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Some combinatorial problems

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Abstract

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There are many interesting and sophisticated problems posed in the IMO, Putnam and domestic Olympiads. Some of these problems have deep mathematical background, nice generalizations, and lead to new areas of research in combinatorics. We investigate several topics in this category and mention some results and open problems.

1. Disjoint simplices

Problem 1.1. Let A be a set of $2n$ points in a general position in the Euclidean plane $\mathbb{R}^2$, and suppose n of the points are colored red and the remaining n are colored blue. Show that there are n pairwise disjoint straight line segments matching the red points with the blue points.

Solution. Consider the set of all $n!$ possible matchings and choose one, M, that minimizes the sum of lengths $l(M)$ of its line segments. It is easy to show that these line segments cannot intersect. Indeed, if two segments r_1, b_1 and r_2, b_2 intersect, where r_1, r_2 are two red points and b_1, b_2 are two blue points, the matching M' obtained from M by replacing r_1b_1 and r_2b_2 by r_1b_2 and r_2b_1 satisfies $l(M')<l(M)$, contradicting the choice of M. $\square$

Akiyama and Alon [1] generalized the result mentioned above to higher dimensional cases.

Theorem 1.1. *Let A be a set of $d \cdot n$ points in a general position in $\mathbb{R}^d$, and let $A = A_1 \cup A_2 \cup \cdots \cup A_d$ be a partition of A into d pairwise disjoint sets, each consisting of*

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n points. Then there are n pairwise disjoint $(d-1)$-dimensional simplices, each containing precisely one vertex from each A_i, $1 \leqslant i \leqslant d$.

2. Alternating simple path problem

In the previous problem, consider 'alternating path' instead of 'line segments' [2]. An alternating path P of A is a sequence $p_1, p_2, \ldots, p_{2n}$ of points of A such that p_{2i} is blue and p_{2i+1} is red. P is called *simple* if it does not intersect itself.

As a natural extension of the matching assertion, we can ask the following question: Does there always exist a simple alternating path P of A? The configuration of 16 points on a circle in Fig. 1 shows that the answer to this question is negative.

Here we can prove the following theorem.

Theorem 2.1. *Let the number of alternating paths with order $2n$ be $P(n)$, and let the number of alternating simple paths with order $2n$ be $P_s(n)$.*

Then $\lim_{n \to \infty} P_s(n)/P(n) = 0$ holds.

3. Balanced colorings

Problem 3.1. Given a finite set of points in the plane, each point having integer coordinates, is it always possible to color some of the points red and the remaining points blue in such a way that, for any straight line L parallel to either one of the coordinate axes, the difference (in absolute value) between the numbers of blue points and red points on L is not greater than 1?

Solution. It is always possible to color the points as required. Take an arbitrary line L parallel to one of the axes and intersecting the given set A. Let $P_1, P_2, \ldots$ be the points of $A \cap L$ arranged in the order of increasing free coordinate. Connect P_1 with P_2, P_3 with P_4 and so on by line segments (there may remain a single point). The same

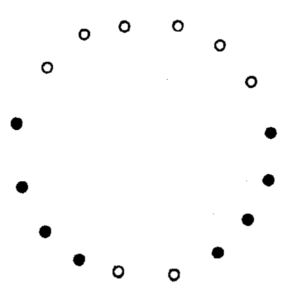

Fig. 1.

is done on every such line L. We obtain a family of segments, each point of A belonging to two segments, at most. Thus the union of these segments splits into polygonal lines, closed or not, without common vertices. Those which are closed are composed of an even number of segments, because any two successive segments are perpendicular. Color the vertices of each polygonal line alternately: red, blue, red, blue etc.; this is possible, in view of the preceding observation on the evenness of cycles. If there have remained any loose points in A, not belonging to any one of the segments under consideration, we color them in an arbitrary way. The coloring thus obtained fulfills the requirement of the problem; points on each horizontal or vertical line, are connected pairwise by segments with endpoints of different colors; if there are an odd number of points on a line, the color of the remaining (rightmost or topmost) point is insignificant. □

4. *m*-Colorings in balanced colorings

In order to extend the result discussed in Problem 3.1, we need to introduce some terminology.

