Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Differential Geometry and its Applications 24 (2006) 271–287

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

www.elsevier.com/locate/difgeo

Surgery and equivariant Yamabe invariant

Chanyoung Sung

Korea Institute for Advanced Study, 207-43 Cheongryangri 2-dong Dongdaemun-gu, Seoul 130-722, South Korea

Received 14 February 2005

Available online 21 October 2005

Communicated by O. Kowalski

Abstract

We consider the equivariant Yamabe problem, i.e., the Yamabe problem on the space of G-invariant metrics for a compact Lie group G. The G-Yamabe invariant is analogously defined as the supremum of the constant scalar curvatures of unit volume G-invariant metrics minimizing the total scalar curvature functional in their G-invariant conformal subclasses. We prove a formula about how the G-Yamabe invariant changes under the surgery of codimension 3 or more, and compute some G-Yamabe invariants.

MSC: 53C20; 58E40; 57R65

Keywords: Yamabe invariant; Scalar curvature; Surgery; Conformal; Equivariant

1. Introduction

By the well-known uniformization theorem, the geometry and topology of compact orientable surfaces have the trichotomy according to the Euler characteristic. The Gauss–Bonnet theorem says that the Euler characteristic is basically the constant scalar curvature of the unit volume. Along this line one can consider the following higher-dimensional generalization, so-called *Yamabe invariant*.

Let M be a smooth compact connected n-manifold. In analogy to the 2-dimension, let us consider the normalized Einstein–Hilbert functional

$$Q(g) = \frac{\int_M s_g \, dV_g}{(\int_M dV_g)^{(n-2)/n}}$$

defined on the space of smooth Riemannian metrics on M, where s_g and dV_g respectively denote the scalar curvature and the volume element of g. The denominator is appropriately chosen for the purpose of the scale invariance. But it turns out that this functional is neither bounded above nor bounded below. In higher dimensions one need to note that there are metrics which are not conformally equivalent to each other. A *conformal class* on M is by definition a collection of smooth Riemannian metrics on M of the form

$$[g] \equiv \{\psi g \mid \psi : M \to \mathbb{R}^+\},$$

E-mail address: cysung@kias.re.kr (C. Sung).

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2005.09.002

where g is a fixed Riemannian metric. In each conformal class $[g]$ the above functional is bounded below and the minimum, called the *Yamabe constant* of $(M, [g])$ and denoted by $Y(M, [g])$, is realized by a so-called *Yamabe metric* which has constant scalar curvature. By Aubin's theorem [3], the Yamabe constant of any conformal class on any n-manifold is always bounded by that of the unit n-sphere $S^n(1) \subset \mathbb{R}^{n+1}$, which is $\Lambda_n \equiv n(n-1)(\mathrm{vol}(S^n(1)))^{2/n}$. The *Yamabe invariant* of M, $Y(M)$, is then defined as the supremum of the Yamabe constant over the set of all conformal classes on M. Note that it is a differential-topological invariant of M depending only on the smooth structure of the manifold.

The computation of the Yamabe invariant has been making notable progress, particularly in low dimensions, due to LeBrun [10,13–15], Bray and Neves [6], Perelman [16], Anderson [2], and etc. But in higher dimensions little is known and noteworthy theorems to this end are the surgery theorems. By the celebrated theorem of Gromov and Lawson [7], also independently by Schoen and Yau [18], the Yamabe invariant of any manifold obtained from the manifolds of positive Yamabe invariant by a surgery of codimension 3 or more is also positive. Moreover we have

Theorem 1.1. (Kobayashi [12], Petean and Yun [17].) *Let M_1, M_2 be smooth compact manifolds of dimension $n \geqslant 3$. Suppose that an $(n-q)$-dimensional smooth compact (possibly disconnected) manifold W embeds into both M_1 and M_2 with trivial normal bundle. Assume $q \geqslant 3$. Let M be any manifold obtained by gluing M_1 and M_2 along W. Then*

$$Y(M) \geqslant \begin{cases} -(|Y(M_1)|^{n/2} + |Y(M_2)|^{n/2})^{2/n} & \text{if } Y(M_i) \leqslant 0\ \forall i, \\ \min(Y(M_1), Y(M_2)) & \text{if } Y(M_1) \cdot Y(M_2) \leqslant 0, \\ \min(Y(M_1), Y(M_2)) & \text{if } Y(M_i) \geqslant 0\ \forall i \text{ and } q = n. \end{cases}$$

When $Y(M_i) \geqslant 0$ and $3 \leqslant q \leqslant n-1$, no estimate has been given even for $W = S^{n-q}$.

Now let us generalize this discussion to the equivariant Yamabe problem. Let G be a compact Lie group acting on (M, g) smoothly as an isometry. We will call such (M, g) as a *Riemannian G-manifold* and $[g]_G$ will denote the set of smooth G-invariant metrics conformal to g. Then we have

Theorem 1.2. (Hebey and Vaugon [8].) *Let (M, g) a smooth compact Riemannian G-manifold. Then there exists a metric $g' \in [g]_G$ of constant scalar curvature realizing*

$$Y\big(M, [g]_G\big) := \inf_{\hat{g} \in [g]_G} \frac{\int_M s_{\hat{g}}\, dV_{\hat{g}}}{(\int_M dV_{\hat{g}})^{(n-2)/n}},$$

and

$$Y\big(M, [g]_G\big) \leqslant \Lambda_n \Big(\inf_{x \in M} |Gx|\Big)^{2/n},$$

where $|Gx|$ denotes the cardinality of the orbit of x.

We will call $Y(M, [g]_G)$ the *G-Yamabe constant* of $(M, [g]_G)$ and such a metric g' will be called as a *G-Yamabe metric*. Obviously $Y(M, [g]_G) \geqslant Y(M, [g])$ for any G-invariant metric g. We also remark that any G-Yamabe metric with the nonpositive G-Yamabe constant is actually a Yamabe metric, and hence the G-Yamabe constant coincides with the Yamabe constant, because the constant scalar curvature metric is unique up to constant in such a conformal class. The *G-Yamabe invariant* $Y_G(M)$ of M is also defined as the supremum of all the G-Yamabe constants. Of course it is an invariant of the G-manifold M. We will show that some standard theorems about the Yamabe constant can be generalized to the G-Yamabe constant and prove the following surgery theorem for the G-Yamabe invariant.

Theorem 1.3. *Let M_1, M_2 be smooth compact manifolds of dimension $n \geqslant 3$ on which a compact Lie group G acts smoothly. Suppose that an $(n-q)$-dimensional smooth compact (possibly disconnected) manifold W with a locally transitive G-action embeds G-equivariantly into both M_1 and M_2 with an equivariant G-action on the trivial normal bundle. Assume $q \geqslant 3$. Let M be any G-manifold obtained by equivariantly gluing M_1 and M_2 along W. Then*

$$Y_G(M) \geqslant \begin{cases} -(|Y_G(M_1)|^{n/2} + |Y_G(M_2)|^{n/2})^{2/n} & \text{if } Y_G(M_i) \leqslant 0\ \forall i, \\ \min(Y_G(M_1), Y_G(M_2)) & \text{otherwise.} \end{cases}$$

In the final section we will use this to compute some G-Yamabe invariants.

2. Approximation of metric for Yamabe invariant

Let's briefly go over the standard setup for the Yamabe problem. Let $p = \frac{2n}{n-2}$, $a = 4\frac{n-1}{n-2}$. Then

$$Q(\varphi^{p-2}g) = \frac{\int_M (a|d\varphi|_g^2 + s_g\varphi^2)\,dV_g}{(\int_M |\varphi|^p\,dV_g)^{2/p}},$$

and

$$Y\big(M,[g]_G\big) = \inf\big\{Q(\varphi^{p-2}g) \mid \varphi \in L_1^2(M) \text{ is nonzero and } G\text{-invariant}\big\},$$

where the Sobolev space $L_1^2(M)$ is the set of $u \in L^2(M)$ such that $du \in L^2(M)$. A smooth function ψ such that $\psi^{p-2}g$ is a G-Yamabe metric will be called a *G-Yamabe minimizer* for $[g]_G$. Generalizing B. Bergery's theorem [5], the G-Yamabe constant also behaves continuously with respect to the conformal class.

Theorem 2.1. *Let g_i, g be G-invariant Riemannian metrics on M such that $g_i \to g$ in the C^1-topology, and $s_{g_i} \to s_g$ in the C^0-topology on M. Then $Y(M,[g_i]_G) \to Y(M,[g]_G)$.*

Proof. By Theorem 1.2, there exists a G-invariant conformal change $\varphi^{p-2}g$ of g making the scalar curvature constant. Since $\varphi^{p-2}g_i \to \varphi^{p-2}g$ and $s_{\varphi^{p-2}g_i} \to s_{\varphi^{p-2}g}$ in the C^0-topology for any positive smooth function φ, we may assume that s_g is constant. We have two cases either $s_g \geqslant 0$, or $s_g < 0$.

Let us consider the first case. Given a sufficiently small $\epsilon > 0$, we can take an integer $N(\epsilon)$ such that for $i \geqslant N(\epsilon)$,

$$(1-\epsilon)g^{-1} \leqslant g_i^{-1} \leqslant (1+\epsilon)g^{-1},$$
$$(1-\epsilon)\,dV_g \leqslant dV_{g_i} \leqslant (1+\epsilon)\,dV_g,$$

and

$$|s_g - s_{g_i}| \leqslant \epsilon.$$

Then for any $\varphi \in L_1^2(M)$

$$\begin{aligned} Q(\varphi^{p-2}g_i) &\leqslant \frac{\int_M (a(1+\epsilon)|d\varphi|_g^2 + (s_g+\epsilon)\varphi^2)(1+\epsilon)\,dV_g}{(\int_M |\varphi|^p(1-\epsilon)\,dV_g)^{2/p}} \\ &\leqslant \frac{(1+\epsilon)\int_M (a|d\varphi|_g^2 + s_g\varphi^2)\,dV_g}{(1-\epsilon)^{2/p}(\int_M |\varphi|^p\,dV_g)^{2/p}} + \frac{\epsilon(1+\epsilon)\int_M (a|d\varphi|_g^2 + \varphi^2)\,dV_g}{(1-\epsilon)^{2/p}(\int_M |\varphi|^p\,dV_g)^{2/p}} \\ &\leqslant \frac{(1+\epsilon)}{(1-\epsilon)^{2/p}} Q(\varphi^{p-2}g) + \frac{\epsilon(1+\epsilon)\bar{C}}{(1-\epsilon)^{2/p}}, \end{aligned}$$

where $\bar{C} > 0$ is a constant satisfying

$$\int_M \big(a|d\psi|_g^2 + \psi^2\big)\,dV_g \leqslant \bar{C}\bigg(\int_M |\psi|^p\,dV_g\bigg)^{2/p}$$

for any $\psi \in L_1^2(M)$, and similarly

$$\begin{aligned} Q(\varphi^{p-2}g_i) &\geqslant \frac{\int_M (a(1-\epsilon)|d\varphi|_g^2 + (s_g-\epsilon)\varphi^2)\,dV_{g_i}}{(\int_M |\varphi|^p\,dV_{g_i})^{2/p}} \\ &\geqslant \frac{(1-\epsilon)\int_M (a|d\varphi|_g^2 + s_g\varphi^2)\,dV_g}{(1+\epsilon)^{2/p}(\int_M |\varphi|^p\,dV_g)^{2/p}} - \frac{\epsilon(1+\epsilon)\int_M (a|d\varphi|_g^2 + \varphi^2)\,dV_g}{(1-\epsilon)^{2/p}(\int_M |\varphi|^p\,dV_g)^{2/p}} \\ &\geqslant \frac{(1-\epsilon)}{(1+\epsilon)^{2/p}} Q(\varphi^{p-2}g) - \frac{\epsilon(1+\epsilon)\bar{C}}{(1-\epsilon)^{2/p}}. \end{aligned}$$

Taking the infimum over φ and letting $\epsilon \to 0$, we get $Y(M,[g_i]_G) \to Y(M,[g]_G)$.

