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Precise large deviations for compound random sums in the presence of dependence structures[☆]

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ABSTRACT

In this paper, we deal with the compound random sums of dependent real-valued random variables with heavy-tailed distributions. We establish a precise large deviation result for a nonstandard renewal risk model in which innovations are extended negatively dependent real-valued random variables with a common dominatedly varying distribution function, their interarrival times are extended negatively dependent nonnegative random variables, and the numbers of innovations caused by individual events are also extended negatively dependent positive random variables. As an illustration of the obtained result, we give two applications related to some insurance risk models.

1. Introduction

In this paper, we investigate a dependent compound renewal risk model driven by three sequences of random variables: the individual innovation sizes $\{X_1, X_2, \ldots\}$ form a sequence of dependent real-valued random variables (r.v.s) with a common distribution function (d.f.) F_X and a finite mean μ_X, the interarrival times $\{\theta_1, \theta_2, \ldots\}$ form another sequence of identically distributed nonnegative r.v.s with a finite positive mean β^{-1}, and $\{Z_1, Z_2, \ldots\}$ constitute a sequence of identically distributed positive integer-valued r.v.s with a common d.f. F_Z and a finite mean μ_Z, where Z_n corresponds to the number of innovations caused by the nth event. We assume that the three sequences above, $\{X_1, X_2, \ldots\}$, $\{\theta_1, \theta_2, \ldots\}$ and $\{Z_1, Z_2, \ldots\}$, are mutually independent, although the r.v.s belonging to any of these sequences can be interdependent in some way.

Let

$$\Theta(t) := \sup\left\{n \ge 1 : \sum_{i=1}^{n} \theta_i \le t\right\} \quad \text{and} \quad \Lambda(t) := \sum_{k=1}^{\Theta(t)} Z_k, \quad t \ge 0 \tag{1.1}$$

denote the renewal counting process and the compound renewal counting process, respectively, and let $\theta(t) := \mathrm{E}\Theta(t)$, $\lambda(t) := \mathrm{E}\Lambda(t) = \mu_Z \theta(t)$. In the case of independent θ_is we have $\theta(t) \sim \beta t$ as $t \to \infty$ by the elementary renewal theorem. This property remains true for the dependence structure considered in the paper (see Lemma 5.5).

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Let $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$, $n \geq 1$. The objective of the paper is to establish precise large deviations for the random sum

$$S_{\Lambda(t)} = \sum_{k=1}^{\Lambda(t)} X_k, \quad t \geq 0$$

with applications to some insurance risk models. Some history and more rigorous formulation of the precise large deviations problem together with the main result of the paper are postponed to Section 3 after we give some preliminaries on the heavy-tailed distributions and dependence structures considered in the paper (Section 2). Section 4 gives two potential applications of the obtained results. Sections 5 and 6 present some auxiliary results and the proof of the main result, respectively.

2. Preliminaries

Throughout this paper, without special statement, all the limit relationships hold for t tending to infinity. For two positive functions $a(t)$ and $b(t)$, we write $a(t) \sim b(t)$ if $\lim a(t)/b(t) = 1$; write $a(t) \lesssim b(t)$ if $\limsup a(t)/b(t) \leq 1$; write $a(t) \gtrsim b(t)$ if $\liminf a(t)/b(t) \geq 1$; write $a(t) = o(b(t))$ if $\lim a(t)/b(t) = 0$; and write $a(t) = O(b(t))$ if $\limsup a(t)/b(t) < \infty$. Furthermore, for two positive bivariate functions $a(t, x)$ and $b(t, x)$, we write $a(t, x) \sim b(t, x)$ uniformly for all x in some nonempty set Ω, if

$$\lim_{t\to\infty} \sup_{x\in\Omega} \left| \frac{a(t, x)}{b(t, x)} - 1 \right| = 0;$$

write $a(t, x) \lesssim b(t, x)$ uniformly for all $x \in \Omega$, if

$$\limsup_{t\to\infty} \sup_{x\in\Omega} \frac{a(t, x)}{b(t, x)} \leq 1;$$

and write $a(t, x) \gtrsim b(t, x)$ uniformly for all $x \in \Omega$, if

$$\liminf_{t\to\infty} \inf_{x\in\Omega} \frac{a(t, x)}{b(t, x)} \geq 1.$$

The indicator function of an event A is denoted by $\mathbb{1}_A$. Also the symbol c always represents a finite and positive constant whose value may vary from line to line.

2.1. Heavy-tailed distribution classes

Here we recall some subclasses of heavy-tailed distributions which we consider in our paper. A d.f. V on $\mathbb{R}$ with the tail $\overline{V} = 1 - V$ is said to have a dominatedly varying tail, denoted by $V \in \mathscr{D}$, if $\limsup \overline{V}(yt)/\overline{V}(t) < \infty$ for any fixed $0 < y < 1$. A slightly smaller class is $\mathscr{C}$. A d.f. V on $\mathbb{R}$ is said to have a consistently varying tail ($V \in \mathscr{C}$) if $\lim_{y\nearrow 1} \limsup_{t\to\infty} \overline{V}(yt)/\overline{V}(t) = 1$. A d.f. V is said to have a long tail ($V \in \mathscr{L}$) if $\lim \overline{V}(y + t)/\overline{V}(t) = 1$ for any fixed $y \in \mathbb{R}$. It is well-known that the following inclusion relationships hold:

$$\mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{L}$$

(see, e.g., [1,2]). For any distribution V, denote

$$J_V^+ = -\lim_{y\to\infty} \frac{\log \overline{V}_*(y)}{\log y}, \quad \text{where} \quad \overline{V}_*(y) := \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} \quad \text{for } y > 1,$$

its upper Matuszewska index. Additionally, denote $L_V := \lim_{y\searrow 1} \overline{V}_*(y)$ (clearly, $0 \leq L_V \leq 1$). The presented definitions yield that the following assertions are equivalent (for details, see [3]):

$$\text{(i) } V \in \mathscr{D}, \qquad \text{(ii) } \overline{V}_*(y) > 0 \quad \text{for some } y > 1, \qquad \text{(iii) } L_V > 0, \qquad \text{(iv) } J_V^+ < \infty.$$

