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PEANO'S CONCEPT OF NUMBER

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SUMMARIES

Giuseppe Peano's development of the real number system from his postulates for the natural numbers and some of his views on definitions in mathematics are presented in order to clarify his concept of number. They show that his use of the axiomatic method was intended to make mathematical theory clearer, more precise, and easier to learn. They further reveal some of his reasons for not accepting the contemporary "philosophies" of logicism and formalism, thus showing that he never tried to found mathematics on anything beyond our experience of the material world.

Lo sviluppo dei numeri reali dai numeri naturali di Giuseppe Peano è qui tracciato, ed alcune sue vedute sulle definizioni matematiche sono presentate allo scopo di chiarire il suo concetto di numero. Esse dimostrano ch'egli adoperò il metodo assiomatico a fin di rendere la teoria della matematica più chiara, più precisa e più facile ad imparare. Esse rivelano, inoltre, alcune sue ragioni per non accettare le "filosofie" contemporanee del logicismo e del formalismo. Cosí è dimostrato ch'egli non ha mai cercato di fondare la sua teoria matematica su altro che la nostra esperienza del mondo materiale.

Развитие системы действительных чисел от натуральных уисел и несколько взглядов у Дж. Пеано представляются об определениях его в математике, чтобы выяснить его понятие о числе. Употреблением аксиоматического метода, Пеано хотел сделать математическую теорию яснее, точнее, и легче учиться. Обнаруживается также несколько причин зачем он не принимал современных "философий" логицизма и формализма, доказывая, таким образом, что он никогда не старался обосновать математику ни на чем кроме опыта у нас материального мира.

1. *Introduction*

In 1891, two years after the publication of his now famous postulates for the natural numbers, Giuseppe Peano published, in the journal founded by him that year, an article with the title "Sul concetto di numero" ["On the concept of number"] [Peano 1959, 80-109]. [1] In it he simplified his system by eliminating the undefined term symbolized by $=$, and the axioms relating to it. This system then consisted of three undefined terms: N (number), 1 (one), and $a+$ (the successor of a, where a is a number). The five axioms were [Peano 1958, 84]:

(1) $1 \; \varepsilon \; N$
(2) $+ \; \varepsilon \; N \backslash N$
(3) $a, b \; \varepsilon \; N \; . \; a+ = b+ \; : \supset . \; a = b$
(4) $1 - \varepsilon \; N+$
(5) $s \; \varepsilon \; K \; . \; 1 \; \varepsilon \; s \; . \; s+ \supset s \; : \supset . \; N \supset s$

These may be read [Peano 1958, 85] [2]:

(1) One is a number.
(2) The sign + placed after a number produces a number.
(3) If a and b are two numbers, and if their successors are equal, then they are also equal.
(4) One is not the successor of any number.
(5) If s is a class containing one, and if the class made up of the successors of s is contained in s, then every number is contained in the class s.

These postulates were not further reduced in number, although their form was changed, due to modifications in Peano's notation.

Because Peano was able to build all of arithmetic on the basis of this set of axioms, his work in this field has become justly famous. (Evidence of continued interest in Peano's ideas is shown by the recent translation of "Sul concetto di numero" into Japanese [Peano 1969].) It is therefore of interest to know just what his conception of number was at that time and whether it later changed. His ideas did indeed evolve in this regard, and even in 1891 he was not as explicit as one would like, so that it is difficult to say exactly what his views were at any one moment. In the present article two lines of development in his work will be traced -- his technical presentation of the real number system and his repeated discussion of definitions in mathematics -- in the hope of exhibiting some constant factors in his mathematical thought and philosophical commitment. Further, although Peano's derivation of the laws of arithmetic from his postulates for the natural numbers has been called "superb" [Birkhoff 1973, 769], I believe there still exist misconceptions about the manner in which the various types of numbers were introduced by him, misconceptions that may be cleared up by this presentation.