In the second case, we have $s_{g_i} < 0$ for all sufficiently large i. Recall O. Kobayashi's lemma [12]:

Lemma 2.2. *Let (M,h) be any Riemannian G-manifold with $Y(M,[h]_G) \leqslant 0$. Then*

$$(\min s_h)\,\mathrm{vol}_h(M)^{2/n} \leqslant Y\big(M,[h]_G\big) \leqslant (\max s_h)\,\mathrm{vol}_h(M)^{2/n}.$$

Proof. The proof should be the same as the nonequivariant case because $Y(M,[h]_G) = Y(M,[h])$ in this case. The case of $n=2$ is immediate from the Gauss–Bonnet theorem. Let us consider the case when $n \geqslant 3$. The right inequality is obvious from

$$\frac{\int_M s_h\,dV_h}{(\int_M dV_h)^{(n-2)/n}} \leqslant (\max s_h)\,\mathrm{vol}_h(M)^{2/n}.$$

For the left inequality, we claim that $\min s_h \leqslant 0$. Otherwise the Sobolev inequality says that there exists a constant $\check{C} > 0$ such that $(\int_M \psi^p\,dV_h)^{2/p} \leqslant \check{C}\int_M (a|d\psi|_h^2 + s_h\psi^2)\,dV_h$ for any $\psi \in L_1^2(M)$. This implies $Y(M,[h]_G) > 0$ which is contradictory to the assumption. Once we have $\min s_h \leqslant 0$, by using the Hölder inequality we get

$$(\min s_h)\,\mathrm{vol}_h(M)^{2/n} \leqslant \frac{\int_M (\min s_h)\varphi^2\,dV_h}{(\int_M |\varphi|^p\,dV_h)^{2/p}} \leqslant Q(\varphi^{p-2}h)$$

for any $\varphi \in L_1^2(M)$, implying that $(\min s_h)\,\mathrm{vol}_h(M)^{2/n} \leqslant Y(M,[h]_G)$. □

By the above lemma,

$$(\min s_{g_i})\,\mathrm{vol}_{g_i}(M)^{2/n} \leqslant Y\big(M,[g_i]_G\big) \leqslant (\max s_{g_i})\,\mathrm{vol}_{g_i}(M)^{2/n}$$

for sufficiently large i. Letting $i \to \infty$, we get $Y(M,[g_i]_G) \to (\max s_g)\,\mathrm{vol}_g(M)^{2/n} = Y(M,[g]_G)$. □

In the light of this, we want to find a sequence of G-invariant metrics which has a nice form to perform a surgery and converges to the given one. Generalizing the results of O. Kobayashi [12], and K. Akutagawa and B. Botvinnik [1], we present:

Theorem 2.3. *Let W be a G-invariant submanifold of a Riemannian G-manifold (M,g) and let $\bar{g}$ be a G-invariant metric defined in an open neighborhood of W, which coincides with g on W up to first derivatives, i.e., $g = \bar{g}$ and $\partial g = \partial\bar{g}$ on W, and has the same scalar curvature as g on W. Then for sufficiently small $\delta > 0$ there exists a G-invariant metric g_δ on M satisfying the following properties.*

(i) $g_\delta \equiv g$ *on* $\{z \in M \mid \mathrm{dist}_g(z,W) > \delta\}$.
(ii) $g_\delta \equiv \bar{g}$ *in an open neighborhood of* W.
(iii) $g_\delta \to g$ *in the* C^1*-topology on* M *as* $\delta \to 0$.
(iv) $s_{g_\delta} \to s_g$ *in the* C^0*-topology on* M *as* $\delta \to 0$.

Proof. Let r be the g-distance from W. Obviously r is G-invariant. The proof goes in the same way as [12] and [1]. We will be content with describing g_δ. Given a $\delta > 0$, take a smooth nonnegative function $w_\delta(r)$, $r \in [0,\infty)$ which satisfies $w_\delta(r) \equiv 1$ on $[0, \frac{1}{4}e^{-1/\delta}]$, $w_\delta(r) \equiv 0$ on $[\delta,\infty)$, $|r\frac{\partial w_\delta}{\partial r}| < \delta$, and $|r\frac{\partial^2 w_\delta}{\partial r^2}| < \delta$. Then $g_\delta = g + w_\delta(r)(\bar{g} - g)$ does the job. □

To apply the above theorem we need to find a metric $\bar{g}$ which approximates g near W in a canonical way. Let us suppose that W has codimension q. Let $(x,y) = (x^1,\ldots,x^{n-q},y^{n-q+1},\ldots,y^n)$ be a local trivialization of the normal bundle of W, where $(x_1,\ldots,x_{n-q})$ is a local coordinate on the base W and $(y_{n-q+1},\ldots,y_q)$ is a coordinate on the fiber vector space. Via the exponential map, this gives a local coordinate near W. Let the indices $i,j,\ldots$ run from 1 to $n-q$, and the indices $\alpha,\beta,\gamma,\ldots$ run from $n-q+1$ to n. Because we have taken the exponential normal coordinate in the normal direction, we have on W

$$\frac{\partial}{\partial y^\alpha} g(\partial_i,\partial_j) = g(\nabla_{\partial_\alpha}\partial_i,\partial_j) + g(\partial_i,\nabla_{\partial_\alpha}\partial_j) = -2\Pi_{ij}^\alpha,$$

$$\frac{\partial}{\partial y^\beta} g(\partial_i, \partial_\alpha) = g(\nabla_{\partial_\beta}\partial_i, \partial_\alpha) + g(\partial_i, \nabla_{\partial_\beta}\partial_\alpha) = -g(\nabla_{\partial_i}\partial_\beta, \partial_\alpha) + g(\partial_i, 0) = -\Gamma^\alpha_{i\beta},$$

and

$$\frac{\partial}{\partial y^\gamma} g(\partial_\alpha, \partial_\beta) = 0,$$

where $\Pi^\alpha_{ij} = g(\partial_i, \nabla_{\partial_j}\partial_\alpha)$ is the second fundamental form of W, and $\Gamma^\alpha_{i\beta}(x)$ is the Christoffel symbol for the g-connection of the normal bundle on W. Therefore near W, g can be written as

$$\begin{aligned} g(x,y) = {} & \sum_{i,j}\Big(g^W_{ij}(x) - 2\sum_\alpha y^\alpha \Pi^\alpha_{ij}(x) + O(r^2)\Big)\,dx^i\,dx^j \\ & + \sum_{i,\alpha,\beta}\big(-\Gamma^\alpha_{i\beta}(x)y^\beta + O(r^2)\big)\,dx^i\,dy^\alpha + \sum_\alpha dy^\alpha\,dy^\alpha + \sum_{\alpha\neq\beta} O(r^2)\,dy^\alpha\,dy^\beta, \end{aligned}$$

where $g^W = g|_W$ and $r = \sum_\alpha (y^\alpha)^2$. We will call the above the canonical coordinate expression of g near W.

Let $\hat{g}$ be the first order approximation of g, i.e.,

$$\hat{g} := \sum_{i,j}\Big(g^W_{ij}(x) - 2\sum_\alpha y^\alpha \Pi^\alpha_{ij}(x)\Big)\,dx^i\,dx^j + \sum_{i,\alpha,\beta}\big(-\Gamma^\alpha_{i\beta}(x)y^\beta\big)\,dx^i\,dy^\alpha + \sum_\alpha dy^\alpha\,dy^\alpha.$$

Since g and r are G-invariant, $\hat{g}$ is also G-invariant. The scalar curvature of $\hat{g}$ is in general different from that of g. For the scalar curvature correction, we want to make a conformal change which is 1 at W up to the first order. Let $\bar{g}(x,y) = u(x,y)^{p-2}\hat{g}$ where u is G-invariant,

$$u(x,0) = 1, \quad \text{and} \quad \frac{\partial}{\partial y^\alpha} u(x,0) = 0 \tag{1}$$

for any α on W. Letting the uppercase Roman indices denote 1 through n and using (1), we have on W

$$\Delta_{\hat{g}} u = -\hat{\nabla}^A \partial_A u = -\hat{g}^{AB}\big(\partial_A\partial_B u - \hat{\Gamma}^C_{AB}\partial_C u\big) = -\hat{g}^{\alpha\beta}\partial_\alpha\partial_\beta u = -\sum_\alpha \frac{\partial}{\partial y^\alpha}\frac{\partial u}{\partial y^\alpha},$$

where $\hat{\nabla}$ and $\hat{\Gamma}$ denote the covariant derivative and Christoffel symbol of $\hat{g}$ respectively. We set

$$u(x,y) := 1 - \frac{r^2}{8aq}(s_g|_W - s_{\hat{g}}|_W).$$

Then on W,

$$s_{\bar{g}} = u^{1-p}(4a\Delta_{\hat{g}} u + s_{\hat{g}} u) = -4a\sum_\alpha \frac{\partial}{\partial y^\alpha}\frac{\partial u}{\partial y^\alpha} + s_{\hat{g}} = s_g.$$

Combined with the above theorem, we obtain:

Theorem 2.4. *Let W be a G-invariant submanifold of a Riemannian G-manifold (M, g). For sufficiently small $\delta > 0$, there exists a G-invariant metric g_δ such that*

(i) $g_\delta \to g$ *in the C^1-topology on M as* $\delta \to 0$.
(ii) $s_{g_\delta} \to s_g$ *in the C^0-topology on M as* $\delta \to 0$.
(iii) $g_\delta \equiv g$ *on* $\{z \in M \mid \mathrm{dist}_g(z, W) > \delta\}$.
(iv) *In an open neighborhood of W, g_δ is conformally equivalent to* $\sum_{i,j}(g^W_{ij}(x) - 2\sum_\alpha y^\alpha \Pi^\alpha_{ij}(x))\,dx^i\,dx^j + \sum_{i,\alpha,\beta}(-\Gamma^\alpha_{i\beta}(x)y^\beta)\,dx^i\,dy^\alpha + \sum_\alpha dy^\alpha\,dy^\alpha$.