Also, $V \in \mathscr{C}$ if and only if $L_V = 1$.

2.2. Dependence structures

In this subsection we describe the dependence structures which we use in our paper. According to Liu [4], r.v.s $\xi_1, \xi_2, \ldots$ are said to be upper extended negatively dependent (UEND(M_ξ)) if there exists some positive constant M_ξ, such that for each $k = 1, 2, \ldots$ and all $y_1, \ldots, y_k$

$$\mathrm{P}\left(\bigcap_{i=1}^{k} \{\xi_i > y_i\}\right) \leq M_\xi \prod_{i=1}^{k} \mathrm{P}(\xi_i > y_i); \tag{2.1}$$

they are said to be lower extended negatively dependent (LEND(M_ξ)) if there exists some positive constant M_ξ, such that for each $k = 1, 2, \ldots$ and all $y_1, \ldots, y_k$

$$\mathrm{P}\left(\bigcap_{i=1}^{k}\{\xi_i \le y_i\}\right) \le M_\xi \prod_{i=1}^{k} \mathrm{P}(\xi_i \le y_i); \tag{2.2}$$

and the r.v.s $\xi_1, \xi_2, \ldots$ are said to be extended negatively dependent (END(M_ξ)) if they are both UEND(M_ξ) and LEND(M_ξ). If $M_\xi = 1$ in (2.1) or (2.2), then the r.v.s $\xi_1, \xi_2, \ldots$ are said to be upper negatively dependent (UND) or lower negatively dependent (LND), respectively; they are said to be negatively dependent (ND) if (2.1) and (2.2) both hold with $M_\xi = 1$ according to Ebrahimi and Ghosh [5] and Block et al. [6].

3. Precise large deviations for the dependent compound renewal risk process

First, we briefly describe the works related to the main result of our paper. Some earlier works on the precise large deviations for nonrandom sums $S_n = \sum_{k=1}^{n} X_k$ can be found in [7–10], among others. All these results are restricted to nonnegative, independent and identically distributed (i.i.d.) r.v.s. Considering the corresponding classes of heavy-tailed distributions, for any fixed $\gamma > 0$, the relation

$$\mathrm{P}(S_n - n\mu_X > x) \sim n\overline{F_X}(x) \tag{3.1}$$

holds uniformly for all $x \ge \gamma n$ as $n \to \infty$. Liu [11] obtained (3.1) with no centering constant $n\mu_X$ for nonnegative negatively associated r.v.s with consistently varying tails. Tang [12] and Liu [4] considered the case of real-valued ND and END r.v.s, respectively, with consistently varying tails and obtained (3.1). Yang and Wang [13] extended Liu's result to the class $\mathscr{D}$ with nonidentically distributed real-valued r.v.s. By doing so, the authors imposed some extra conditions under which the distributions of $X_1, X_2, \ldots$ do not differ too much from each other. Below we restate Theorem 2.1 of Yang and Wang [13] for identically distributed r.v.s.

Theorem 3.1 *([13]). Let $X_1, X_2, \ldots$ be END(M) r.v.s with common d.f. F_X and mean $\mu_X = 0$. If $F_X \in \mathscr{D}$, then for any $\gamma > 0$*

$$\mathrm{P}(S_n > x) \lesssim L_{F_X}^{-1}\, n\overline{F_X}(x) \tag{3.2}$$

holds uniformly for all $x \ge \gamma n$ as $n \to \infty$. If, in addition, $F_X(-x) = o\big(\overline{F_X}(x)\big)$ as $x \to \infty$ and $\mathrm{E}|X_1|^{\alpha_X}\mathbb{1}_{\{X_1 \le 0\}} < \infty$ for some $\alpha_X > 1$, then for any $\gamma > 0$

$$\mathrm{P}(S_n > x) \gtrsim L_{F_X} n\overline{F_X}(x) \tag{3.3}$$

holds uniformly for all $x \ge \gamma n$ as $n \to \infty$.

