setminus [\lambda]$ such that $\mathrm{res}(x_k) = \mathrm{res}(y_k)$ $(k = 1, 2)$. By the above result, we have $|i_k - j_k| = 1$ or $\{i_k, j_k\} = \{1, r\}$. If $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$, we have $n \geqslant g(Q_1, \dots, Q_r) + 2$ by considering the residues contained in $R(\mathcal{B})$ of (3.18.3). This contradicts the condition (3.18.1). For example, in the case where $r = 6$, $i_1 = 1$, $j_1 = 2$, $i_2 = 3$, $j_2 = 4$, we have

$$R(\mathcal{B}) \supset \{f_1, f_1 + 1, \dots, f_2\} \cup \{f_3, f_3 + 1, \dots, f_4\}.$$

Thus we may assume that

$$\{i_1, j_1\} \cap \{i_2, j_2\} \neq \emptyset. \tag{3.18.5}$$

If the two sets contain exactly one common element, we have $n \geqslant f^{+2}(Q_1, \ldots, Q_r) + 1$ by considering the residues contained in $R(\mathcal{B})$ of (3.18.3). This contradicts the condition (3.18.1). For example, in the case where $r = 6$, $i_1 = 1$, $j_1 = 2$, $i_2 = 2$, $j_2 = 3$, we have

$$R(\mathcal{B}) \supset \{f_1, f_1 + 1, \ldots, f_2\} \cup \{f_2, f_2 + 1, \ldots, f_3\}.$$

As a conclusion, we have $\{i_1, j_1\} = \{i_2, j_2\}$.

Finally, we show that such a pair $\{i, j\}$ is independent of the choice of $\lambda, \mu \in \mathcal{B}$. For $\lambda, \mu, \nu \in \mathcal{B}$, take $x = (a, b, i) \in [\lambda] \setminus [\mu]$, $y = (c, d, j) \in [\mu] \setminus [\lambda]$ such that $\mathrm{res}(x) = \mathrm{res}(y)$, and take $x' = (a', b', i') \in [\mu] \setminus [\nu]$, $y' = (c', d', j') \in [\nu] \setminus [\mu]$ such that $\mathrm{res}(x') = \mathrm{res}(y')$. If $\{i, j\} \cap \{i', j'\} = \emptyset$, we have $n \geqslant g(Q_1, \ldots, Q_r) + 2$, and if $\{i, j\}$ and $\{i', j'\}$ contain exactly one common element, we have $n \geqslant f^{+2}(Q_1, \ldots, Q_r) + 1$ in a similar way as above. Thus we have $\{i, j\} = \{i', j'\}$. The lemma is proved. □

Lemma 3.18 implies the following proposition which shows the "if" part of Theorem 3.13.

Proposition 3.19. *Under the condition* (CP), *if* $n < \min\{2f^{+1}(Q_1, \ldots, Q_r) + 4, f^{+2}(Q_1, \ldots, Q_r) + 1, g(Q_1, \ldots, Q_r) + 2\}$ $(r \geqslant 3)$ *or if* $n < \min\{e, 2f^{+1}(Q_1, Q_2) + 4\}$ $(r = 2)$, *then the decomposition matrix of a block of* $\mathscr{S}_{n,r}(q, Q_1, \ldots, Q_r)$ *is given as*

$$D = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix} \quad (\textit{all omitted entries are zero}).$$

Moreover, any projective indecomposable $\mathscr{S}_{n,r}$*-module has the simple socle.*

In particular, $\mathscr{S}_{n,r}(q, Q_1, \ldots, Q_r)$ *is of finite type. Thus,* $\mathscr{H}_{n,r}(q, Q_1, \ldots, Q_r)$ *is also of finite type.*

Proof. Take a block $\mathcal{B}$ of $\mathscr{S}_{n,r}(q, Q_1, \ldots, Q_r)$. We compute the decomposition matrix of $\mathcal{B}$ by the Jantzen sum formula in Theorem 3.3.