Peano first presented his postulates for the natural numbers in 1889 in *Arithmetices principia, nova methodo exposita* [Peano 1958, 20-55; English translation in Peano 1973, 101-134]. With the simplification of 1891 he was able to prove that the five postulates were mutually independent (of which he was "morally certain" in 1889, because of their substantial agreement with Dedekind's analysis in *Was sind und was sollen die Zahlen?* [Dedekind 1888], a work that Peano read just as his own was going to press). In 1891 Peano also constructed a more complete system than in 1889, for although he often claimed to have been the first, in *Arithmetices principia*, to develop a complete system in symbols (the *nova methodo* of the title), in fact negative numbers were not defined there, rather only positive integers, positive rationals, and positive reals -- and few operations were defined for these. In later developments, the various classes of numbers were defined in the order: natural numbers N, integers n, positive rationals R, rationals r, positive reals Q, reals q. Each of these will be treated separately.

2. *Development of the Real Number System*

2.1 *Natural numbers N*

In 1889 and 1891 the sequence of natural numbers began with 1, and the set of natural numbers was designated by N. This was modified in 1898 [Peano 1959, 216] so that the sequence began with 0, the set being designated by N_o. The set of five postulates for N, 1, $a + 1$ (or 0, N_o, $a+$) was increased to six in 1901 with the addition of: $N_o \ \varepsilon \ Cls$, i.e. the natural numbers form a class. With the addition of this last, the postulates have received their final form, as follows (where $a + 1$ is identified with $a+$) [Peano 1901, 41-43]:

(0) $N_o \ \varepsilon \ Cls$
(1) $0 \ \varepsilon \ N_o$
(2) $a \ \varepsilon \ N_o \ . \supset . \ a+ \ \varepsilon \ N_o$
(3) $s \ \varepsilon \ Cls \ . \ 0 \ \varepsilon \ s \ : x \ \varepsilon \ s \ . \supset_x . \ x+ \ \varepsilon \ s \ : \supset . \ N_o \supset s$
(4) $a, b \ \varepsilon \ N_o \ . \ a + 1 = b + 1 \ . \supset . \ a = b$
(5) $a \ \varepsilon \ N_o \ . \supset . \ a + 1 -= 0$

The change to N_o from N (or N_1 as it was now designated) brought other changes with it. For example, in 1891 (as already in 1889) addition of natural numbers was defined by the single equation: $a + (b + 1) = (a + b) + 1$. (This is to be understood, as Peano explained, in the sense that if a and b are numbers and the right hand member of the equation has meaning, but the left has to this point been undefined, then the expression on the left has the meaning of the expression on the right.) By the mathematical induction postulate, addition is then defined for all pairs of natural numbers. In 1898 the recursive definition of addition required two equations: $a + 0 = a$, $a + (b+) = (a + b)+$.

(This definition was early criticized by K. Grandjot and others, who believed that these equations should be taken as postulates. Peano has been ably defended against these criticisms by Ugo Cassina [1961, 291-298]. The remarks of later critics, such as J. van Heijenoort [1967, 83] have faulted this as part of Peano's overall lack of a deductive scheme. These criticisms are more to the point, although Peano's pioneer role in the use of recursive definitions should be emphasized.)

2.2 *Integers n*

With the natural numbers available, it is a relatively simple matter to define the integers and rational numbers in terms of ordered pairs. W.O. Quine, for example, writes [Quine 1963, 120]:

Another way would be to take 2/3 as the class of all pairs $\langle 2,3\rangle$, $\langle 4,6\rangle$, $\langle 6,9\rangle$, *etc.; this would mean defining x/y in general as*

$$\{\langle z,w\rangle : z,w \in N \,.\, x\cdot w = z\cdot y\}.$$

Such was Peano's version (1901).

But such was, in fact, *not* Peano's version. Neither for rationals nor for integers did he use ordered pairs. Rather, in each case these were defined as operations (or somewhat ambiguously as result of operations). In 1891, after defining $\alpha-$ as the operation inverse to α, he wrote [Peano 1959, 94]:

The combination of the sign of the inverse - and of the positive number b is what is called negative number. *Hence the sign -5 has the meaning "invert, and then repeat five times." Thus, if xp indicates "the father of x," the expression xp-2 means "the son of the son of x"; and if x is a [natural] number, x+-5 means "that which is obtained by carrying out the operation inverse to successor of x five times" or "the number preceding x by five places." We shall have no need to introduce any new symbol to indicate "negative number," since the notation -N is sufficient.*

This is then followed by the definition: $n = N \cup 0 \cup -N$. (In which, as Cassina has observed [Peano 1959, 94], the symbol 0 stands for $\iota 0$, i.e. the set composed of the single element 0. This lapse is surprising here, since Peano uses the symbol ι, introduced the previous year [Peano 1957, 130], later in this same article.)