The precise large deviation results for random sums $S_{N(t)}$ are closely related to the results of types (3.1)–(3.3). The study of precise large deviations for random sums was initiated by Klüppelberg and Mikosch [14], who presented several applications of such sums in insurance and finance. The study of precise large deviations focuses on the quantity

$$S_{N(t)} = \sum_{k=1}^{N(t)} X_k, \quad t \ge 0,$$

where $N(t)$ is an integer-valued counting process with the mean function $\nu(t) = \mathrm{E}N(t)$, independent of the sequence $\{X_1, X_2, \ldots\}$, and usually deals with asymptotic relation

$$\mathrm{P}(S_{N(t)} - \mu_X \nu(t) > x) \sim \nu(t)\overline{F_X}(x) \tag{3.4}$$

as $t \to \infty$, uniformly for all $x \ge \gamma\nu(t)$. Recent advances on precise large deviations for random sums can be found in [15,16,10,11,17,18], among others. Note that, Chen et al. [19] and Yang and Wang [13] obtained the relations of type (3.4) in the presence of consistent variation and dominated variation, respectively, under the END structure and the following condition on the counting process $N(t)$:

$$\mathrm{E}\big((N(t))^p\, \mathbb{1}_{\{N(t) > (1+\delta)\nu(t)\}}\big) = O(\nu(t)) \tag{3.5}$$

for some $p > J_{F_X}^{+}$ and for all $\delta > 0$. Note that every renewal counting process $\{N(t), t \ge 0\}$ generated by i.i.d. nonnegative r.v.s $Y_1, Y_2, \ldots$, such that $\mathrm{E}Y_1 < \infty$, fulfills (3.5) with any $p > 0$ and $\delta > 0$, see [15,20].

Kaas and Tang [21] and Konstantinides and Loukissas [22] derived the precise large deviations results for the compound random sums $S_{\Lambda(t)}$ with nonnegative consistently varying-tailed increments and the compound renewal counting process $\Lambda(t)$ defined in (1.1). Concerning dependence structures, Konstantinides and Loukissas [22] assumed that all three sequences $\{X_1, X_2, \ldots\}$, $\{\theta_1, \theta_2, \ldots\}$, $\{Z_1, Z_2, \ldots\}$, generating the process $S_{\Lambda(t)}$, consist of i.i.d. r.v.s., while Kaas and Tang [21] assumed some negative dependence structure among $Z_1, Z_2, \ldots$. In the following theorem, we extend the above mentioned results to the case of real-valued increments with dominatedly varying tails, together allowing some dependence within the sequences $\{X_1, X_2, \ldots\}$, $\{\theta_1, \theta_2, \ldots\}$, $\{Z_1, Z_2, \ldots\}$.

Theorem 3.2. *Let $\{X_1, X_2, \ldots\}$ be a sequence of END(M) real-valued r.v.s with common d.f. $F_X \in \mathscr{D}$ and finite mean μ_X; let $\{\theta_1, \theta_2, \ldots\}$ be a sequence of END(M) and identically distributed nonnegative r.v.s with positive mean $1/\beta$; let $\{Z_1, Z_2, \ldots\}$ be a sequence of END(M) positive integer-valued r.v.s with common d.f. F_Z and $\mathrm{E}Z_1^{\alpha_Z} < \infty$ with some $\alpha_Z > J_{F_X}^+ + 2$. Suppose that $\{\Lambda(t), t \ge 0\}$ is a compound renewal counting process defined in (1.1) and assume that the sequences $\{X_1, X_2, \ldots\}$, $\{\theta_1, \theta_2, \ldots\}$ and $\{Z_1, Z_2, \ldots\}$ are mutually independent.*

(i) *If $\mu_X < 0$ then*

$$\mathrm{P}\left(S_{\Lambda(t)} - \mu_X \lambda(t) > x\right) \lesssim \left(L_{F_X}\right)^{-2} \lambda(t) \overline{F_X}(x) \tag{3.6}$$

uniformly for all $x \ge \gamma\lambda(t)$ and any $\gamma > |\mu_X|$. If, in addition, $F_X(-x) = o(\overline{F_X}(x))$ as $x \to \infty$ and $\mathrm{E}|X_1|^{\alpha_X} \mathbb{1}_{\{X_1 \le 0\}} < \infty$ for some $\alpha_X > 1$ then

$$\mathrm{P}\big(S_{\Lambda(t)} - \mu_X \lambda(t) > x\big) \gtrsim L_{F_X}^2 \lambda(t) \overline{F_X}(x) \tag{3.7}$$

in the same uniformity region, i.e. for all $x \ge \gamma\lambda(t)$ and any $\gamma > |\mu_X|$.

(ii) *If $\mu_X \ge 0$, then (3.6) and (3.7) hold uniformly for all $x \ge \gamma\lambda(t)$ and any $\gamma > 0$.*

The proof of Theorem 3.2 is presented in Section 6.

4. Applications

The obtained precise large deviations result provides a method of computing probabilities of rare events related to the sums of heavy-tailed random variables, by providing an approximate expression for the corresponding probabilities in terms of characteristics of the underlying random variables: constants L_{F_X}, μ_Z, mean function $\theta(t)$ and tail probability $\overline{F_X}(x)$.

In this section we present two applications of the main result illustrating how asymptotic relations (3.6) and (3.7) can be used for computation of some risk characteristics in the insurance business. Nice introductions to the research on precise large deviations with the focus on applications to insurance are Klüppelberg and Mikosch [14] and Mikosch and Nagaev [9]. In particular, some problems related to large claims, such as total claim amount, proportional reinsurance, excess-of-loss reinsurance and stop-loss reinsurance, are considered therein.

Compound renewal risk model. The first application concerns the so-called compound renewal risk model which can be used in the multi-risk insurance business. Such a model was introduced by Tang et al. [15] (see also [23]). The construction of the compound renewal risk model is given via the following assumptions.

Assumption A_1. The accidents' interarrival times $\theta_1, \theta_2, \ldots$ are nonnegative identically distributed r.v.s with finite positive mean β^{-1}.