Let P_n be a subset of n elements of the lattice points L of $\mathbb{R}^2$, i.e. $(x, y) \in P_n$ implies $x, y \in \mathbb{N}$. For every $i \in \mathbb{N}$, let R_i and C_i be the rows and columns of P_n, i.e. $R_i = \{(x, y) \in P_n : y = i\}$ and $C_i = \{(x, y) \in P_n : x = i\}$. An m-coloring of P_n is a partitioning of P_n into m subsets $S_i, \ldots, S_m$. Given an m-coloring of P_n, let $R_{ij} = R_i \cap S_j$ and $C_{ij} = C_i \cap S_j$.

An m-coloring of P_n is called *almost balanced* if for any row or column of $\mathbb{R}^2$, we have: $\| R_{i,j} | - | R_{ik} \| \leqslant 1$ and $\| C_{ij} | - | C_{ik} \| \leqslant 1$, i.e. an m-coloring of P_n is almost balanced if for every row and column of $\mathbb{R}^2$ the number of elements colored j differs from the number of elements colored k by at most one.

We can now extend the result of Problem 3.1 [3].

Theorem 4.1. *Let $P_n \subset L$. Then P_n can always be m-colored with an almost balanced m-coloring, $1 \leqslant m \leqslant n$.*

Remark 4.1. Note that there exists a polynomial time algorithm to find almost balanced m-colorings of P_n. The complexity of such an algorithm equals that of finding m-edge colorings in bipartite graphs.

5. Configuration and discrepancy

As a generalization of the previous section, we consider a concept called 'Discrepancy' which is defined on a family of sets.

Let n, k be natural numbers greater than or equal to 2 and $\mathbb{Z}$ be the set of integers. We call a finite subset A of $\mathbb{Z}^n$ a *configuration*, and a family $\mathscr{A}$ ($\subset 2^A$) of A, where any member of $\mathscr{A}$ has no common element of A with other members a *partition of the configuration* A. We define a k-coloring C of a configuration A as the mapping C from a configuration A to the set $\{1, 2, \ldots, k\}$, and denote as *label* the value assigned to an element of A by the mapping C. For a configuration A with a k-coloring C and a partition $\mathscr{A}$ of A, we consider the number of points having label i in a member S of $\mathscr{A}$, (subset of A), i.e., $n(i) := |\{s \in S \mid C(s) = i\}|$. Moreover we define $D(S)$:= the maximum value of $|n(i) - n(j)|(1 \leqslant i < j \leqslant k)$ for any member S of $\mathscr{A}$, and *discrepancy* of k-coloring C, of a partition $\mathscr{A}$ of a configuration A, as the maximum value of $D(S)$ for any S. We consider the orthogonal coordinate axes for $\mathbb{Z}^n$ in the usual way.

Definition 5.1. Let $\mathscr{A}$ be the set consisting of the intersection set of A and the lines parallel to the coordinate axes. We call this set the *line partition* $\mathscr{A}$ (L.P. $\mathscr{A}$) of a configuration A.

Definition 5.2. Let $\mathscr{A}$ be the set consisting of the intersection set of A and the modular class of hyperplanes parallel to the $(n-1)$-coordinate axes. We call this set the *hyperplane modular class partition* $\mathscr{A}$ (H.M.C.P. $\mathscr{A}$) of a configuration A.

Then the following propositions hold.

Proposition 5.1. *Let $A \subset Z^2$. Then there exists a k-coloring C with discrepancy at most* 1 *for any $k(\geqslant 2)$, any configuration A and the L.P. $\mathscr{A}$ of the configuration A. (Now the L.P. $\mathscr{A}$ coincides with the H.M.C.P. $\mathscr{A}$.)*

Proposition 5.2. *Let $A \subset Z^3$. Then there exists a configuration A with the L.P. $\mathscr{A}$ for which no* 2*-coloring C has discrepancy at most* 1.