In the case where $r \geqslant 3$, assume that $n < \min\{2f^{+1}(Q_1, \ldots, Q_r) + 4, f^{+2}(Q_1, \ldots, Q_r) + 1, g(Q_1, \ldots, Q_r) + 2\}$. Then by Lemma 3.18, there exist $i, j \in \{1, \ldots, r\}$ such that, $\lambda^{(k)} = \mu^{(k)}$ for any $\lambda, \mu \in \mathcal{B}$ and $k \neq i, j$. Put $n' = |\lambda^{(i)}| + |\lambda^{(j)}|$ for $\lambda \in \mathcal{B}$, which is independent of $\lambda \in \mathcal{B}$. By Theorem 3.15 combined with Lemma 3.18, one can find a block $\mathcal{B}'$ of $\mathscr{S}_{n',2}(q, Q_i, Q_j)$ which contains $\{(\lambda^{(i)}, \lambda^{(j)}) \mid \lambda \in \mathcal{B}\}$. In order to compute the decomposition matrix of $\mathcal{B}$ by the Jantzen sum formula, we take the following modular system. Let $F[t]$ be a polynomial ring over F with indeterminate t, and $R = F[t]_{\langle t \rangle}$ be the localization of $F[t]$ by the prime ideal $\langle t \rangle$ generated by the polynomial t. Let K be the quotient field of R. Put $\widehat{q} = q$, $\widehat{Q}_i = q^{f_i}$, $\widehat{Q}_j = t + q^{f_j}$ and $\widehat{Q}_k = t^{2k} + q^{f_k}$ for $k \neq i, j$ as elements in R. Then, (K, R, F) becomes a modular system. Under this modular system, we see that $\mathscr{S}^K_{n,r}(\widehat{q}, \widehat{Q}_1, \ldots, \widehat{Q}_r)$ is semisimple. Thus, we can apply the Jantzen sum formula (Theorem 3.3). By the definition (3.2.1) combined with Lemma 3.18, the Janzten coefficient $J_{\lambda\mu}$ for $\lambda, \mu \in \mathcal{B}$ is determined by only the informations of $(\lambda^{(i)}, \lambda^{(j)})$ and $(\mu^{(i)}, \mu^{(j)})$ since $\lambda^{(k)} = \mu^{(k)}$ for $k \neq i, j$. Moreover, this Jantzen coefficient $J_{\lambda\mu}$ coincides with the Jantzen coefficient $J_{(\lambda^{(i)}, \lambda^{(j)}), (\mu^{(i)}, \mu^{(j)})}$ in $\mathscr{S}_{n',2}(q, Q_i, Q_j)$, where we take the same modular system (K, R, F) with the parameters $\widehat{q}, \widehat{Q}_i, \widehat{Q}_j$.

This means that the Jantzen sum formula for $\mathcal{B}$ coincides with the Jantzen sum formula for $\mathcal{B}'$. Moreover, $\mathscr{S}_{n',2}(q, Q_i, Q_j)$ satisfies the assumption of the proposition for the case where $r = 2$. Thus, we have only to compute the decomposition matrix in the case where $r = 2$.

Note that, the Jantzen sum formula (more precisely the Jantzen coefficient) for $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ coincides with the Jantzen sum formula for $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ (see Theorem 3.3). For $\mathscr{H}_{n,2}(q, Q_1, Q_2)$ satisfying the condition $n < \min\{e, 2f^{+1}(Q_1, Q_2) + 4\}$, the Jantzen coefficient has been computed by [4, Theorem 6.2]. Thus, we can compute the decomposition matrix of $\mathcal{B}$ in a similar way as in the proof of [4, Theorem 6.2], and we obtain the matrix as given in the proposition.