Assumption A_2. Individual claim sizes and the number of individual claims caused by the nth accident at time $\theta_1 + \cdots + \theta_n$ are $X_1^{(n)}, X_2^{(n)}, \ldots$ and Z_n, respectively.

Assumption A_3. The sequences $\{X_1^{(n)}, X_2^{(n)}, \ldots\}$, $n \ge 1$, are independent copies of the sequence $\{X_1, X_2, \ldots\}$ of identically distributed nonnegative r.v.s with common d.f. F_X and finite mean μ_X. $\{Z_1, Z_2, \ldots\}$ is a sequence of identically distributed positive integer-valued r.v.s with finite mean μ_Z.

Assumption A_4. The sequences $\{\theta_1, \theta_2, \ldots\}$, $\{Z_1, Z_2, \ldots\}$ and $\{X_1^{(n)}, X_2^{(n)}, \ldots\}$, $n \ge 1$, are mutually independent.

In the described model, $\Theta(t)$ is the number of accidents up to time t, $\Lambda(t) = \sum_{k=1}^{\Theta(t)} Z_k$ is the total number of claims up to time t, $\sum_{n=1}^{\Theta(t)} \sum_{k=1}^{Z_n} X_k^{(n)}$ is the total claim amount up to time t. Obviously, if $X_1, X_2, \ldots$ are i.i.d., then under Assumptions A_1–A_4 it holds that

$$\sum_{n=1}^{\Theta(t)} \sum_{k=1}^{Z_n} X_k^{(n)} \overset{d}{=} \sum_{k=1}^{\Lambda(t)} X_k,$$

where $\overset{d}{=}$ denotes the equality in distribution, and Theorem 3.2 can be applied in order to estimate the probability that the total claim amount up to time t exceeds a certain level. For instance, if $F_X \in \mathscr{C}$, $\{\theta_1, \theta_2, \ldots\}$ and $\{Z_1, Z_2, \ldots\}$ are END(M) sequences, $\mathrm{E}Z_1^{\alpha_Z} < \infty$ for $\alpha_Z > J_{F_X}^+ + 2$ and Assumptions A_1–A_4 are satisfied, then

$$\mathrm{P}\big(S_{\Lambda(t)} > x + \mu_X \mu_Z \theta(t)\big) \sim \mu_Z \theta(t) \overline{F_X}(x)$$

uniformly for all $x > \gamma\mu_Z\theta(t)$ with any $\gamma > 0$.

Compound customer-arrival-based insurance risk model. Another example concerns the model suggested by Ng et al. [16], where the number of customers rather than the number of claims is counted in order to describe the surplus of an insurance company. We give a rigorous description of such a compound customer-arrival-based insurance risk model (CCIRM) via the following assumptions.

Assumption B_1. The interarrival times between the customers' arrivals $\theta_1, \theta_2, \ldots$ are identically distributed nonnegative r.v.s with positive mean β^{-1}.

Assumption B_2. At the time $\theta_1 + \cdots + \theta_n$, the nth customer purchases some random number Z_n of insurance contracts. Assume that $Z_1, Z_2, \ldots$ are identically distributed positive integer-valued r.v.s with common d.f. F_Z and finite mean μ_Z. For each insurance policy, the insurance company will bear a risk from this policy holder within a fixed term.

Assumption B_3. The potential claim due to the kth insurance policy of the nth customer is $Y_k^{(n)}$. For each $n = 1, 2, \ldots$, the sequence $\{Y_1^{(n)}, Y_2^{(n)}, \ldots\}$ is an independent copy of the sequence of identically distributed nonnegative r.v.s $\{Y_k, k = 1, 2, \ldots\}$ with common d.f. F_Y and finite mean μ_Y. Assume that $\{Y_k^{(n)},\ k = 1, 2, \ldots\}_{n=1,2,\ldots}$, $\{\theta_n,\ n = 1, 2, \ldots\}$ and $\{Z_n, n = 1, 2, \ldots\}$ are mutually independent.

Assumption B_4. The premium of each policy is $(1 + \rho)\mu_Y$, where the positive constant ρ can be interpreted as the safety loading coefficient.

In such a model, $\Theta(t)$ is the number of customers arriving by time t, $\Lambda(t)$ is the total number of purchased insurance policies by time t and

$$S_{\Lambda(t)} = \sum_{n=1}^{\Theta(t)} \sum_{k=1}^{Z_n} \big(Y_k^{(n)} - (1+\rho)\mu_Y\big), \quad t \geq 0$$

is the prospective loss process. If $Z_1 = Z_2 = \cdots = 1$, then the model reduces to the standard customer-arrival-based insurance risk model (CIRM), which, in the independence structure, was introduced by Ng et al. [16]. The proposed *dependent compound CIRM* is based on the following two motivations. Firstly, in such a model we can regard each customer as a company, who may buy more than one insurance policy for its employers and the number of the employers is a positive integer-valued r.v.; secondly, this model considers all the companies coming from a certain category, hence, the interarrival times for the customer-arrivals and the numbers of insurance policies purchased by the companies may be dependent, but all the companies within this category are indifferent in terms of their total potential claims.