For the conformal classes which are close in a G-invariant subset, we can obtain a common upper bound.

Proposition 2.5. *Let $\{g_\alpha \mid \alpha \in I\}$ be a collection of smooth G-invariant metrics on a compact G-manifold X. Suppose that there exists a constant D_1 and D_2 such that $|g_\alpha - g_\beta| \leqslant D_1$ and $|s_{g_\alpha} - s_{g_\beta}| \leqslant D_2$ in some G-invariant open subset $U \subset X$ for any $\alpha, \beta \in I$. Then there exists a constant D such that $Y(X, [g_\alpha]_G) \leqslant D$ for any $\alpha \in I$.*

Proof. Take a smooth bump function $\phi(x) \geqslant 0$ supported in U. In general ϕ is not G-invariant. Let $d\mu$ be the unit-volume bi-invariant measure on G. Define $\bar{\phi}(x) := \int_G \phi(gx)\, d\mu(g)$. Then $\bar{\phi}$ is G-invariant and also supported in U. Now $Q(\bar{\phi}^{p-2} g_\alpha)$ is bounded above and by definition $Y(X, [g_\alpha]_G) \leqslant Q(\bar{\phi}^{p-2} g_\alpha)$ for any $\alpha \in I$. □

3. Proof of main theorem

We start with the equivariant version of O. Kobayashi's lemma [12].

Lemma 3.1. *Let $(M_1 \cup M_2, g_1 \cup g_2)$ be the disjoint union of (M_1, g_1) and (M_2, g_2). Then $Y(M_1 \cup M_2, [g_1 \cup g_2]_G)$ is given by*

$$\begin{cases} -(|Y(M_1, [g_1]_G)|^{n/2} + |Y(M_2, [g_2]_G)|^{n/2})^{2/n} & \text{if } Y(M_i, [g_i]_G) \leqslant 0\ \forall i, \\ \min(Y(M_1, [g_2]_G), Y(M_2, [g_2]_G)) & \text{otherwise}, \end{cases}$$

and

$$Y_G(M_1 \cup M_2) = \begin{cases} -(|Y_G(M_1)|^{n/2} + |Y_G(M_2)|^{n/2})^{2/n} & \text{if } Y_G(M_i) \leqslant 0\ \forall i, \\ \min(Y_G(M_1), Y_G(M_2)) & \text{otherwise}. \end{cases}$$

Proof. Suppose $Y(M_1, [g_1]_G) \geqslant Y(M_2, [g_2]_G) \geqslant 0$. Then for any $c^2 g_1' \cup g_2' \in [g_1 \cup g_2]_G$ where $c > 0$ is a constant,

$$\begin{aligned} Q\big(c^2 g_1' \cup g_2'\big) &= \frac{\int_{M_1} c^{n-2} s_{g_1'}\, dV_{g_1'} + \int_{M_2} s_{g_2'}\, dV_{g_2'}}{(\int_{M_1} c^n\, dV_{g_1'} + \int_{M_2} dV_{g_2'})^{(n-2)/n}} \\ &\geqslant \frac{(\int_{M_1} c^n\, dV_{g_1'})^{(n-2)/n} Y(M_1, [g_1]_G) + (\int_{M_2} dV_{g_2'})^{(n-2)/n} Y(M_2, [g_2]_G)}{(\int_{M_1} c^n\, dV_{g_1'} + \int_{M_2} dV_{g_2'})^{(n-2)/n}} \\ &\geqslant \frac{(\int_{M_1} c^n\, dV_{g_1'})^{(n-2)/n} Y(M_2, [g_2]_G) + (\int_{M_2} dV_{g_2'})^{(n-2)/n} Y(M_2, [g_2]_G)}{(\int_{M_1} c^n\, dV_{g_1'})^{(n-2)/n} + (\int_{M_2} dV_{g_2'})^{(n-2)/n}} \\ &= Y\big(M_2, [g_2]_G\big), \end{aligned}$$

and $Q(c^2 g_1' \cup g_2') \to Y(M_2, [g_2]_G)$ if $c \to 0$ and g_2' is a G-Yamabe metric on M_2.

Suppose $Y(M_1, [g_1]_G) \geqslant 0 \geqslant Y(M_2, [g_2]_G)$. Also for any $c^2 g_1' \cup g_2' \in [g_1 \cup g_2]_G$,

$$\begin{aligned} Q\big(c^2 g_1' \cup g_2'\big) &= \frac{\int_{M_1} c^{n-2} s_{g_1'}\, dV_{g_1'} + \int_{M_2} s_{g_2'}\, dV_{g_2'}}{(\int_{M_1} c^n\, dV_{g_1'} + \int_{M_2} dV_{g_2'})^{(n-2)/n}} \\ &\geqslant \frac{0 + (\int_{M_2} dV_{g_2'})^{(n-2)/n} Y(M_2, [g_2]_G)}{(\int_{M_1} c^n\, dV_{g_1'} + \int_{M_2} dV_{g_2'})^{(n-2)/n}} \\ &\geqslant \frac{(\int_{M_2} dV_{g_2'})^{(n-2)/n} Y(M_2, [g_2]_G)}{(\int_{M_2} dV_{g_2'})^{(n-2)/n}} \\ &= Y\big(M_2, [g_2]_G\big), \end{aligned}$$

and $Q(c^2 g_1' \cup g_2') \to Y(M_2, [g_2]_G)$ if $c \to 0$ and g_2' is a G-Yamabe metric on M_2.

For the last remaining case, suppose $Y(M_i, [g_i]_G) \leqslant 0$ and we assume g_i is a G-Yamabe metric for $(M_i, [g_i]_G)$ for each i such that $s_{g_1} = s_{g_2} < 0$. Now note that Lemma 2.2 still holds true for the nonconnected manifolds and its corollary is that any G-invariant metric of nonpositive constant scalar curvature is a G-Yamabe metric. Thus $g_1 \cup g_2$

is a Yamabe metric and

$$\begin{aligned} Y\big(M_1 \cup M_2, [g_1 \cup g_2]_G\big) &= s_{g_1 \cup g_2} \operatorname{vol}_{g_1 \cup g_2}(M_1 \cup M_2)^{2/n} \\ &= -\big(|s_{g_1 \cup g_2}|^{n/2} \operatorname{vol}_{g_1 \cup g_2}(M_1 \cup M_2)\big)^{2/n} \\ &= -\big(|s_{g_1}|^{n/2} \operatorname{vol}_{g_1}(M_1) + |s_{g_2}|^{n/2} \operatorname{vol}_{g_2}(M_2)\big)^{2/n} \\ &= -\big(\big|Y\big(M_1, [g_1]_G\big)\big|^{n/2} + \big|Y\big(M_2, [g_2]_G\big)\big|^{n/2}\big)^{2/n}. \end{aligned}$$

The second assertion is immediately obtained by taking the supremum of the first equality. □

By the above lemma, we only need to prove the following theorem.

Theorem 3.2. *Let M_0 be a smooth compact (possibly disconnected) manifold of dimension $n \geqslant 3$ on which a compact Lie group G acts smoothly, and W be an $(n-q)$-dimensional smooth compact (possibly disconnected) manifold with a locally transitive G-action. Suppose that two copies of W embed G-equivariantly into M_0 with an equivariant G-action on the trivial normal bundle. Assume $q \geqslant 3$. Let M be any G-manifold obtained by an equivariant surgery on M_0 along W. Then*

$$Y_G(M) \geqslant Y_G(M_0).$$

Proof. The idea of proof when $q = n$ is the same as the well-known result of Osamu Kobayashi [12], which considers a gluing with a long neck. When $q < n$, the idea is inspired by Dominic Joyce's method in [11]. We construct M with the volume of the gluing region very small. This forces the G-Yamabe minimizer of M to concentrate away from the gluing region, otherwise the value of Yamabe functional gets too big. Then the G-Yamabe constant of M is basically expressed by that of M_0. Although we can simplify our proof a little bit by restricting to the case $Y_G(M_0) > 0$, we will prove the general case for completeness. By abuse of notation W will also denote the submanifolds embedded in M.

Let $0 < \epsilon_1, \epsilon_2 \ll 1$. Take a conformal class $[g_0]_G$ on M_0 such that $Y(M_0, [g_0]_G) \geqslant Y_G(M_0) - \frac{\epsilon_1}{2}$. Applying Theorems 2.1 and 2.4, we can find a G-invariant metric g satisfying $Y(M_0, [g]_G) \geqslant Y(M_0, [g_0]_G) - \frac{\epsilon_1}{2}$ and g near W is the canonical first order approximation of g_0, i.e., $g = \sum_{i,j}((g_0^W)_{ij}(x) - 2\sum_\alpha y^\alpha \Pi_{ij}^\alpha(x))\,dx^i\,dx^j + \sum_{i,\alpha,\beta}(-\Gamma_{i\beta}^\alpha(x) y^\beta)\,dx^i\,dy^\alpha + \sum_\alpha dy^\alpha\,dy^\alpha$, where $(y_{n-q+1}, \ldots, y_n)$ is the g_0-exponential normal coordinate in the normal direction. Since $\operatorname{dist}_g((x,y), W) = \sum_\alpha (y^\alpha)^2$, it turns out that $(y_{n-q+1}, \ldots, y_n)$ is also the g-exponential normal coordinate, and so the above expression of g is the canonical coordinate expression for g itself by the uniqueness. So we may assume that

$$Y\big(M_0, [g]_G\big) \geqslant Y_G(M_0) - \epsilon_1$$

and

$$g = \sum_{i,j}\left(g_{ij}^W(x) - 2\sum_\alpha y^\alpha \Pi_{ij}^\alpha(x)\right) dx^i\,dx^j + \sum_{i,\alpha,\beta}\big(-\Gamma_{i\beta}^\alpha(x) y^\beta\big)\,dx^i\,dy^\alpha + \sum_\alpha dy^\alpha\,dy^\alpha$$

on $N(r_0) := \{r = (\sum_\alpha y_\alpha^2)^{1/2} \leqslant r_0\}$. Also keep in mind that the G-action fixes r, and acts on x as in W.