Next, we show that any projective indecomposable $\mathscr{S}_{n,r}$-module has the simple socle. Let $\{\lambda_1, \dots, \lambda_m\} = \{\lambda \in \mathcal{B}\}$ be such that $i < j$ if $\lambda_i \lhd \lambda_j$. Then P^{λ_i} has the radical series as in (1.7.1). It is clear that P^{λ_m} has the simple socle from the radical series. We claim that

$$F\big(L^{\lambda_i}\big) \neq 0 \quad \text{for } i = 1, \dots, m-1. \tag{3.19.1}$$

By Lemma 2.8(i), we see that $F(L^{\lambda_1}) \neq 0$ since $L^{\lambda_1} = W^{\lambda_1}$. Assume that $F(L^{\lambda_i}) = 0$ for some $i = 2, \dots, m-1$. Then we have that $F(L^{\lambda_{i-1}}) \neq 0$ or $F(L^{\lambda_{i+1}}) \neq 0$ since $F(P^{\lambda_i}) \neq 0$ by Corollary 3.16(i). If $F(L^{\lambda_{i-1}}) \neq 0$ and $F(L^{\lambda_{i+1}}) \neq 0$, then $F(P^{\lambda_{i-1}})$ and $F(P^{\lambda_{i+1}})$ are projective indecomposable $\mathscr{H}_{n,r}$-modules by Lemma 2.8(iii). Moreover, $F(P^{\lambda_{i-1}})$ and $F(P^{\lambda_{i+1}})$ belong to the same block of $\mathscr{H}_{n,r}$ by Corollary 3.16(ii). However, $F(P^{\lambda_{i-1}})$ and $F(P^{\lambda_{i+1}})$ are not linked since $F(L^{\lambda_i}) = 0$. This is a contradiction. If $F(L^{\lambda_{i-1}}) = 0$, we have that $F(L^{\lambda_{i+1}}) \neq 0$, and also $F(L^{\lambda_{i-2}}) \neq 0$ since $F(P^{\lambda^{i-1}}) \neq 0$. Then $F(P^{\lambda_{i-2}})$ and $F(P^{\lambda_{i+1}})$ belong to the same block of $\mathscr{H}_{n,r}$, but these are not linked. This is a contradiction. In the case where $F(L^{\lambda_{i+1}}) = 0$, we have a similar contradiction. Hence, we have the claim (3.19.1).

By (3.19.1) combined with Lemma 2.8(iii), $F(P^{\lambda_i})$ $(1 \leqslant i \leqslant m-1)$ is a projective indecomposable $\mathscr{H}_{n,r}$-module, and it is also injective since $\mathscr{H}_{n,r}$ is self-injective by [19]. Thus, $F(P^{\lambda_i})$ has the simple socle. Combining with (3.19.1), we see that P^{λ_i} $(1 \leqslant i \leqslant m-1)$ has the simple socle.

Finally, we note that the block $\mathcal{B}$ of $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ is also a quasi-hereditary cellular algebra. Then we conclude that $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ is of finite type by Proposition 1.7, and $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ is also of finite type by Lemma 2.6. □

As a corollary of the proof of the proposition, we have the following.

Corollary 3.20. *Suppose that $r \geqslant 3$, and that $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ satisfies the assumption of Proposition* 3.19*. Then, for each block $\mathcal{B}$ of $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$, there exist $i, j \in \{1, \dots, r\}$ such that $\mathcal{B}$ is Morita equivalent to a block $\mathcal{B}'$ of $\mathscr{S}_{n',2}(q, Q_i, Q_j)$, where $n' = |\lambda^{(i)}| + |\lambda^{(j)}|$ for some $\lambda \in \mathcal{B}$, and $\mathcal{B}'$ contains the partitions $\{(\lambda^{(i)}, \lambda^{(j)}) \mid \lambda \in \mathcal{B}\}$.*

Proof. By the proof of Proposition 3.19 and Proposition 1.7, both of $\mathcal{B}$ and $\mathcal{B}'$ are Morita equivalent to the algebra $\mathcal{A}_m$ in 1.6, where $m = \sharp\{\lambda \in \mathcal{B}\}$. □

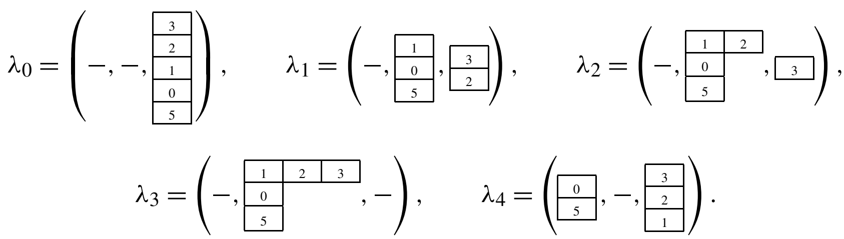
Next, we discuss the "only if" part of Theorem 3.13.