If in the latter model the r.v.s $Y_1, Y_2, \ldots$ are *independent* (it is common, for example, in the medical malpractice insurance or in transport company's insurance), then

$$S_{\Lambda(t)} \stackrel{d}{=} \sum_{k=1}^{\Lambda(t)} \big(Y_k - (1+\rho)\mu_Y\big),$$

so that Theorem 3.2 can be used to estimate the surplus of the insurance company within the period $[0, t]$, which can be written as $U_x(t) = x - S_{\Lambda(t)}$. For instance, if $F_Y \in \mathscr{C}$, $\{\theta_1, \theta_2, \ldots\}$ and $\{Z_1, Z_2, \ldots\}$ are END(M) sequences, $\mathrm{E}Z_1^{\alpha_Z} < \infty$ for $\alpha_Z > J_{F_Y}^+ + 2$ and Assumptions B_1–B_4 are satisfied, then

$$\mathrm{P}(U_x(t) < 0) \sim \mu_Z \theta(t) \overline{F_Y}\big(x + \rho\mu_Y\mu_Z\theta(t)\big)$$

uniformly for all $x \geq \gamma\mu_Z\theta(t)$ with any $\gamma > 0$.

5. Auxiliary results

In this section we formulate the auxiliary lemmas which will be used in the proof of the main result.

We start with one property of the class $\mathscr{D}$. The lemma below can be verified by using the results in [3, Chapter 2]. See also Lemma 3.5 in [24].

Lemma 5.1. *For a d.f.* $F \in \mathscr{D}$ *it holds that* $x^{-p} = o(\overline{F}(x))$, *as* $x \to \infty$, *for any* $p > J_F^+$.

The next two lemmas present two inequalities for UEND r.v.s, proved by Chen et al. [19] (see their Lemmas 2.2 and 2.3, respectively).

Lemma 5.2. *Let* $\{\xi_1, \xi_2, \ldots\}$ *be a sequence of UEND(*M_ξ*) r.v.s with common d.f.* F_ξ *and finite mean* μ_ξ. *If* $\mathrm{E}\xi_1^\alpha \mathbb{1}_{\{\xi_1 \geq 0\}} < \infty$ *for some* $\alpha > 1$ *then for every fixed* $\gamma > 0$ *and* $p > 0$ *there exist positive numbers* $v = v(\alpha, p)$ *and* $c = c(v, \gamma, M_\xi)$ *such that*

$$\mathrm{P}(\xi_1 + \cdots + \xi_n - n\mu_\xi > x) \leq n\overline{F_\xi}(vx) + cx^{-p}$$

for all $n = 1, 2, \ldots$ *and* $x \geq \gamma n$.

Lemma 5.3. *Let* $\{\xi_1, \xi_2, \ldots\}$ *be a sequence of UEND(*M_ξ*) r.v.s with common d.f.* F_ξ*. If* $0 < \mathrm{E}\xi_1 \mathbb{1}_{\{\xi_1 \ge 0\}} < \infty$ *then for every fixed* $v > 0$ *there exists a positive number* $c = c(v, M_\xi)$ *such that*

$$\mathrm{P}(\xi_1 + \cdots + \xi_n > x) \le n\overline{F_\xi}(vx) + c(n/x)^{1/v}$$

for all $n = 1, 2, \ldots$ *and* $x > 0$.

The statement below about the uniform estimate of the sum of UEND(M_ξ) random variables was proved by Tang [12]. (In fact, his Corollary 3.1 was proved for UND r.v.s. But, obviously, the change of the UND structure to the UEND structure can only affect the positive constant in the estimate.)

Lemma 5.4. *Let* $\{\xi_1, \xi_2, \ldots\}$ *be a sequence of UEND(*M_ξ*) r.v.s with common d.f.* $F_\xi \in \mathscr{D}$ *and mean* $\mu_\xi = 0$*. Then for each fixed* $\gamma > 0$ *and some* $c = c(\gamma, M_\xi)$*, irrespective to x and n, the inequality*

$$\mathrm{P}(\xi_1 + \cdots + \xi_n > x) \le c\, n\overline{F_\xi}(x)$$

holds for all $x \ge \gamma n$ *and* $n = 1, 2, \ldots$*.*

The following statement is an elementary renewal theorem for the compound renewal counting process defined in (1.1). The proof of the lemma uses the recent results of Chen et al. [25].

Lemma 5.5. *Let* $\Lambda(t)$ *be a compound renewal counting process defined in (1.1), let* $\{\theta_1, \theta_2, \ldots\}$ *be a sequence of END(M) identically distributed nonnegative r.v.s with positive mean* $1/\beta$*, and let* $\{Z_1, Z_2, \ldots\}$ *be a sequence of END(M) identically distributed, positive integer-valued r.v.s with common d.f.* F_Z *and finite mean* μ_Z*. Assume that the sequence* $\{Z_1, Z_2, \ldots\}$ *is independent of* $\{\theta_1, \theta_2, \ldots\}$*. Then* $\lambda(t) \sim \mu_Z \beta t$ *and*

$$\frac{\Lambda(t)}{\lambda(t)} \xrightarrow{\text{a.s.}} 1. \tag{5.1}$$

Proof. The first statement of the lemma follows from Theorem 4.2 (b) of Chen et al. [25]. To show the second statement, for positive t write

$$\frac{\Lambda(t)}{\lambda(t)} = \frac{\sum_{k=1}^{\Theta(t)} Z_k}{\Theta(t)} \frac{\Theta(t)}{\mu_Z \theta(t)}. \tag{5.2}$$