We first consider the case when $q = n$, i.e., W is a finite set of points. In this case g is the Euclidean metric near W. Since r is G-invariant, by multiplying a conformal factor $f(r)$ which is $\frac{1}{r^2}$ near W, $(M_0 - (W \cup W), g)$ is conformal to a Riemannian G-manifold (M_0', g') whose end is two copies of an infinite cylinder $W \times S^{n-1}(1) \times [0, \infty)$. Cut off both infinite cylinders at a large integer $l \in [0, \infty)$ and glue them along the boundary to get a Riemannian G-manifold $(M_l, \bar{g}_l)$ which contains a cylinder $W \times S^{n-1}(1) \times [0, 2l]$. Note that the complement of the cylindrical region in M_l is G-invariant and the same for any l. Thus by Proposition 2.5, $\{Y(M_l, [\bar{g}_l]_G) \mid l \in [0, \infty)\}$ is bounded above. This is an important fact to be used below.

To estimate a lower bound of $Y(M_l, [\bar{g}_l]_G)$, let ψ_l be a G-Yamabe minimizer satisfying $\int_{M_l} \psi_l^p\,dV_{\bar{g}_l} = 1$. Since $\{Y(M_l, [\bar{g}_l]_G) \mid l \in [0, \infty)\}$ is bounded above, there exists a constant $A > 0$ independent of l such that

$$\int_{W \times S^{n-1}(1) \times [0, 2l]} \big(a|d\psi_l|_{\bar{g}_l} + 2(n-1)(n-2)\psi_l^2\big)\,dV_{\bar{g}_l} \leqslant A.$$

Combined with $\int_{W\times S^{n-1}(1)\times[0,2l]} \psi_l^p\, dV_{\bar{g}_l} < 1$, it implies that there exists an integer $N_l \in [0, l-1]$ such that

$$\int\limits_{W\times S^{n-1}(1)\times[2N_l,2N_l+2]} \left(a|d\psi_l|_{\bar{g}_l} + 2(n-1)(n-2)\psi_l^2\right) dV_{\bar{g}_l} \leqslant \frac{A+1}{l}, \tag{2}$$

and

$$\int\limits_{W\times S^{n-1}(1)\times[2N_l,2N_l+2]} \psi_l^p\, dV_{\bar{g}_l} \leqslant \frac{A+1}{l}. \tag{3}$$

Let $\xi(t):\mathbb{R}\to[0,1]$ be a smooth function such that

$$\xi(t) = \begin{cases} 1 & \text{for } t\in(-\infty,0]\cup[2,\infty), \\ 0 & \text{for } t\in[\frac{2}{3},\frac{4}{3}]. \end{cases}$$

Define a smooth function Ψ_l on M_l as

$$\Psi_l = \begin{cases} \psi_l(z,t)\xi(t-2N_l) & \text{for } (z,t)\in(W\times S^{n-1}(1))\times[0,2l], \\ \psi_l & \text{elsewhere}. \end{cases}$$

Cut M_l at $W \times S^{n-1}(1) \times \{2N_l + 1\}$ and glue two half infinite cylinders to get back (M_0', g'). Extend Ψ_l to M_0' by defining it to be zero on the additional half infinite cylinders. Noting (2), (3), and the fact that $\{Y(M_l, [\bar{g}_l]_G) \mid l \in [0,\infty)\}$ is bounded above, we can get

$$Q\left(\Psi_l^{p-2}g'\right) \leqslant Y\left(M_l,[\bar{g}_l]_G\right) + \frac{B}{l},$$

where B is a constant independent of l. This implies $Y_G(M_0) - \epsilon_1 \leqslant Y(M_0,[g]_G) \leqslant Y(M_l,[\bar{g}_l]_G) + \frac{B}{l}$. Letting $l\to\infty$ and $\epsilon_1 \to 0$, we finally obtain $Y_G(M_0) \leqslant Y_G(M)$.

Now we turn to the case of $q < n$ which will be needed at the last stage. We will perform a refined version of the well-known Gromov–Lawson bending [7,19] on $N(r_0)$. The manifold is constructed as a hypersurface in the Riemannian product $\mathbb{R}\times M_0$ in accordance with an appropriate smooth curve γ in $\{(t,r)\in\mathbb{R}^2\}$, which starts tangentially to the r-axis at $t=0$ and ends up parallel to the t-axis as in Fig. 1. We extend the isometric G-action to $\mathbb{R}\times M_0$ in an obvious way that t is invariant. Since r is G-invariant, the constructed manifold is a G-invariant submanifold of the

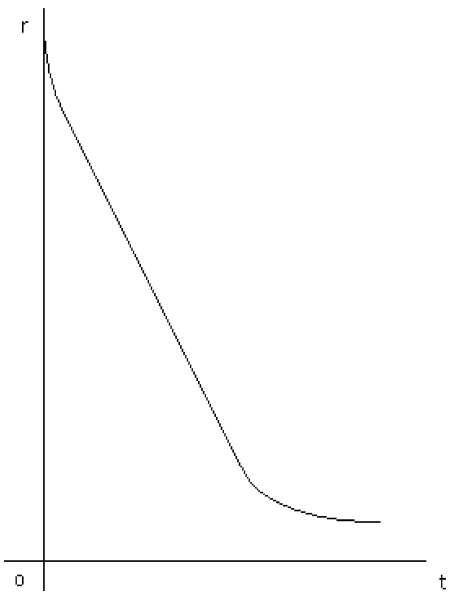

Fig. 1. Curve γ.

Riemannian G-manifold, and hence also a Riemannian G-manifold. The angle of bending at each radius is denoted by θ, and $k \geqslant 0$ denotes the geodesic curvature. The scalar curvature s is given by

$$s = s_g - 2\,\mathrm{Ric}_g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)\sin^2\theta + \left(-\frac{2(q-1)}{r} + O(1)\right)k\sin\theta + (q-1)(q-2)\frac{\sin^2\theta}{r^2} + O(1)\frac{\sin^2\theta}{r}$$
$$\geqslant s_g + \frac{(q-1)(q-2)}{2}\frac{\sin^2\theta}{r^2} - 3(q-1)\frac{k\sin\theta}{r},$$

for sufficiently small $r > 0$, where s_g and Ric_g denote the scalar curvature and the Ricci curvature of g respectively.

The construction of γ is done in 3 steps. First, by continuity we make a bending of small θ_0 keeping

$$\frac{(q-1)(q-2)}{2}\frac{\sin^2\theta}{r^2} - 3(q-1)\frac{k\sin\theta}{r} > -\epsilon_2$$

so that $s > s_g - \epsilon_2$. Let r_1 be the radius at the end and take r_1' such that $0 < r_1' \ll r_1$. As a second step γ goes down to $r = r_2$ straight, i.e., $k = 0$. Since $k = 0$, we have in this step

$$s \geqslant s_g + \frac{(q-1)(q-2)}{2}\frac{\sin^2\theta_0}{r^2} > s_g.$$

Here $r_2 > 0$ is chosen small enough so that there exists a C^∞ function $\eta(r): \mathbb{R}^+ \to [0, 1]$ such that

$$\eta(r) = \begin{cases} 0 & \text{for } r \leqslant r_2, \\ 1 & \text{for } r \geqslant r_1', \end{cases}$$

and

$$|d\eta| \leqslant \sqrt{\frac{(q-1)(q-2)}{2}}\frac{\sin\theta_0}{r}.$$

(Consider the graph of $y = (\sqrt{\frac{(q-1)(q-2)}{2}}\sin\theta_0)\ln x$.) This $\eta(r)$ will be used later as a radial cut-off function on (M_0, g). Now the third step proceeds. We bend γ after the following prescription of the curvature function $k(L)$ parameterized by the arc length L (see Fig. 2). Here, k_0, the maximum of k, is defined as $\frac{(q-2)\sin\theta_0}{6r_2}$ so that

$$\frac{(q-1)(q-2)}{2}\frac{\sin^2\theta}{r^2} - 3(q-1)\frac{k\sin\theta}{r} \geqslant 0 \tag{4}$$

is ensured during this process and hence $s \geqslant s_g$. The amount of the bend $\Delta\theta$ is

$$\Delta\theta = \int k\,dL \approx k_0 \cdot \frac{r_2}{2} = \frac{(q-2)\sin\theta_0}{12}.$$

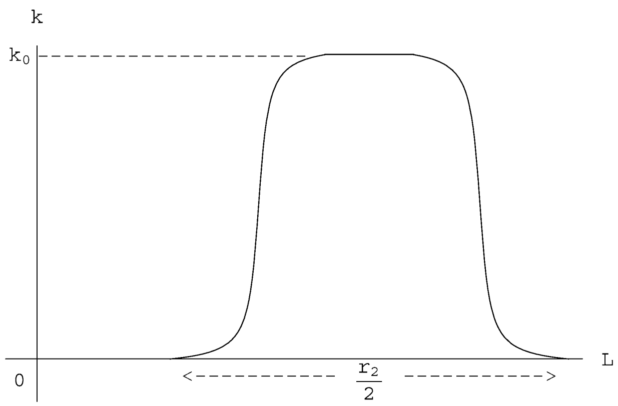

Fig. 2. Curvature function $k(L)$.

Repeat this process with the curvature prescription completely determined only by the ending radius of the previous process until we achieve a total bend of $\frac{\pi}{2}$. So the length of γ during this step is less than

$$\frac{r_2}{2}\left(\left[\frac{\pi}{2}/\Delta\theta\right]+1\right) \leqslant \frac{3\pi r_2}{(q-2)\sin\theta_0}+\frac{r_2}{2}. \tag{5}$$

Let r_3 be the final radius.

To smoothly glue two bent regions along the boundary $W \times S^{q-1}$, we have to homotope the metrics on the boundaries. Let h_r be the metric on $W \times S^{q-1}$ induced from the boundary of $(N(r), g)$. On $W \times S^{q-1}$ we define a G-invariant product metric $\bar{h}_r := \sum_{i,j} \bar{g}^W + g_{\mathrm{std}}(r)$ where $\bar{g}^W$ is a fixed G-invariant metric on W and $g_{\mathrm{std}}(r)$ denotes the round metric of $S^{q-1}(r)$. Obviously the scalar curvature $s_{\bar{h}_r}$ of $\bar{h}_r$ is $\frac{(q-1)(q-2)}{r^2} + O(1)$. Moreover

Lemma 3.3. *Let h_r^ν for $\nu \in [0,1]$ be the convex combination $\nu h_r + (1-\nu)\bar{h}_r$ of h_r and $\bar{h}_r$. Then there exists a constant $C > 0$ such that the scalar curvature $s_{h_r^\nu}$ of h_r^ν is bounded below $\frac{C}{r^2}$ for any ν and any sufficiently small $r > 0$.*

Proof. This is basically because h_r^ν is very close to a Riemannian submersion with totally geodesic fibers $S^{q-1}(r)$, and hence the O'Neill's formula [4] gives such an estimate of $s_{h_r^\nu}$. It's enough to show that the difference between $s_{h_r^\nu}$ and $s_{\bar{h}_r}$ is at most $O(\frac{1}{r})$.