Example 5.1. For a configurtion of 7 black points see Fig. 2.

Proposition 5.3. *Let $A \subset Z^3$. Then there exists a configuration A with the L.P. $\mathscr{A}$ for which no* 3*-coloring C has discrepancy at most* 1.

Example 5.2. For a configuration of 22 black points see Fig. 3.

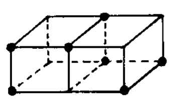

Fig. 2.

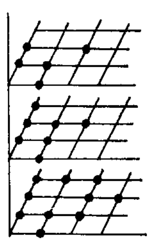

Fig. 3.

There are essentially two ways of coloring the points of the plane A. See Fig. 4. After the colorings of A are determined, there are only a few possibilities for the colorings of B and C.

The following result is an extension of Proposition 5.3.

Proposition 5.4. *Let $A \subset Z^3$. Then there exists a configuration A with the L.P. $\mathscr{A}$ for which no k-coloring C has discrepancy at most 1 for any $k(\geqslant 2)$.*

Example 5.3. Let x, y, z be integers and (x, y, z) be a point in $\mathbb{R}^3$. Then the following set $F_x \cup F_z \cup F_z \cup F'_x$ which has $(3k^2-3k+1)$ points is an example of this case (see Fig. 5):

$$F_x = \{(x, y, z) \mid x=0,\ 1 \leqslant y \leqslant k-2,\ 1 \leqslant z \leqslant k-1\},$$

$$F_y = \{(x, y, z) \mid 0 \leqslant x \leqslant k-1,\ y=0,\ 0 \leqslant z \leqslant k-1\} \setminus \{(0, 0, 0)\},$$

$$F_z = \{(x, y, z) \mid 0 \leqslant x \leqslant k-1,\ 1 \leqslant y \leqslant k-1,\ z=0\},$$

$$F'_x = \{(x, y, z) \mid x=0,\ y=k,\ 0 \leqslant z \leqslant k-1\}.$$

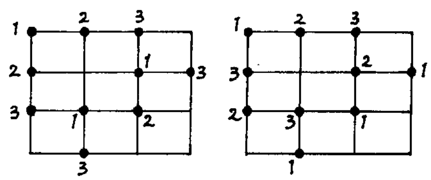

Fig. 4.

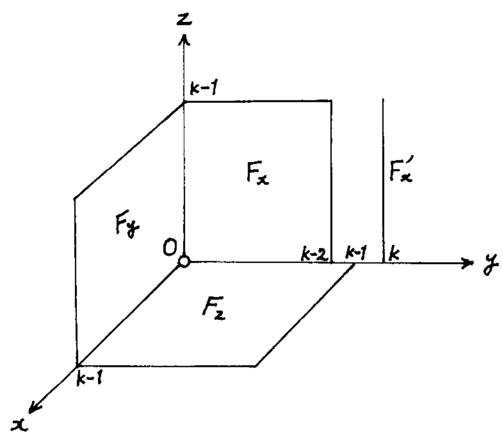

Fig. 5.

If there exists a k-coloring with discrepancy at most 1 in the square of (edge) length k except for one point, we can find a k-coloring with discrepancy at most 1 in the square of (edge) length k by properly labelling that point.

Therefore we can propose the following conjectures.

Conjecture 5.1. Let $A \subset Z^3$. Then there exists a 3-coloring C with discrepancy at most 2, for any configuration A and the *L.P.* $\mathscr{A}$ of the configuration A.

Conjecture 5.2. Let $A \subset Z^3$. Then there exists a 3-coloring C with discrepancy at most 2, for any configuration A and the H.M.C.P. $\mathscr{A}$ of the configuration A.

6. Assigned numbers on the pentagon

Problem 6.1. To each vertex of a regular pentagon, an integer is assigned in such a way that the sum of all the five integers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y<0$ then the following operation is allowed: the numbers x, y, z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily terminates after a finite number of steps.