By Theorem 1.1 of Chen et al. [25], we have

$$\frac{1}{n}\sum_{k=1}^{n} Z_k \xrightarrow{\text{a.s.}} \mu_Z \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n} \theta_k \xrightarrow{\text{a.s.}} \frac{1}{\beta} \quad \text{as } n \to \infty. \tag{5.3}$$

These two relations and the Anscombe's theorem yield the convergence

$$\frac{1}{\Theta(t)}\sum_{k=1}^{\Theta(t)} Z_k \xrightarrow{\text{a.s.}} \mu_Z,$$

since the last relation in (5.3) implies

$$\Theta(t) = \sum_{n=1}^{\infty} \mathbb{1}_{\{\theta_1 + \cdots + \theta_n \le t\}} \xrightarrow{\text{a.s.}} \infty.$$

This, together with (5.2) and Theorem 4.2 of Chen et al. [25], implies (5.1). □

The following lemma provides some property of the renewal counting process $\Theta(t)$ defined in (1.1). Having in mind that $\theta(t) \sim \beta t$ according to Lemma 5.5, the proof of this lemma differs only slightly from the proof of Theorem 1 in [20] and thus is omitted.

Lemma 5.6. *Let* $\Theta(t)$ *be a renewal counting process in (1.1) driven by a sequence of LEND(*M_θ*), identically distributed nonnegative r.v.s* $\theta_1, \theta_2, \ldots$ *with positive mean* $1/\beta$*. Then for any* $\delta > 0$ *there exists* $\Delta > 0$ *such that*

$$\sum_{k > (1+\delta)\theta(t)} (1 + \Delta)^k \mathrm{P}(\Theta(t) \ge k) = o(1).$$

6. Proof of Theorem 3.2

Write

$$\begin{aligned}\mathrm{P}(S_{\Lambda(t)} - \mu_X\lambda(t) > x) &= \sum_{k=1}^{\infty} \mathrm{P}\big(S_k - \mu_X\lambda(t) > x\big)\mathrm{P}(\Lambda(t) = k)\\ &= \left(\sum_{k<(1-\delta)\lambda(t)} + \sum_{|k-\lambda(t)|\le\delta\lambda(t)} + \sum_{k>(1+\delta)\lambda(t)}\right)\mathrm{P}\big(S_k - \mu_X\lambda(t) > x\big)\mathrm{P}(\Lambda(t) = k)\\ &=: I_1 + I_2 + I_3,\end{aligned} \tag{6.1}$$

where δ is an arbitrarily chosen constant in the interval $(0, \min\{\gamma/(2|\mu_X|), 1/2\})$.

The rest of the proof deals with the summands I_1, I_2 and I_3 separately.

(I) Consider the term I_1. We will prove that under conditions of the theorem it holds that

$$\limsup_{t\to\infty} \sup_{x\ge\gamma\lambda(t)} \frac{I_1}{\lambda(t)\overline{F_X}(x)} = 0, \tag{6.2}$$

where $\gamma > |\mu_X|$ if $\mu_X < 0$ and $\gamma > 0$ if $\mu_X \ge 0$.

In the case $\mu_X < 0$ we have that for any $\gamma > |\mu_X|$ and all $x \ge \gamma\lambda(t)$

$$\begin{aligned} I_1 &\le \sum_{k<(1-\delta)\lambda(t)} \mathrm{P}\big(S_k - k\mu_X > x - |\mu_X|\lambda(t)\big)\mathrm{P}(\Lambda(t) = k)\\ &\le \sum_{k<(1-\delta)\lambda(t)} \mathrm{P}\big(S_k - k\mu_X > (1 - |\mu_X|/\gamma)x\big)\mathrm{P}(\Lambda(t) = k).\end{aligned}$$

Since $F_X \in \mathscr{D}$, by Lemma 5.4 we have

$$\begin{aligned} I_1 &\le \sum_{k<(1-\delta)\lambda(t)} c\,\overline{F_X}\big((1 - |\mu_X|/\gamma)x\big)\,k\,\mathrm{P}(\Lambda(t) = k)\\ &\le c\,(1-\delta)\lambda(t)\overline{F_X}\big((1 - |\mu_X|/\gamma)x\big)\,\mathrm{P}(\Lambda(t) < (1-\delta)\lambda(t)),\end{aligned}$$

where c is a positive constant irrespective to x and k. Hence, using Lemma 5.5, we obtain relation (6.2) for any $\gamma > |\mu_X|$.

In the case $\mu_X \ge 0$ we have that $x - k\mu_X + \mu_X\lambda(t) \ge x$ for all $k < (1-\delta)\lambda(t)$. Therefore,

$$I_1 \le \sum_{k<(1-\delta)\lambda(t)} \mathrm{P}(S_k - k\mu_X > x)\mathrm{P}(\Lambda(t) = k),$$

and, similarly, we derive relation (6.2) for any $\gamma > 0$.