As before we let $i, j, k, \ldots$ denote the indices of coordinates of W in $W \times S^{q-1}$ and $\alpha, \beta, \gamma, \ldots$ denote the indices of coordinates of S^{q-1} in $W \times S^{q-1}$, and $A, B, C, \ldots$ will denote the indices of coordinates of both W and S^{q-1}. Writing an $(n-1) \times (n-1)$ matrix (M_{AB}) as

$$\left(\begin{array}{c|c} M_{ij} & M_{i\alpha} \\ \hline M_{\alpha i} & M_{\alpha\beta} \end{array}\right),$$

we have

$$(h_r^\nu) = (\bar{h}_r) + (h_r^\nu - \bar{h}_r) = \left(\begin{array}{c|c} O(1) & 0 \\ \hline 0 & O(r^2) \end{array}\right) + \left(\begin{array}{c|c} O(r) & O(r^2) \\ \hline O(r^2) & 0 \end{array}\right), \tag{6}$$

and

$$(h_r^\nu)^{-1} = (\bar{h}_r)^{-1} + \left((h_r^\nu)^{-1} - (\bar{h}_r)^{-1}\right) = \left(\begin{array}{c|c} O(1) & 0 \\ \hline 0 & O(\frac{1}{r^2}) \end{array}\right) + \left(\begin{array}{c|c} O(r) & O(1) \\ \hline O(1) & O(\frac{1}{r}) \end{array}\right). \tag{7}$$

The same estimates also hold for their derivatives.

Recall that Christoffel symbols of a metric h are given by

$$\Gamma_{AB}^C = \frac{1}{2}\sum_D h^{CD}\left\{\frac{\partial h_{AD}}{\partial x_B} + \frac{\partial h_{BD}}{\partial x_A} - \frac{\partial h_{AB}}{\partial x_D}\right\}, \tag{8}$$

and the Riemann curvature tensor R is given by

$$R_{ABC}^D = \partial_A \Gamma_{BC}^D - \partial_B \Gamma_{AC}^D + \Gamma_{BC}^E \Gamma_{AE}^D - \Gamma_{AC}^E \Gamma_{BE}^D. \tag{9}$$

Denote the Christoffel symbol of $\bar{h}_r$ and h_r^ν by $\bar{\Gamma}_r$ and Γ_r^ν respectively. Then the direct computations show that

$$(\bar{\Gamma}_r)_{AB}^C = O(1) = \partial(\bar{\Gamma}_r)_{AB}^C,$$

and

$$(\Gamma_r^\nu)_{AB}^C - (\bar{\Gamma}_r)_{AB}^C = O(r) = \partial(\Gamma_r^\nu)_{AB}^C - \partial(\bar{\Gamma}_r)_{AB}^C$$

except

$$(\Gamma_r^\nu)_{ij}^\alpha - (\bar{\Gamma}_r)_{ij}^\alpha = O\left(\frac{1}{r}\right) = \partial(\Gamma_r^\nu)_{ij}^\alpha - \partial(\bar{\Gamma}_r)_{ij}^\alpha.$$

Also denote the Riemann curvature tensor of $\bar{h}_r$ and h_r^ν by $\bar{R}_r$ and R_r^ν respectively. Then

$$(R_r^\nu)^\alpha_{\alpha ij} - (\bar{R}_r)^\alpha_{\alpha ij} = O\left(\frac{1}{r}\right) = (R_r^\nu)^l_{ijk} - (\bar{R}_r)^l_{ijk},$$

and

$$(R_r^\nu)^\alpha_{\alpha\beta\gamma} - (\bar{R}_r)^\alpha_{\alpha\beta\gamma} = O(r).$$

Thus the difference between sectional curvatures of $\bar{h}_r$ and h_r^ν is bounded above by $O(\frac{1}{r})$, and hence so is the differences of two scalar curvatures, completing the proof. □

Now we have the metric h_{r_3} on the boundary. We have to homotope h_{r_3} to a G-invariant product metric $\bar{h}_{r_3}$. Consider a smooth homotopy $H_{r_3}(z,t) := \varphi(t)h_{r_3} + (1-\varphi(t))\bar{h}_{r_3}$ for $(z,t) \in (W \times S^{q-1}) \times [0,1]$, where $\varphi : [0,1] \to [0,1]$ is a smooth decreasing function which is 1 near 0 and 0 near 1. In the above lemma we have seen that $(W \times S^{q-1}, H_{r_3}(z,t))$ for each $t \in [0,1]$ has positive scalar curvature. Then by the Gromov–Lawson lemma in [7], there exists a constant $d > 0$ such that the metric $H_{r_3}(z,t/d) + dt^2$ on $W \times S^{q-1} \times [0,d]$ has positive scalar curvature for sufficiently small $r_3 > 0$. Obviously $H_{r_3}(z,t/d) + dt^2$ is also G-invariant and we now glue to get a smooth G-invariant metric with scalar curvature bigger than $s_g - \epsilon_2$ on M.

An important fact about the bending of γ is that if we can take r_1' and r_2 further small, we only need to shrink the remaining part of γ homothetically. Let $\{(t, f(t))\}$ be the graph of γ in step 3 and τ_1 be $f^{-1}(r_2)$. For $0 < \mu \leqslant 1$, let us take $\mu r_1'$ and μr_2 instead of r_1' and r_2 respectively, and let τ_μ be the t-coordinate corresponding to μr_2. Then we shrink the step 3 part of γ homothetically by μ and concatenate it to $(\tau_\mu, \mu r_2)$. Indeed the equation of this portion of the curve is given by $(t, \mu f(\frac{t-\tau_\mu+\mu\tau_1}{\mu}))$. Moreover, noting that the geodesic curvature k is dilated by $\frac{1}{\mu}$ without changing θ, the scalar curvature at $(t, \mu y)$ satisfies

$$s(t,\mu y) \geqslant s_g(t,\mu y) + \frac{(q-1)(q-2)}{2}\frac{\sin^2\theta}{(\mu|y|)^2} - 3(q-1)\frac{k\sin\theta}{\mu|y|} \geqslant s_g(t,\mu y),$$

where we used (4) in the second inequality. We denote the curve with $\mu r_1'$ and μr_2 instead of r_1' and r_2 by γ_μ.

We also claim that the metric on the homotopy region $W \times S^{q-1} \times [0,d]$ can be accordingly shrunk to $H_{\mu r_3}(z,t/d) + \mu^2\,dt^2$ still having positive scalar curvature for any $\mu \in (0,1]$, once r_2 and hence r_3 was chosen sufficiently small.

Lemma 3.4. *The scalar curvature of the manifold $W \times S^{q-1} \times [0,d]$ with the metric $H_{\mu r_3}(z,t/d) + \mu^2\,dt^2$ is bounded below by $\frac{C}{(\mu r_3)^2} + \frac{C'}{\mu^2}$ for any $\mu \in (0,1]$, and any sufficiently small $r_3 > 0$, where $C > 0$ is given in Lemma 3.3 and C' is a constant.*

Proof. The proof continues from the above lemma. Using the estimates (6) and (7), $H_{\mu r_3}(z,t/d) + \mu^2\,dt^2$ is given by

$$\left(\begin{array}{c|c} H_{\mu r_3}(z,t/d) & 0 \\ \hline 0 & \mu^2 \end{array}\right) = \left(\begin{array}{c|c} \begin{array}{c|c} O(1) & O((\mu r_3)^2) \\ \hline O((\mu r_3)^2) & O((\mu r_3)^2) \end{array} & 0 \\ \hline 0 & \mu^2 \end{array}\right),$$

and its inverse is given by

$$\left(\begin{array}{c|c} (H_{\mu r_3}(z,t/d))^{-1} & 0 \\ \hline 0 & \frac{1}{\mu^2} \end{array}\right) = \left(\begin{array}{c|c} \begin{array}{c|c} O(1) & O(1) \\ \hline O(1) & O(\frac{1}{(\mu r_3)^2}) \end{array} & 0 \\ \hline 0 & \frac{1}{\mu^2} \end{array}\right).$$

The same estimates also hold for their derivatives. We let Γ^μ and R^μ be the Christoffel symbol and the Riemann curvature tensor of $H_{\mu r_3}(z,t/d) + \mu^2\,dt^2$ respectively. As before $A, B, C, \ldots$ run from 1 to $n-1$, and N denotes the index of the last coordinate function t. The direct computations show that

$$(\Gamma^\mu)^N_{NN} = (\Gamma^\mu)^N_{AN} = 0 = \partial(\Gamma^\mu)^N_{NN} = \partial(\Gamma^\mu)^N_{AN},$$
$$(\Gamma^\mu)^N_{AB} = \frac{1}{\mu^2}O(1) = \partial(\Gamma^\mu)^N_{AB}, \qquad (\Gamma^\mu)^C_{AN} = O(1) = \partial(\Gamma^\mu)^C_{AN},$$

and

$$(R^\mu)^N_{NBC} = \frac{1}{\mu^2}O(1).$$

Let X_t be the hypersurface $W \times S^{q-1} \times \{t\}$. Then the second fundamental form of X_t is given by $(\Gamma^\mu)^N_{AB} = \frac{1}{\mu^2}O(1)$, and hence its norm is of the form $\frac{1}{\mu}O(1)$. Denote the scalar curvature of the hypersurface X_t with the induced metric by s_{X_t}. It follows from the Gauss curvature equation and the above lemma that the scalar curvature is given by

$$s_{X_t} + \frac{1}{\mu^2}O(1) + 2\sum_{B=1}^{n}(R^\mu)^N_{NBB} = s_{X_t} + \frac{1}{\mu^2}O(1) \geqslant \frac{C}{(\mu r_3)^2} + \frac{1}{\mu^2}O(1). \qquad \square$$

Therefore the scalar curvature of $H_{\mu r_3}(z, t/d) + \mu^2\,dt^2$ is positive for sufficiently small $r_3 > 0$. From now on we assume that r_2 was taken small enough to ensure this, and the Riemannian G-manifold obtained by γ_μ and $H_{\mu r_3}(z, t/d) + \mu^2\,dt^2$ is denoted by $(M_\mu, \tilde{g}_\mu)$.