Solution. Let $u_1, \ldots, u_5$ be the integers assigned to the vertices of the pentagon at a certain instant and suppose that at least one of them is negative; say $u_j<0$ (if there are more negative u_i's, choose any of them to be u_j). Let $V=(v_1, \ldots, v_5)$ be the lattice vector obtained from $U=(u_1, \ldots, u_5)$ by means of the operation defined in the problem, applied to $x=u_{j-1}$, $y=u_j$, $z=u_{j+1}$ (cyclic numbering; $u_0=u_5$, $u_6=u_1$). Consider the function $F(U)=\sum(u_{i+1}-u_{i-1})^2$. It is easily verified that $F(V)-F(U)=2u_jS$, where $S=\sum u_i=\sum v_i>0$; hence $F(V)<F(U)$. Thus the values of F in successive steps of the procedure constitute a strictly decreasing sequence of non-negative integers. Any such sequence is necessarily finite. $\square$

Clearly, we can replace the condition 'integer' by 'rational number' for C_n ($n \geqslant 3$) in this problem. Alon et al. generalized this problem for 'real number' [4].

Theorem 6.1. *To each vertex of C_n ($n \geqslant 3$) a real number is assigned in such a way that the sum of all the n numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y<0$ then the following operation is allowed: the numbers x, y, z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the n numbers is negative. This procedure terminates after a finite number of steps.*

Moreover, neither the number of steps nor the last state changes for all choices of the vertices.

Problem 6.1. To each vertex of an r-regular graph G an integer is assigned in such a way that the sum of those integers is positive. Suppose that a vertex and its neighbors are assigned the numbers $x, u_1, u_2, \ldots, u_r$ respectively. Then if $x<0$, the following operation is performed: the numbers x, u_i are replaced by $-x, u_i+2x/r$ $(i=1,2,\ldots,r)$ respectively. Such an operation is performed repeatedly as long as at least one of the $|V(G)|$ numbers is negative. Determine whether this procedure necessarily terminates after a finite number of steps. In particular what happens if the graphs are restricted to the complete graphs K_n?

7. Unit cube's problem

Problem 7.1. A finite number of unit squares are placed on a plane in such a way that each edge of a unit square is parallel to the coordinate axes, and no point of the plane is covered by more than two squares. Prove that the minimum number of colors required, to color the unit squares in such a way that no two squares with the same color have a common point, is at most 3.

Solution. We construct a graph G as follows: let each vertex v of G represent a square s, join two vertices by an edge if and only if the squares s_i and s_j that they represent have a common point. Note that the graph G obtained in this manner is a triangle-free planar graph with the maximum degree $\Delta(G)\leqslant 4$. Then it follows from the theorem stated below that G is 3-colorable. □

Theorem 7.1 (Grünbaum [5]). *A planar graph with no more than three triangles is 3-colorable.*

We now consider a finite number of 'unit cubes' in d-dimensional Euclidean space $\mathbb{E}^d$. Let S be a nonempty set consisting of a finite number of unit cubes. Assume that each edge of a unit cube is parallel to the d-axes. A set S is said to be have *multiplicity* k if any point in $\mathbb{E}^d$ is contained in at most k unit cubes.

An assignment of colors to the unit cubes of S is called a *coloring* of S if no two overlapping cubes have the same color. When n colors are used, such a coloring is called an *n-coloring.* The minimum number n which gives an n-coloring of S is denoted by $f(d,k)$. It is obvious that $f(d,k)\geqslant k$. Using this terminology we can state the result given in Problem 7.1 as $f(2,2)=3$.

Proposition 7.1. (1) $f(1,k)=k$,

(2) $f(3,2)\leqslant 5$.

Proof. The proof is by induction on $|S|$. □

Conjectures and Problems 7.2. We state the following conjectures.
Prove that:

(1) $f(d, 2) \leqslant d+1$,

(2) $f(2, k) \leqslant k+1$.

An interesting problem is to determine the value of $f(d, k)$ for arbitrary integers d, k.

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