Proposition 3.21. *Under the condition* (CP)*, suppose that $n < \min\{e, 2f^{+1}(Q_1, Q_2, Q_3) + 4\}$ and $n \geqslant f^{+2}(Q_1, Q_2, Q_3) + 1$. Then $\mathscr{H}_{n,3}(q, Q_1, Q_2, Q_3)$ is of infinite type.*

Proof. By Remarks 2.4(iii), we may assume that $0 = f_1 \leqslant f_2 \leqslant f_3 \leqslant e-1$ without loss of generality. First, we assume that $n \geqslant 4$. Then, by the condition $n < 2f^{+1}(Q_1, Q_2, Q_3) + 4$, we have $0 = f_1 < f_2 < f_3$. Moreover, we can take $f^{+2}(Q_1, Q_2, Q_3) = f_3 - f_1 = f_3$ by permuting and multiplying Q_1, Q_2, Q_3 by a common scalar if necessary. Let $\mathcal{B}$ be the block of $\mathscr{H}_{n,3}(q, Q_1, Q_2, Q_3)$ with the residue $(f_3, f_3 - 1, \dots, 1, 0, e-1, \dots, e-f')$, where $f' = n - (f_3 + 1)$. The condition $n \geqslant f^{+2}(Q_1, Q_2, Q_3) + 1 = f_3 + 1$ implies that $f' \geqslant 0$, and the condition $n < e$ implies that $e - f' > f_3 + 1$.

Put $\lambda_0 = (-, -, (1^n))$, $\lambda_i = (-, (i, 1^{f_2+f'}), (1^{f_3-f_2-i+1}))$ for $i = 1, \dots, k-1$ and $\lambda_k = ((1^{n-f_3}), -, (1^{f_3}))$, where $k = f_3 - f_2 + 2$.

Example. (In the case where $n = 5$, $e = 6$, $f_1 = 0$, $f_2 = 1$, $f_3 = 3$.)



It is easy to see that $\lambda_0, \dots, \lambda_k$ all lie in the block $\mathcal{B}$, and let $\lambda_{k+1}, \dots, \lambda_m$ denote the remaining multipartitions in $\mathcal{B}$, in any order. Then $\mathcal{B}$ is a cellular algebra with respect to the poset $(\Lambda^+_{\mathcal{B}}, \trianglerighteq)$, where $\Lambda^+_{\mathcal{B}} = \{\lambda_i \mid 0 \leqslant i \leqslant m\}$. Let $\mathcal{B}^\vee$ be the F-subspace of $\mathcal{B}$ spanned by the cellular basis elements indexed by λ_i $(i \neq 0, 1, k)$. Note that $e - f' > f_3 + 1$ and the definitions of multipartitions $\lambda_0, \lambda_1, \dots, \lambda_k$, one sees that $\lambda_i \ntrianglerighteq \lambda_j$ for $i = 0, 1, k$ and $j \neq 0, 1, k$. This implies that $\mathcal{B}^\vee$ is a two-sided ideal of $\mathcal{B}$. Thus $\overline{\mathcal{B}} = \mathcal{B}/\mathcal{B}^\vee$ becomes a cellular algebra with respect to the poset $(\{\lambda_0, \lambda_1, \lambda_k\}, \trianglerighteq)$. One can easily check that λ_i $(i = 0, 1, k)$ is a Kleshchev multipartition, thus $D^{\lambda_i} \neq 0$ for $i = 0, 1, k$. Now, we can compute the decomposition matrix of $\overline{\mathcal{B}}$ by the Jantzen sum formula, and its matrix is given as follows.