We define three Riemannian manifolds with boundary $(S_{\delta,\varepsilon}, \tilde{g}_{\delta,\varepsilon}) \subset (T_{\delta,\varepsilon}, \tilde{g}_{\delta,\varepsilon}) \subset (N_{\delta\varepsilon}, \tilde{g}_{\delta\varepsilon})$ by

$$S_{\delta,\varepsilon} \equiv M_{\delta\varepsilon} - \big(M_0 - \{r \leqslant \delta\varepsilon r_1'\}\big),$$
$$T_{\delta,\varepsilon} \equiv M_{\delta\varepsilon} - \big(M_0 - \{r \leqslant \varepsilon r_1\}\big),$$

and

$$N_{\delta\varepsilon} \equiv M_{\delta\varepsilon} - \big(M_0 - \{r \geqslant r_0\}\big)$$

with the induced metric. (In fact, $(S_{\delta,\varepsilon}, \tilde{g}_{\delta,\varepsilon})$ depends only on $\delta\varepsilon$.) To investigate the relation between $T_{\delta,1}$ and $T_{\delta,\varepsilon}$, let x be any point in W and define a q-dimensional Riemannian submanifold $(T_{\delta,\varepsilon,x}, \tilde{g}_{\delta,\varepsilon,x}) \subset (T_{\delta,\varepsilon}, \tilde{g}_{\delta,\varepsilon})$ by $T_{\delta,\varepsilon,x} \equiv T_{\delta,\varepsilon} \cap (\{x\} \times S^{q-1} \times [0,d])$ with the induced metric. Taking into account that g is C^0-near to the product metric on $N(r_0)$, i.e., $g = g^W + g_E + O(r_0)$, where g_E is the Euclidean metric on $\mathbb{R}^q$, we have

$$\tilde{g}_{\delta,\varepsilon} = g^W + \tilde{g}_{\delta,\varepsilon,x} + O(\varepsilon r_1)$$

on $T_{\delta,\varepsilon}$. The obvious shrinking map from γ_δ for $r \leqslant r_1$ onto $\gamma_{\delta\varepsilon}$ for $r \leqslant \varepsilon r_1$ and the identity map in the homotopy region induces a diffeomorphism $\Phi_{\delta,\varepsilon}$ from $T_{\delta,1}$ to $T_{\delta,\varepsilon}$, which gives $\Phi^*(\tilde{g}_{\delta,\varepsilon,x}) = \varepsilon^2\tilde{g}_{\delta,1,x}$. Thus we have on $T_{\delta,\varepsilon}$,

$$\begin{aligned}\Phi^*(dV_{\tilde{g}_{\delta,\varepsilon}}) &= \Phi^*\big(\big(1 + O(\varepsilon r_1)\big)\,dV_{g^W}\,dV_{\tilde{g}_{\delta,\varepsilon,x}}\big) = \varepsilon^q\big(1 + O(\varepsilon r_1)\big)\,dV_{g^W}\,dV_{\tilde{g}_{\delta,1,x}}\\ &\lessgtr \varepsilon^q(1 \pm C_1 r_1)\,dV_{\tilde{g}_{\delta,1}},\end{aligned} \tag{10}$$

where $C_1 > 0$ is a constant. From now on C_i's will denote some positive constants. Let $\langle\cdot,\cdot\rangle_{\tilde{g}_{\delta,\varepsilon}}$ and $\langle\cdot,\cdot\rangle_{\tilde{g}_{\delta,\varepsilon,x}}$ denote the inner product on $(T_{\delta,\varepsilon}, \tilde{g}_{\delta,\varepsilon})$ and $(T_{\delta,\varepsilon,x}, \tilde{g}_{\delta,\varepsilon,x})$ respectively. Then we also have on $T_{\delta,\varepsilon}$,

$$\begin{aligned}\Phi^*\langle\omega,\omega\rangle_{\tilde{g}_{\delta,\varepsilon}} &= \Phi^*\big(\big(1 + O(\varepsilon r_1)\big)\langle\omega,\omega\rangle_{\tilde{g}_{\delta,\varepsilon,x}}\big) = \frac{1}{\varepsilon^2}\big(1 + O(\varepsilon r_1)\big)\langle\omega,\omega\rangle_{\tilde{g}_{\delta,1,x}}\\ &\lessgtr \frac{1}{\varepsilon^2}(1 \pm C_2 r_1)\langle\omega,\omega\rangle_{\tilde{g}_{\delta,1}}\end{aligned} \tag{11}$$

for any 1-form ω belonging to $T^*(S^{q-1} \times [0,d])$ in $T^*(W \times S^{q-1} \times [0,d])$. It's important that C_1 and C_2 are uniform constants independent of any choices we made such as θ_0, r_2, δ, and etc., as long as r_0 is sufficiently small, which we always assume. From now on we will omit Φ^* for convenience. Also note that for any choice of r_0 and θ_0, the length of the step 3 part of $\gamma_{\delta\varepsilon}$ and the volume of the homotopy region can be made arbitrarily small by taking r_2

much smaller, which we always assume from now on. This means that there exist constants $C_3, C_4, C_5 > 0$ such that

$$\mathrm{vol}_{\tilde{g}_{\delta\varepsilon}}(N_{\delta\varepsilon}) \leqslant C_3 r_0^q, \qquad \mathrm{vol}_{\tilde{g}_{\delta,\varepsilon}}(T_{\delta,\varepsilon}) \geqslant C_4(\varepsilon r_1)^q,$$

and

$$\mathrm{vol}_{\tilde{g}_{\delta,\varepsilon}}(S_{\delta,\varepsilon}) \leqslant C_5(\delta\varepsilon r_1')^q,$$

where C_i's are also uniform constants when r_0, θ_0 and r_2 are chosen small by the above way. As the last preparation, we have

Lemma 3.5. *There is a constant $\hat{C} > 0$ independent of $\delta \in (0, 1]$ satisfying the Sobolev inequality*

$$\left(\frac{\int_{S_{\delta,1}} \varphi^p \, dV_{\tilde{g}_{\delta,1}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/p} \leqslant \hat{C}\left(\left(\frac{\int_{S_{\delta,1}} \varphi^2 \, dV_{\tilde{g}_{\delta,1}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/2} + \left(\frac{\int_{S_{\delta,1}} |d\varphi|^2_{\tilde{g}_{\delta,1}} \, dV_{\tilde{g}_{\delta,1}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/2}\right) \tag{12}$$

for any $\varphi \in L_1^2(S_{\delta,1})$.

Proof. For a fixed θ_0, r_1' and r_2, get $(S_{1,1}, \tilde{g}_{1,1})$ and choose a $\hat{C}$ satisfying the above inequality. In the same way as above, consider a diffeomorphism Ψ from $S_{1,1}$ onto $S_{\delta,1}$ such that

$$\Psi^*(dV_{\tilde{g}_{\delta,1}}) \lesseqgtr \delta^q(1 \pm C_6 r_1')\, dV_{\tilde{g}_{1,1}},$$

$$\Psi^*\langle\omega, \omega\rangle_{\tilde{g}_{\delta,1}} \lesseqgtr \frac{1}{\delta^2}(1 \pm C_7 r_1')\langle\omega, \omega\rangle_{\tilde{g}_{1,1}},$$

and

$$\Psi^*\langle\sigma, \sigma\rangle_{\tilde{g}_{\delta,1}} \lesseqgtr (1 \pm C_8 r_1')\langle\sigma, \sigma\rangle_{\tilde{g}_{1,1}}$$

for any 1-forms ω and σ belonging to $T^*(S^{q-1} \times [0, d])$ and T^*W in $T^*(W \times S^{q-1} \times [0, d])$ respectively. Then the result follows immediately. □

Although it is not necessary for our further discussion, we remark that

Remark. In fact $\hat{C}$ may depend only on θ_0, r_1', and r_2. Notice that $\hat{C}$ is a continuous function of the metric in C^0-norm. Since the ambiguity of the step 3 construction of γ can be made very small, any possible $(S_{1,1}, \tilde{g}_{1,1})$ is C^0-close, once θ_0, r_1', r_2 are determined. As a final note, actually we will not need the δ-independence of $\hat{C}$, because we will use $\hat{C}$ for a fixed δ.

Now let us get down to estimating the G-Yamabe constant of $(M_{\delta\varepsilon}, [\tilde{g}_{\delta\varepsilon}]_G)$. Let $\varphi_{\delta\varepsilon}$ be a G-Yamabe minimizer satisfying $\int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} = 1$. We have two cases, either

$$\int\limits_{S_{\delta,\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta,\varepsilon}} \leqslant \frac{2^{p+1}\hat{C}^p \, \mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(T_{\delta,1} - S_{\delta,1})} \int\limits_{M_{\delta\varepsilon} - S_{\delta,\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}}$$

or not.

Assume the first case. Let $\eta_{\delta\varepsilon}(r)$ be defined by $\eta(\frac{r}{\delta\varepsilon})$. On the support of $\eta_{\delta\varepsilon}$, $\tilde{g}_{\delta\varepsilon}$ is very close to g when θ_0 is very small. To compare these two metrics on this region, let $i : M_0 - N(\delta\varepsilon r_3) \to M_{\delta\varepsilon}$ be the obvious inclusion map. Then i is isometric on the outside of $N(r_0)$. On $N(r_0) - N(\delta\varepsilon r_3)$, i is isometric in the direction orthogonal to the radial direction, and $\frac{\partial}{\partial r}$ gets dilated by $\frac{1}{\sqrt{1-\sin^2\theta}}$. In particular on the support of $\eta_{\delta\varepsilon}$,

$$dV_{\tilde{g}_{\delta\varepsilon}} \geqslant dV_g \geqslant \sqrt{1 - \sin^2\theta_0}\, dV_{\tilde{g}_{\delta\varepsilon}},$$

and

$$|\omega|_{\tilde{g}_{\delta\varepsilon}} \leqslant |\omega|_g \leqslant \frac{1}{\sqrt{1 - \sin^2\theta_0}}|\omega|_{\tilde{g}_{\delta\varepsilon}}$$

for any 1-form ω. This gives us that

$$\int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} \geqslant \int_{M_0} (\eta_{\delta\varepsilon}\varphi_{\delta\varepsilon})^p \, dV_g,$$

and

$$\begin{aligned}
\int_{M_0} (\eta_{\delta\varepsilon}\varphi_{\delta\varepsilon})^p \, dV_g &\geqslant \sqrt{1-\sin^2\theta_0} \int_{M_{\delta\varepsilon}} (\eta_{\delta\varepsilon}\varphi_{\delta\varepsilon})^p \, dV_{\tilde{g}_{\delta\varepsilon}} \\
&\geqslant \sqrt{1-\sin^2\theta_0} \Big(\int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} - \int_{S_{\delta,\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta,\varepsilon}} \Big) \\
&\geqslant \sqrt{1-\sin^2\theta_0} \Big(\int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} - \frac{2^{p+1}\hat{C}^p \operatorname{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}{\operatorname{vol}_{\tilde{g}_{\delta,1}}(T_{\delta,1}-S_{\delta,1})} \int_{M_{\delta\varepsilon}-S_{\delta,\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} \Big) \\
&\geqslant \sqrt{1-\sin^2\theta_0} \Big(\Big(1 - \frac{2^{p+1}\hat{C}^p C_5(\delta r_1')^q}{C_4 r_1^q - C_5(\delta r_1')^q} \Big) \int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} \Big) \\
&= \sqrt{1-\sin^2\theta_0} (1 - C_9\delta^q) \int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}}.
\end{aligned}$$