(II) Next we deal with I_2. First we will prove that for any $\gamma > 0$

$$\lim_{\delta\downarrow 0}\limsup_{t\to\infty} \sup_{x\ge\gamma\lambda(t)} \frac{I_2}{\lambda(t)\overline{F_X}(x)} \le \frac{1}{L_{F_X}^2}. \tag{6.3}$$

In the case $\mu_X < 0$, if $\gamma > 0$, $x \ge \gamma\lambda(t)$, $|k - \lambda(t)| \le \delta\lambda(t)$ with $\delta \in (0, \min\{\gamma/(2|\mu_X|), 1/2\})$, then

$$x - k\mu_X + \mu_X\lambda(t) \ge \big(1 - \delta|\mu_X|/\gamma\big)\,x \ge (\gamma - \delta|\mu_X|)\lambda(t) \ge (\gamma - \delta|\mu_X|)(1+\delta)^{-1}k$$

and we obtain

$$\begin{aligned}&\limsup_{t\to\infty} \sup_{x\ge\gamma\lambda(t)} \frac{I_2}{\sum_{|k-\lambda(t)|\le\delta\lambda(t)} k\overline{F_X}((1 - \delta|\mu_X|/\gamma)x + \mu_X)\,\mathrm{P}(\Lambda(t) = k)}\\ &\quad\le \limsup_{t\to\infty} \max_{|k-\lambda(t)|\le\delta\lambda(t)} \sup_{x\ge\gamma\lambda(t)} \frac{\mathrm{P}(S_k - k\mu_X > x(1 - \delta|\mu_X|/\gamma))}{k\overline{F_X}((1 - \delta|\mu_X|/\gamma)x + \mu_X)}\\ &\quad\le \limsup_{t\to\infty} \max_{|k-\lambda(t)|\le\delta\lambda(t)} \sup_{y\ge(\gamma-\delta|\mu_X|)(1+\delta)^{-1}k} \frac{\mathrm{P}(S_k - k\mu_X > y)}{k\overline{F_X}(y + \mu_X)} \le \frac{1}{L_{F_X}},\end{aligned}$$

where the last inequality follows from (3.2). Hence, uniformly for $x \ge \gamma\lambda(t)$, relation

$$I_2 \lesssim (1+\delta)\left(L_{F_X}\right)^{-1}\lambda(t)\,\overline{F}_X((1 - 2\delta|\mu_X|/\gamma)x)\,\mathrm{P}(|\Lambda(t) - \lambda(t)| \le \delta\lambda(t))$$

holds because $|\mu_X|/x \le |\mu_X|/(\gamma\lambda(t)) \le \delta|\mu_X|/\gamma$ for sufficiently large t. This estimate and Lemma 5.5 imply that for the above δ, the following inequality holds

$$\limsup_{t\to\infty} \sup_{x\ge\gamma\lambda(t)} \frac{I_2}{\lambda(t)\overline{F_X}(x)} \le \frac{1+\delta}{L_{F_X}} \limsup_{t\to\infty} \sup_{x\ge\gamma\lambda(t)} \frac{\overline{F_X}((1-2\delta|\mu_X|/\gamma)\,x)}{\overline{F_X}(x)}.$$

As $F_X \in \mathscr{D}$, the desired estimate (6.3) follows. Similar arguments show that in the case $\mu_X \ge 0$ estimate (6.3) holds for every positive γ.

Next we will show that, under additional requirements

$$\mathrm{E}|X_1|^{\alpha_X}\mathbb{1}_{\{X_1\le 0\}} < \infty \quad \text{for some } \alpha_X > 1; \qquad F_X(-x) = o\left(\overline{F_X}(x)\right) \quad \text{as } x\to\infty, \tag{6.4}$$

the lower estimate

$$\lim_{\delta\downarrow 0}\liminf_{t\to\infty} \inf_{x\ge\gamma\lambda(t)} \frac{I_2}{\lambda(t)\overline{F_X}(x)} \ge L_{F_X}^2 \tag{6.5}$$

holds for every positive $\gamma > 0$.

If $\mu_X < 0$, then similarly as above we have that

$$\liminf_{t\to\infty} \inf_{x\ge\gamma\lambda(t)} \frac{I_2}{\sum_{|k-\lambda(t)|\le\delta\lambda(t)} k\overline{F_X}(x(1+\delta|\mu_X|/\gamma)+\mu_X)\,\mathrm{P}(\Lambda(t)=k)}$$
$$\ge \liminf_{t\to\infty} \min_{|k-\lambda(t)|\le\delta\lambda(t)} \inf_{y\ge(\gamma+\delta|\mu_X|)(1+\delta)^{-1}k} \frac{\mathrm{P}(S_k-k\mu_X>y)}{k\overline{F_X}(y+\mu_X)}.$$

Hence, according to Theorem 3.1, we obtain that the relation

$$I_2 \gtrsim (1-\delta)L_{F_X}\lambda(t)\,\overline{F_X}((1+2\delta|\mu_X|/\gamma)x)\,\mathrm{P}(|\Lambda(t)-\lambda(t)|\le\delta\lambda(t))$$

holds uniformly for all $x \ge \gamma\lambda(t)$. This estimate and Lemma 5.5 imply that, under additional conditions in (6.4), relation (6.5) holds. The proof in the case $\mu_X \ge 0$ is similar.

(III) Consider now term I_3. We will prove that

$$\limsup_{t\to\infty} \sup_{x\ge\gamma\lambda(t)} \frac{I_3}{\overline{F_X}(x)} = 0 \tag{6.6}$$

for any $\delta \in (0, 1/2)$ and any $\gamma > |\mu_X|$ if $\mu_X < 0$ and any $\gamma > 0$ if $\mu_X \ge 0$.