	D^{λ_0}	D^{λ_1}	D^{λ_k}
S^{λ_0}	1	0	0
S^{λ_1}	a	1	0
S^{λ_k}	b	0	1

$(a, b > 0)$.

This implies that $\overline{\mathcal{B}}$ is of infinite type by Lemma 1.8, thus $\mathcal{B}$ and so $\mathscr{H}_{n,3}(q, Q_1, Q_2, Q_3)$ is of infinite type.

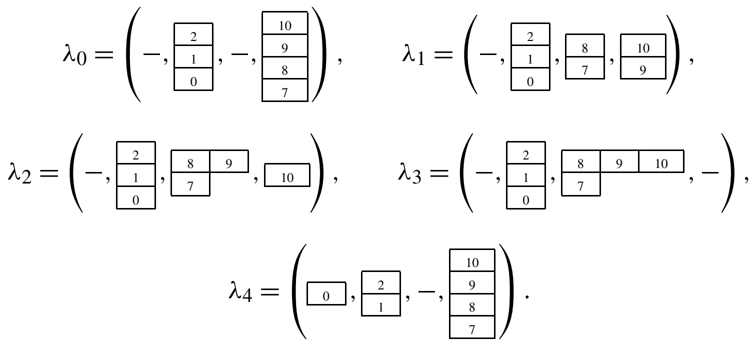
In the case where $n \leqslant 3$, we have the following three cases, $0 = f_1 = f_2 = f_3$, $0 = f_1 = f_2 < f_3$ or $f_1 < f_2 < f_3$. For each case, we can prove in a similar way as above by taking the appropriate block. $\square$

Proposition 3.22. *Under the condition* (CP), *suppose that* $n < \min\{2f^{+1}(Q_1, \dots, Q_4) + 4, f^{+2}(Q_1, \dots, Q_4) + 1\}$ *and* $n \geqslant g(Q_1, \dots, Q_4) + 2$. *Then* $\mathscr{H}_{n,4}(q, Q_1, \dots, Q_4)$ *is of infinite type.*

Proof. By Remarks 2.4(iii), we may assume that $0 = f_1 \leqslant f_2 \leqslant f_3 \leqslant e-1$ without loss of generality. First, we suppose that $n \geqslant 4$. Then, by the condition $n < 2f^{+1}(Q_1, \dots, Q_4) + 4$, we have $0 = f_1 < f_2 < f_3 < f_4$. If $g(Q_1, \dots, Q_4) = g_i + g_{i+1}$ for some $i \in \{1, 2, 3, 4\}$, we have $g(Q_1, \dots, Q_4) = f^{+2}(Q_1, \dots, Q_4)$. In this case, the condition $n \geqslant g(Q_1, \dots, Q_4) + 2$ contradicts the condition $n < f^{+2}(Q_1, \dots, Q_4) + 1$. Thus one can assume that $g(Q_1, \dots, Q_4) = g_1 + g_3 = (f_2 - f_1) + (f_4 - f_3)$ (by permuting the parameters, and by a scalar multiplication if necessary).

Let $\mathcal{B}$ be the block of $\mathscr{H}_{n,4}(q, Q_1, \dots, Q_4)$ with the residue $(0, 1, 2, \dots, f_2, f', f' + 1, \dots, f_4 - 1, f_4)$, where $f' = f_4 - (n - (f_2 + 1)) + 1$. Note that $(f_2 + 1) + (f_4 - f' + 1) = n$ and that $f' \leqslant f_3 < f_4$ by the condition $n \geqslant g_1 + g_3 + 2$. Moreover, by the condition $n < f^{+2}(Q_1, \dots, Q_r) + 1$, we have $n < f_4 - f_2 + 1 < f_4 + 1$. This implies that $f_2 < f' - 1$. Put $\lambda_0 = (-, (1^{f_2+1}), -, (1^{f_4-f'+1}))$, $\lambda_i = (-, (1^{f_2+1}), (i, 1^{f_3-f'}), (1^{f_4-f_3-i+1}))$ for $i = 1, \dots, k$ and $\lambda_{k+1} = ((1), (1^{f_2}), -, (1^{f_4-f'+1}))$, where $k = f_4 - f_3 + 1$.