Using the fact that $s_{\tilde{g}_{\delta\varepsilon}} \geqslant s_g + \frac{(q-1)(q-2)}{2}\frac{\sin^2\theta_0}{r^2} \geqslant s_g + |d\eta_{\delta\varepsilon}|_g^2$ on the support of $d\eta_{\delta\varepsilon}$, and $s_{\tilde{g}_{\delta\varepsilon}}$ is bounded below by $(\min s_g) - \epsilon_2$, we get

$$\begin{aligned}
&\int_{M_0} \big(\big| d(\eta_{\delta\varepsilon}\varphi_{\delta\varepsilon}) \big|_g^2 + s_g (\eta_{\delta\varepsilon}\varphi_{\delta\varepsilon})^2 \big) \, dV_g \\
&\quad = \int_{\{r \geqslant \delta\varepsilon r_2\}} \big(\eta_{\delta\varepsilon}^2 |d\varphi_{\delta\varepsilon}|_g^2 + |d\eta_{\delta\varepsilon}|_g^2 \varphi_{\delta\varepsilon}^2 + s_g(\eta_{\delta\varepsilon}\varphi_{\delta\varepsilon}^2) \big) \, dV_g \\
&\quad \leqslant \int_{M_{\delta\varepsilon}} \frac{1}{1-\sin^2\theta_0} \big(|d\varphi_{\delta\varepsilon}|_{\tilde{g}_{\delta\varepsilon}}^2 + s_{\tilde{g}_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^2 \big) \, dV_{\tilde{g}_{\delta\varepsilon}} + C_{10} \int_{\{\delta\varepsilon r_2 \leqslant r \leqslant \delta\varepsilon r_1'\}} \varphi_{\delta\varepsilon}^2 \, dV_{\tilde{g}_{\delta\varepsilon}} + C_{11} \sin^2\theta_0 \int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^2 \, dV_{\tilde{g}_{\delta\varepsilon}},
\end{aligned}$$

where C_{10} and C_{11} are constants depending only on $\min s_g$. By using the Hölder inequality the second term is bounded above by

$$C_{10} \big(\operatorname{vol}(S_{\delta\varepsilon})_{\tilde{g}_{\delta\varepsilon}} \big)^{2/n} \Big(\int_{S_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} \Big)^{2/p} \leqslant C_{10} \big(C_5 (\delta\varepsilon r_1')^q \big)^{2/n} \Big(\int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} \Big)^{2/p},$$

and the third term is bounded above by

$$C_{11} \sin^2\theta_0 \big(\operatorname{vol}(M_{\delta\varepsilon})_{\tilde{g}_{\delta\varepsilon}} \big)^{2/n} \Big(\int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} \Big)^{2/p} \leqslant C_{12} \sin^2\theta_0 \Big(\int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}} \Big)^{2/p},$$

where we used the fact that

$$\operatorname{vol}_{\tilde{g}_{\delta\varepsilon}}(M_{\delta\varepsilon}) = \operatorname{vol}_g \big(M_0 - N(r_0) \big) + \operatorname{vol}_{\tilde{g}_{\delta\varepsilon}}(N_{\delta\varepsilon}) \leqslant \operatorname{vol}_g \big(M_0 - N(r_0) \big) + C_3 r_0^q.$$

Thus

$$Q_g(\eta_{\delta\varepsilon}\varphi_{\delta\varepsilon}) \geqslant Y\big(M_0, [g]_G \big) \geqslant Y_G(M_0) - \epsilon_1$$

is bounded above by

$$(1-\sin^2\theta_0)^{-1}Y\big(M_{\delta\varepsilon},[\tilde g_{\delta\varepsilon}]_G\big)+C_{10}\big(C_5(\delta\varepsilon r_1')^q\big)^{2/n}+C_{12}\sin^2\theta_0$$

or

$$\frac{(1-\sin^2\theta_0)^{-1}Y(M_{\delta\varepsilon},[\tilde g_{\delta\varepsilon}]_G)+C_{10}(C_5(\delta\varepsilon r_1')^q)^{2/n}+C_{12}\sin^2\theta_0}{(1-\sin^2\theta_0)^{1/p}(1-C_9\delta^q)^{2/p}}$$

for any δ and ε. Recall that $C_{10}C_5^{2/n}$ and C_{12} are uniform constants independent of any choices and C_9 is independent of δ and ε. Taking first θ_0 and then δ arbitrarily small, we have

$$Y_G(M)\geqslant Y_G(M_0)-\epsilon_1.$$

Since $\epsilon_1>0$ is arbitrary, it follows that

$$Y_G(M)\geqslant Y_G(M_0).$$

In the second case, we want to derive a contradiction when $\varepsilon>0$ gets sufficiently small for any fixed $\delta>0$.

Lemma 3.6. *Suppose that (X,h) is a compact Riemannian manifold with smooth boundary and that $f\in L^2(X)$ satisfying $\int_X f\,dV_h=0$. Then there exists a function $\xi\in L^2_2(X)$ unique up to the addition of constant such that $\Delta\xi=f$ and in addition $\vec n\cdot\nabla\xi$ vanishes at the boundary, where $\vec n$ is the unit outward normal to the boundary.*

Proof. See [9]. □

Consider a step function f_δ on $T_{\delta,1}$ defined by

$$f_\delta=\begin{cases}\operatorname{vol}_{\tilde g_{\delta,1}}(S_{\delta,1})^{-1} & \text{on } S_{\delta,1},\\ (\operatorname{vol}_{\tilde g_{\delta,1}}(S_{\delta,1})-\operatorname{vol}_{\tilde g_{\delta,1}}(T_{\delta,1}))^{-1} & \text{on } T_{\delta,1}-S_{\delta,1}.\end{cases}$$

Then $\int_{T_{\delta,1}} f_\delta\,dV_{\tilde g_{\delta,1}}=0$, so by the above lemma, there exists a function $\xi_\delta\in L^2_2(T_{\delta,1})$ satisfying $\Delta\xi_\delta=f_\delta$, and that $\nabla\xi_\delta$ vanishes normal to the boundary. For any $\varphi\in L^2_1(T_{\delta,1})$ the integration by parts yields

$$\begin{aligned}&\frac{1}{\operatorname{vol}_{\tilde g_{\delta,1}}(S_{\delta,1})}\int_{S_{\delta,1}}\varphi\,dV_{\tilde g_{\delta,1}}-\frac{1}{\operatorname{vol}_{\tilde g_{\delta,1}}(T_{\delta,1}-S_{\delta,1})}\int_{T_{\delta,1}-S_{\delta,1}}\varphi\,dV_{\tilde g_{\delta,1}}\\&\quad=\int_{T_{\delta,1}}\varphi\Delta\xi_\delta\,dV_{\tilde g_{\delta,1}}=\int_{T_{\delta,1}}\langle d\varphi,d\xi_\delta\rangle_{\tilde g_{\delta,1}}\,dV_{\tilde g_{\delta,1}},\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{\operatorname{vol}_{\tilde g_{\delta,1}}(S_{\delta,1})}\left|\int_{S_{\delta,1}}\varphi\,dV_{\tilde g_{\delta,1}}\right|-\frac{1}{\operatorname{vol}_{\tilde g_{\delta,1}}(T_{\delta,1}-S_{\delta,1})}\left|\int_{T_{\delta,1}-S_{\delta,1}}\varphi\,dV_{\tilde g_{\delta,1}}\right|\\&\quad\leqslant\left(\int_{T_{\delta,1}}|d\xi_\delta|^2_{\tilde g_{\delta,1}}\,dV_{\tilde g_{\delta,1}}\right)^{1/2}\left(\int_{T_{\delta,1}}|d\varphi|^2_{\tilde g_{\delta,1}}\,dV_{\tilde g_{\delta,1}}\right)^{1/2}\end{aligned}\tag{13}$$

by the Hölder inequality. Since $S_{\delta,1}$ is connected, the constants are the only eigenvectors of Δ on $S_{\delta,1}$ with eigenvalue 0 and derivative vanishing normal to the boundary. By the discreteness of the spectrum of Δ on $S_{\delta,1}$ with these boundary conditions, we have

$$\left(\frac{\int_{S_{\delta,1}}\varphi^2\,dV_{\tilde g_{\delta,1}}}{\operatorname{vol}_{\tilde g_{\delta,1}}(S_{\delta,1})}\right)^{1/2}\leqslant\frac{1}{\operatorname{vol}_{\tilde g_{\delta,1}}(S_{\delta,1})}\left|\int_{S_{\delta,1}}\varphi\,dV_{\tilde g_{\delta,1}}\right|+\left(\frac{\int_{S_{\delta,1}}|d\varphi|^2_{\tilde g_{\delta,1}}\,dV_{\tilde g_{\delta,1}}}{C_{13}\operatorname{vol}_{\tilde g_{\delta,1}}(S_{\delta,1})}\right)^{1/2}.\tag{14}$$

Also, by the Sobolev inequality (12),

$$\frac{1}{\hat{C}}\left(\frac{\int_{S_{\delta,1}}\varphi^p\,dV_{\tilde{g}_{\delta,1}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/p}-\left(\frac{\int_{S_{\delta,1}}|d\varphi|^2_{\tilde{g}_{\delta,1}}\,dV_{\tilde{g}_{\delta,1}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/2}\leqslant\left(\frac{\int_{S_{\delta,1}}\varphi^2\,dV_{\tilde{g}_{\delta,1}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/2}. \tag{15}$$

On the other hand, the Hölder inequality gives

$$\frac{1}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(T_{\delta,1}-S_{\delta,1})}\left|\int\limits_{T_{\delta,1}-S_{\delta,1}}\varphi\,dV_{\tilde{g}_{\delta,1}}\right|\leqslant\left(\frac{\int_{T_{\delta,1}-S_{\delta,1}}\varphi^p\,dV_{\tilde{g}_{\delta,1}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(T_{\delta,1})-\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/p}. \tag{16}$$