If $\mu_X < 0$, then for γ, δ as above and for all $x \ge \gamma\lambda(t)$ we have

$$I_3 \le \sum_{k>(1+\delta)\lambda(t)} \mathrm{P}\big(S_k > (1-|\mu_X|/\gamma)x\big)\mathrm{P}(\Lambda(t)=k). \tag{6.7}$$

Since μ_X is finite, by Lemma 5.1 we have that $J_{F_X}^+ \ge 1$. Let $p \in (J_{F_X}^+, \alpha_Z - 2)$. According to Lemmas 5.1 and 5.3, for all $k = 1, 2, \dots$ and $x > 0$ we have that

$$\mathrm{P}(S_k > (1-|\mu_X|/\gamma)x) \le k\overline{F_X}((1-|\mu_X|/\gamma)x/p) + ck^px^{-p} \le c\,k^p\,\overline{F_X}(x),$$

where c is a positive constant irrespective to k and x. Hence, we will prove (6.6) if we show that

$$K := \sum_{k>(1+\delta)\lambda(t)} k^p\,\mathrm{P}(\Lambda(t)=k) = o(1) \tag{6.8}$$

for any $\delta \in (0, 1/2)$.

Take any $\epsilon > 0$ such that $(1+\delta)\mu_Z > \epsilon + \mu_Z$ and split K as follows:

$$K = \sum_{k>(1+\delta)\lambda(t)} k^p \left(\sum_{n>k/(\epsilon+\mu_Z)} + \sum_{n\le k/(\epsilon+\mu_Z)}\right)\mathrm{P}\left(\sum_{i=1}^n Z_i = k\right)\mathrm{P}(\Theta(t)=n)$$
$$=: K_1 + K_2. \tag{6.9}$$

For sufficiently large t,

$$K_1 \le \sum_{k>(1+\delta)\lambda(t)} k^p\,\mathrm{P}\left(\Theta(t) > \frac{k}{\epsilon+\mu_Z}\right)$$
$$\le (\epsilon+\mu_Z)^p \sum_{m>(1+\tilde{\delta})\theta(t)} m^p\,\mathrm{P}(\Theta(t)\ge m),$$

where $1+\tilde{\delta} := (1+\delta)\mu_Z/(\epsilon+\mu_Z) > 1$. Therefore, by Lemma 5.6 we have

$$K_1 = o(1). \tag{6.10}$$

Since $EZ_1^{\alpha_Z} < \infty$ for some $\alpha_Z > J_{F_X}^{+} + 2$, Lemma 5.2 implies that for fixed $\tilde{\gamma} > 0$ and $\tilde{p} > 0$ there exist positive constants $v = v(\alpha_Z, \tilde{p})$ and $c = c(\tilde{\gamma}, \tilde{p}, M)$ such that

$$P\left(\sum_{i=1}^{m}(Z_i - \mu_Z) > y\right) \le m\overline{F_Z}(vy) + \frac{c}{y^{\tilde{p}}}$$

for all $m = 1, 2, \ldots$ and $y \ge \tilde{\gamma} m$. Choose $\tilde{\gamma} = \epsilon$ and $\tilde{p} > p + 1$ in the last estimate. Then for sufficiently large t

$$\begin{aligned} K_2 &= \sum_{k>(1+\delta)\lambda(t)} k^p\, P\left(\sum_{i=1}^{\Theta(t)} Z_i = k, \Theta(t) \le \frac{k}{\epsilon + \mu_Z}\right) \\ &\le \sum_{k>(1+\delta)\lambda(t)} k^p\, P\left(\sum_{i \le k/(\epsilon+\mu_Z)} Z_i \ge k\right) \\ &\le \sum_{k>(1+\delta)\lambda(t)} k^p\, P\left(\sum_{i \le k/(\epsilon+\mu_Z)} (Z_i - \mu_Z) \ge \frac{\epsilon k}{\epsilon + \mu_Z}\right) \\ &\le \sum_{k>(1+\delta)\lambda(t)} k^p\, P\left(\frac{k}{\epsilon + \mu_Z}\, \overline{F_Z}\left(\frac{\epsilon v k}{\epsilon + \mu_Z}\right) + c\left(\frac{\epsilon k}{\epsilon + \mu_Z}\right)^{-\tilde{p}}\right). \end{aligned}$$

By Markov's inequality,

$$\overline{F_Z}\left(\frac{\epsilon v k}{\epsilon + \mu_Z}\right) \le \left(\frac{\epsilon + \mu_Z}{\epsilon v\, k}\right)^{\alpha_Z} EZ_1^{\alpha_Z},$$

implying that

$$K_2 \le \sum_{k>(1+\delta)\lambda(t)} \left(\frac{(\epsilon + \mu_Z)^{\alpha_Z - 1} EZ_1^{\alpha_Z}}{(\epsilon v)^{\alpha_Z}}\, k^{-(\alpha_Z - p - 1)} + \frac{c\,(\epsilon + \mu_Z)^{\tilde{p}}}{\epsilon^{\tilde{p}}}\, k^{-(\tilde{p} - p)}\right) = o(1) \tag{6.11}$$

because $\alpha_Z - p - 1 > 1$ and $\tilde{p} - p > 1$. The estimate (6.8) follows now from (6.9)–(6.11).

If $\mu_X \ge 0$ then, instead of (6.7), we obtain

$$I_3 \le \sum_{k>(1+\delta)\lambda(t)} P(S_k > x) P(\Lambda(t) = k)$$

with $\gamma > 0$ and $\delta \in (0, 1/2)$. From this estimate, we derive (6.6) similarly as in the case $\mu_X < 0$.

Finally, the statement of the theorem follows from estimates (6.2), (6.3), (6.5) and (6.6).

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