Example. (In the case where $n = 7, e = 16, f_1 = 0, f_2 = 2, f_3 = 8, f_4 = 10$.)



In a similar way as in the proof of Proposition 3.21, one can consider the quotient algebra $\overline{\mathcal{B}} = \mathcal{B}/\mathcal{B}^{\vee}$ of $\mathcal{B}$ which is a cellular algebra with respect to the poset $(\{\lambda_0, \lambda_1, \lambda_{k+1}\}, \trianglerighteq)$. One can easily check that λ_i $(i = 0, 1, k + 1)$ is a Kleshchev multipartition. Now we can compute the decomposition matrix of $\overline{\mathcal{B}}$ by the Jantzen sum formula, and its matrix is completely the same as the decomposition matrix of $\overline{\mathcal{B}}$ in Proposition 3.21, replacing k by $k + 1$. Thus, by Lemma 1.8, $\overline{\mathcal{B}}$ is of infinite type, thus $\mathcal{B}$ and so $\mathscr{H}_{n,4}(q, Q_1, \dots, Q_4)$ is of infinite type.

For the case where $n \leqslant 3$, one can check case by case as in the proof of Proposition 3.21. □

Now, we can prove Theorem 3.13.

Proof of Theorem 3.13. The "if" part is already shown in Proposition 3.19. We show the "only if" part, and we prove only the statement for $\mathscr{H}_{n,r}$ since the statement for $\mathscr{S}_{n,r}$ follows from one for $\mathscr{H}_{n,r}$ by Lemma 2.6.

By Remarks 2.4(iii), we may assume that $0 \leqslant f_1 \leqslant f_2 \leqslant \cdots \leqslant f_r \leqslant e - 1$.

First, we consider the case where $r = 2$. If $n \geqslant \min\{e, 2f^{+1}(Q_1, Q_2) + 4\}$, then $\mathscr{H}_{n,2}(q, Q_1, Q_2)$ is of infinite type by [4, Theorem 1.4].

Next, we consider the case where $r \geqslant 3$. If $n \geqslant \min\{e, 2f^{+1}(Q_1, \dots, Q_r) + 4\}$, then we have $n \geqslant \min\{e, 2(f_{i+1} - f_i) + 4\}$ for some $i = 1, \dots, r - 1$ or $n \geqslant \min\{e, 2(e - f_r) + 4\}$. Take such i (put $i = r$ if the last case occurs), then $\mathscr{H}_{n,2}(q, Q_i, Q_{i+1})$ (put $i + 1 = 1$ if $i = r$) has infinite type by the above arguments. Thus, $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ has infinite type by Corollary 2.13.

If $n < \min\{e, 2f^{+1}(Q_1, \dots, Q_r) + 4\}$ and $n \geqslant f^{+2}(Q_1, \dots, Q_r) + 1$, then there exist $i \in \{1, \dots, r\}$ such that $\mathscr{H}_{n,3}(q, Q_i, Q_{i+1}, Q_{i+2})$ (put $r + 1 = 1, r + 2 = 2$ if $i = r - 1$ or $i = r$)

satisfies the assumption in Proposition 3.21 (by adjusting the parameters by a scalar multiplication if necessary). By Proposition 3.21, such $\mathscr{H}_{n,3}(q, Q_i, Q_{i+1}, Q_{i+2})$ has infinite type. Thus, we see that $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ has infinite type by Corollary 2.13.