Adding together (13), (14), (15), and (16) yields

$$\frac{1}{\hat{C}}\left(\frac{\int_{S_{\delta,1}}\varphi^p\,dV_{\tilde{g}_{\delta,1}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/p}-\left(\frac{\int_{T_{\delta,1}-S_{\delta,1}}\varphi^p\,dV_{\tilde{g}_{\delta,1}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(T_{\delta,1})-\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/p}\leqslant C_{14}\left(\int\limits_{T_{\delta,1}}|d\varphi|^2_{\tilde{g}_{\delta,1}}\,dV_{\tilde{g}_{\delta,1}}\right)^{1/2}.$$

Now if φ is G-invariant, then $\frac{\partial\varphi}{\partial x_i}=0$ for any $i=1,\dots,n-q$, because the G-action on W is locally transitive. Then using (10) and (11), we get

$$\frac{1}{\hat{C}}\left(\frac{1-C_{15}r_1}{\varepsilon^q}\frac{\int_{S_{\delta,\varepsilon}}\varphi^p\,dV_{\tilde{g}_{\delta,\varepsilon}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/p}-\left(\frac{1+C_{16}r_1}{\varepsilon^q}\frac{\int_{T_{\delta,\varepsilon}-S_{\delta,\varepsilon}}\varphi^p\,dV_{\tilde{g}_{\delta,\varepsilon}}}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(T_{\delta,1}-S_{\delta,1})}\right)^{1/p}$$
$$\leqslant C_{14}\left(\frac{1+C_{17}r_1}{\varepsilon^{q-2}}\int\limits_{T_{\delta,\varepsilon}}|d\varphi|^2_{\tilde{g}_{\delta,\varepsilon}}\,dV_{\tilde{g}_{\delta,\varepsilon}}\right)^{1/2}.$$

Since C_{15}, C_{16}, and C_{17} are uniform constants, we get for sufficiently small $r_1>0$,

$$\left(\int\limits_{S_{\delta,\varepsilon}}\varphi^p\,dV_{\tilde{g}_{\delta,\varepsilon}}\right)^{1/p}-\left(\frac{2\hat{C}^p\,\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(T_{\delta,1}-S_{\delta,1})}\int\limits_{T_{\delta,\varepsilon}-S_{\delta,\varepsilon}}\varphi^p\,dV_{\tilde{g}_{\delta,\varepsilon}}\right)^{1/p}\leqslant C_{18}\varepsilon^{1-q/n}\left(\int\limits_{T_{\delta,\varepsilon}}|d\varphi|^2_{\tilde{g}_{\delta,\varepsilon}}\,dV_{\tilde{g}_{\delta,\varepsilon}}\right)^{1/2}.$$

Under the assumption that

$$\int\limits_{S_{\delta,\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta,\varepsilon}}>\frac{2^{p+1}\hat{C}^p\,\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}{\mathrm{vol}_{\tilde{g}_{\delta,1}}(T_{\delta,1}-S_{\delta,1})}\int\limits_{M_{\delta\varepsilon}-S_{\delta,\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta\varepsilon}},$$

we have

$$\frac{1}{2}\left(\int\limits_{S_{\delta,\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta,\varepsilon}}\right)^{1/p}\leqslant C_{18}\varepsilon^{1-q/n}\left(\int\limits_{T_{\delta,\varepsilon}}|d\varphi_{\delta\varepsilon}|^2_{\tilde{g}_{\delta,\varepsilon}}\,dV_{\tilde{g}_{\delta,\varepsilon}}\right)^{1/2},$$

and

$$\begin{aligned}\left(\int\limits_{M_{\delta\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta\varepsilon}}\right)^{1/p}&=\left(\int\limits_{S_{\delta,\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta,\varepsilon}}+\int\limits_{M_{\delta\varepsilon}-S_{\delta,\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta\varepsilon}}\right)^{1/p}\\&\leqslant\left(\int\limits_{S_{\delta,\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta,\varepsilon}}\right)^{1/p}+\left(\int\limits_{M_{\delta\varepsilon}-S_{\delta,\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta\varepsilon}}\right)^{1/p}\\&\leqslant\left(\int\limits_{S_{\delta,\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta,\varepsilon}}\right)^{1/p}\left(1+\left(\frac{\mathrm{vol}_{\tilde{g}_{\delta,1}}(T_{\delta,1}-S_{\delta,1})}{2^{p+1}\hat{C}^p\,\mathrm{vol}_{\tilde{g}_{\delta,1}}(S_{\delta,1})}\right)^{1/p}\right),\end{aligned}$$

which yield

$$C_{19}\varepsilon^{-2+\frac{2q}{n}}\leqslant\frac{\int_{T_{\delta,\varepsilon}}|d\varphi_{\delta\varepsilon}|^2_{\tilde{g}_{\delta,\varepsilon}}\,dV_{\tilde{g}_{\delta,\varepsilon}}}{(\int_{M_{\delta\varepsilon}}\varphi^p_{\delta\varepsilon}\,dV_{\tilde{g}_{\delta\varepsilon}})^{2/p}}.$$

On the other hand, using the fact that $s_{\tilde{g}_{\delta\varepsilon}}$ is bounded below and $\mathrm{vol}_{\tilde{g}_{\delta\varepsilon}}(M_{\delta\varepsilon})$ is bounded above for any $\varepsilon \in (0, 1]$, a simple application of the Hölder inequality gives

$$\frac{\int_{M_{\delta\varepsilon}} s_{\tilde{g}_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^2 \, dV_{\tilde{g}_{\delta\varepsilon}}}{(\int_{M_{\delta\varepsilon}} \varphi_{\delta\varepsilon}^p \, dV_{\tilde{g}_{\delta\varepsilon}})^{2/p}} \geqslant \min\bigl(0, (\min s_g) - \epsilon_2\bigr)\bigl(\mathrm{vol}_{\tilde{g}_{\delta\varepsilon}}(M_{\delta\varepsilon})\bigr)^{1-2/p} \geqslant -C_{20}$$

for any ε. Now by letting $\varepsilon \to 0$, $Q(\varphi_{\delta\varepsilon}^{p-2}\tilde{g}_{\delta\varepsilon}) \to \infty$. By the way, Proposition 2.5 says that $Y(M_{\delta\varepsilon}, [\tilde{g}_{\delta\varepsilon}]_G)$ is bounded above for any $\varepsilon \in (0, 1]$, because a G-invariant open set $(M_0 - N(r_0), g)$ is isometrically embedded into $(M_{\delta\varepsilon}, \tilde{g}_{\delta\varepsilon})$ under the identity map. This leads to a contradiction, completing the proof. □

4. Examples

Consider the unit n-sphere $S^n(1) \subset \mathbb{R}^{n+1}$ for $n \geqslant 3$ and an isometric G-action where $G = SO(n - q + 1)$ with $q \geqslant 3$ acts on the first $n - q + 1$ coordinates of $\mathbb{R}^{n+1}$ fixing the last q coordinates of $\mathbb{R}^{n+1}$. Then the complement of fixed point set $S^n(1) \cap (\{0\} \times \mathbb{R}^q)$ is foliated by G-invariant $(n - q)$-spheres on each of which the G-action is transitive. Since the round metric is a G-invariant Yamabe metric and the G-action has fixed points,

$$Y_G(S^n) = Y(S^n) = \Lambda_n.$$

Take two copies of S^n and perform a surgery along such S^{n-q} to get a Riemannian G-manifold $S^{n-q+1} \times S^{q-1}$. By our surgery theorem,

$$Y_G(S^{n-q+1} \times S^{q-1}) \geqslant Y_G(S^n \cup S^n) = Y_G(S^n) = \Lambda_n.$$

Since $S^{n-q+1} \times S^{q-1}$ has fixed points, $Y_G(S^{n-q+1} \times S^{q-1}) \leqslant \Lambda_n$ and hence

$$Y_G(S^{n-q+1} \times S^{q-1}) = \Lambda_n.$$

Taking connected sums of $S^{n-q+1} \times S^{q-1}$ along fixed points, we also have

$$Y_G\bigl(l(S^{n-q+1} \times S^{q-1}) \sharp\, m\overline{S^{n-q+1} \times S^{q-1}}\bigr) = \Lambda_n$$

for any integers $l, m \geqslant 0$.

References

[1] K. Akutagawa, B. Botvinnik, Relative Yamabe invariant, Comm. Anal. Geom. 10 (5) (2002) 935–969.
[2] M. Anderson, Remarks on Perelman's papers, Preprint.
[3] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976) 269–296.
[4] A. Besse, Einstein Manifolds, Springer-Verlag, Berlin, 1987.
[5] L. Bérard Bergery, Scalar curvature and isometry groups, in: M. Berger, S. Murakami, T. Ochiai (Eds.), Spectra of Riemannian Manifolds, Tokyo, 1983, pp. 9–28.
[6] H. Bray, A. Neves, Prime 3-manifolds with Yamabe invariant greater than $\mathbb{R}P^3$, Ann. of Math. 159 (1) (2004) 407–424.
[7] M. Gromov, H.B. Lawson, The classification of simply connected manifolds of positive scalar curvature, Ann. of Math. 111 (1980) 423–434.
[8] E. Hebey, M. Vaugon, Le problème de Yamabe équivariant, Bull. Sci. Math. 117 (1993) 241–286.
[9] L. Hörmander, Linear Partial Differential Operators, Grund. Math. Wiss., vol. 116, Springer-Verlag, Berlin, 1963.
[10] M. Ishida, C. LeBrun, Curvature, connected sums, and Seiberg–Witten theory, Comm. Anal. Geom. 11 (5) (2003) 809–836.
[11] D. Joyce, Constant scalar curvature metrics on connected sums, Int. J. Math. Math. Sci. 7 (2003) 405–450.
[12] O. Kobayashi, Scalar curvature of a metric with unit volume, Math. Ann. 279 (1987) 253–265.
[13] C. LeBrun, Four manifolds without Einstein metrics, Math. Res. Lett. 3 (1996) 133–147.
[14] C. LeBrun, Yamabe constants and the perturbed Seiberg–Witten equations, Comm. Anal. Geom. 5 (1997) 535–553.
[15] C. LeBrun, Kodaira dimension and the Yamabe problem, Comm. Anal. Geom. 7 (1999) 133–156.
[16] G. Perelman, Ricci flow with surgery on three-manifolds, math.DG/0303109.
[17] J. Petean, G. Yun, Surgery and the Yamabe invariant, GAFA 9 (1999) 1189–1199.
[18] R. Schoen, S.T. Yau, On the structure of manifolds with positive scalar curvature, Manuscripta Math. 28 (1979) 159–183.
[19] C. Sung, Surgery, curvature, and minimal volume, Ann. Global Anal. Geom. 26 (2004) 209–229.

Further reading

[20] J. Lee, T. Parker, The Yamabe problem, Bull. Amer. Soc. 17 (1987) 37–81.