In 1898 (the next time he treated negative numbers), Peano commented more explicitly on this way of introducing negative numbers, without, however, removing the ambiguities inherent in his definition. He wrote [Peano 1959, 227]:

P·7. 'Let a be a [natural] number; then -a is an operation that, applied to a number not less than a, produces a number.'

This is only different in form from P·2. It may be *compared with P015·4, which says that +a, i.e. the operation + repeated a times, is an* $N_0 \mathfrak{J} N_0$ *[i.e. a function that maps* N_0 *into* N_0*].*

The symbols $+N_0$ *and* $-N_0$ *(P024) correspond, more or less, to the words 'positive numbers' and 'negative numbers'; they appear as operations* $\mathfrak{J}$*. The symbol +a is equivalent to the expression 'add a' and -a means 'subtract a'. Following common usage, however, we call these operations 'whole numbers' (P030) and by definitions P025, 037, and 039·1, we have the formal coincidence of the numbers* N_0 *and the positive numbers* $+N_0$*.*

In an article commenting on this section of the *Formulaire*, he wrote [Peano 1959, 243]:

The so-called negative numbers and fractions are defined as the operations of subtraction and division. They thus go back to remotest antiquity, and well represent the use that we make of them; the resulting theory is very simple. They are considered this way in a number of texts, in a more or less clear manner.

If all this seems inadequate as *definition*, Peano does at least give a definition of the equality of two integers [Peano 1959, 227]:

P031. 'Two whole numbers x and y are equal, by definition, when for each positive number u, we have u + x = u + y, so long as these operations are possible on positive numbers.'

$$031.\quad x,y \,\varepsilon\, n \,.\, \supset \therefore x = y \,.=:\, u \,\varepsilon\, N_0 \,.\, u+x,\, u+y \,\varepsilon\, N_0 \,.\, \supset_u .\, u + x = u + y \qquad Df.$$

The above explanations were repeated the following year, with the additional comment [Peano 1899, 38]: "This manner of considering positive and negative numbers is found more or less clearly in several authors," and Peano quotes from Maclaurin and Cauchy. This is again repeated in the 1901 *Formulaire de Mathématiques* [Peano 1901, 49], but the explanation is greatly reduced in the last two editions of the *Formulaire* [Peano 1903, Peano 1908]. Integers continued, however, to be defined as operations, and this process was continued in the case of rational numbers.

2.3 *Positive Rationals R*

Peano defined positive rational numbers in 1889 as follows [Peano 1958, 46; Peano 1973, 126]:

$$R = \,::\, [x\,\varepsilon] \therefore p,q \,\varepsilon\, N,\ \frac{p}{q} = x \ :-=_{p,q} \Lambda$$

This may be read: *R* is the non-empty set of all *x* such that *p* and q are positive integers and $x = \frac{p}{q}$. But what is something of the form $\frac{p}{q}$? This has not been directly defined, but Peano did define the operation of the "ratio" $\frac{p}{q}$ on a positive integer as: $m,p,q \,\varepsilon\, N \,.\, \supset \,.\, m(\frac{p}{q}) = mp/q$. This must be understood in the sense that, if mp/q has meaning for some m,p,q the $m(\frac{p}{q})$ has the same meaning. But again, what is mp/q? This symbolism was previously defined by:

$$a,b \,\varepsilon\, N \,.\, \supset \,.\, b/a = N[x \,\varepsilon](xa = b).$$

This may be read: If a and b are positive integers, then b/a is that positive integer that when multiplied by a equals b. Thus $m(\frac{p}{q})$ is a positive integer.