Finally, we consider the case where $n < \min\{2f^{+1}(Q_1, \dots, Q_r) + 4, f^{+2}(Q_1, \dots, Q_r) + 1\}$ and $n \geqslant g(Q_1, \dots, Q_r) + 2$. If $g(Q_1, \dots, Q_r) = g_i + g_{i+1}$ for some $i \in \{1, \dots, r\}$, we have $g(Q_1, \dots, Q_r) = f^{+2}(Q_1, \dots, Q_r)$. In this case, the conditions $n < f^{+2}(Q_1, \dots, Q_r) + 1$ and $n \geqslant g(Q_1, \dots, Q_r) + 2$ are not compatible. Thus, there exist $i, j \in \{1, \dots, r\}$ such that $j - i > 1$, $\{i, j\} \neq \{1, r\}$, and that $\mathscr{H}_{n,4}(q, Q_i, Q_{i+1}, Q_j, Q_{j+1})$ (put $r + 1 = 1$ if $j = r$) satisfies the assumption in Proposition 3.22 (by adjusting the parameters by a scalar multiplication if necessary). By Proposition 3.22, such $\mathscr{H}_{n,4}(q, Q_i, Q_{i+1}, Q_j, Q_{j+1})$ has infinite type. Thus $\mathscr{H}_{n,r}(q, Q_1, \dots, Q_r)$ is of infinite type by Corollary 2.13.

This proves the “only if” part. We have completed the proof of Theorem 3.13. □

Remarks 3.23.

(i) The Poincaré polynomial of the complex reflection group $W = \mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ is given as

$$P_W(t) = \prod_{i=1}^{n} \frac{t^{ir} - 1}{t - 1}.$$

Now, we consider the Ariki–Koike algebra $\mathscr{H}_{n,r}$ with one parameter, namely, in the case where the first relation in the definition of Ariki–Koike algebra (see 2.2) is

$$(T_0 - q)(T_0 - \zeta)(T_0 - \zeta^2) \cdots (T_0 - \zeta^{r-1}) = 0, \tag{3.23.1}$$

where ζ is a primitive r-th root of unity. We assume that q is a primitive e-th root of unity and that r divides e. Then we have $\zeta = q^{\frac{e}{r}}$. In order to apply Theorem 3.13, we rewrite the relation (3.23.1) by changing the generator T_0 by $q^{-1}T_0$ as follows.

$$(T_0 - 1)(T_0 - q^{\frac{e}{r}-1})(T_0 - q^{\frac{2e}{r}-1}) \cdots (T_0 - q^{\frac{(r-1)e}{r}-1}).$$

In this case, the condition $n < \min\{2f^{+1}(Q_1, \dots, Q_r) + 4, f^{+2}(Q_1, \dots, Q_r) + 1, g(Q_1, \dots, Q_r) + 2\}$ is equivalent to the condition $n \leqslant \frac{2e}{r}$. Thus, by Theorem 3.13, the Ariki–Koike algebra $\mathscr{H}_{n,r}$ is of finite type if and only if $n \leqslant \frac{2e}{r}$. Moreover, if $\mathscr{H}_{n,r}$ is not semisimple then we have $\frac{e}{r} \leqslant n$. The condition $\frac{e}{r} \leqslant n \leqslant \frac{2e}{r}$ is equivalent to the condition that q (a primitive e-th root of unity) is a simple root of the Poincaré polynomial $P_W(t)$. This result is compatible with a generalization of Uno’s conjecture for Hecke algebras [23].

(ii) One can check that, if $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ is of finite type, then the weight of each block of $\mathscr{S}_{n,r}(q, Q_1, \dots, Q_r)$ (in the sense of Fayers [14]) is less than or equal to one by [14, Proposition 3.5] and the definition of the weight of a block [14, 2.1] combined with Lemma 3.18. On the other hand, if the weight of a block of $\mathscr{S}_{n,r}$ is 0, then such a block is semisimple by [14, Theorem 4.1]. If the weight of a block of $\mathscr{S}_{n,r}$ is one, then such a block is of finite type by [14, Theorem 4.12] combined with Proposition 1.7. (These facts give an alternate proof of Proposition 3.19.) Hence, it is likely that a block of cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ is of (non-semisimple) finite type if and only if the weight of the block is equal to one.

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