This ambiguous state of affairs was clarified only slightly in "Sul concetto di numero" where, after defining p/q only if p is a multiple of q, Peano notes that if m and p are integers and q is a positive integer (or at least not 0), then $m \times (p/q) = (m \times p)/q$. Then he comments that if p is not a multiple of q, the expression $m \times (p/q)$ has no meaning. The expression $(m \times p)/q$ may, however, have meaning [i.e. if $m \times p$ is a multiple of q]. In this case he simply proposed to define $m \times (p/q)$ as $(m \times p)/q$. This still leaves open the question of whether he is thinking in this case of p/q merely as an ordered pair. I believe the most likely explanation is given in his next treatment of this subject, in 1898 [Peano 1898, 16] and repeated the following year [Peano 1899, 43], in which he gives the definition:

$$R = x \ni \exists\, (a;b) \ni [a,b \,\varepsilon\, N_1 \,.\, x = (\times b)(/a) \qquad Df.$$

His understanding of this is explained by the comment [Peano 1899, 42]:

> *According to ordinary language, b/a precedes the number, or magnitude, on which one operates, and means 'divide by a and multiply by b'. E.g., '3/5 of 15 francs' means 'that which is obtained by dividing 15 francs by 5 and multiplying the result by 3'. But we prefer to give to b/a, which follows the number on which one operates, the meaning 'multiply by b and divide by a', in order to make the operation possible in a larger number of cases.*

It seems clear from this that Peano was already thinking of a rational number as a (double) operation, and he confirmed this explicitly in 1901, in a footnote added to the publication of the paper he read at the International Congress of Philosophy in Paris (1900) [Peano 1958, 367]:

> *Now, the fraction a/b is introduced 'by abstraction' [by others]. They have no equality of the form:*
>
> *a/b = (expression composed of the preceding ideas),*
> *but they give only a nominal definition of the relation a/b = c/d.*

We prefer to consider a/b as representing the composite operation $\times a/b$, *i.e. 'multiply by a and divide by b'. The two operators a/b and c/d are equal whenever, applied to the same number (one that makes the two operations possible), they give equal results....*

One also finds this way of considering fractions as operators in Méray, Leçons sur l'Analyse Infinitésimale *(1894), p. 2, and already published by him in* Nouvelles Annales de Mathématiques *(1889), p. 421. This idea is the most natural.*

Peano did not, however, insist on being followed in this way of defining rationals. Indeed, with regard to those, just mentioned, who defined the equality $a/b = c/d$ and so introduced the rationals "by abstraction", he wrote in 1902 [Peano 1902, 64]: "The choice of one or the other of the possible definitions to be the actual definition depends on reasons of convenience."

2.4 *Rationals r*

Peano did not define the set of rational numbers in *Arithmetices principia* (1889), and in "Sul concetto di numero" (1891) there is only the abbreviated definition: $R = N/N$; $r = n/N$. This follows the introduction of positive rationals, and the first half of this is meant as a definition of the symbol R. However one understands R, then, r is to be understood similarly, replacing N in the "numberator" by n. This article contains no development of the rationals. This definition of r is repeated in all editions of the *Formulaire de Mathématiques* from 1898 on, except in the last edition (1908), where it is given as a "possible definition" and r is defined by: $r = +R \cup -R \cup \iota 0$ (which was given as a possible definition in 1901 and and 1903, and as *the* definition in the *Aritmetica generale e algebra elementare* of 1902.)

2.5 *Positive real numbers Q*

Peano defined the set of positive real numbers in *Arithmetices principia* by:

$$Q = [x\,\varepsilon](a\,\varepsilon\,KR : a -= \Lambda : R \ni > Ta \,.-= \Lambda : Ta = x \therefore -= \Lambda)\,.$$

This may be read: Q is the non-empty set of all x such that x is a least upper bound of a non-empty set of positive rational numbers and there are rational numbers larger than this least upper bound. Thus, a positive real number *is* a "least upper bound", which Peano had just introduced *without* a definition! He merely said [Peano 1958, 49; Peano 1973, 128]:

If $a\,\varepsilon\,KR$ *[i.e. a is a set of positive rational numbers], the symbol Ta is read* upper boundary *or* upper limit of the class *a [i.e., least upper bound]. We shall define only a few relations and operations on this new entity.*

[In fact, he defines only the relations $x < Ta$, $x = Ta$, $x > Ta$, for $x \in R$.]

This time the explanation of his views came only a decade later in 1899 in "Sui numeri irrazionali", in which he reviews the above definition, transcribing it into his current symbolism. He then discusses the method of introducing irrationals of no less than eleven authors, quoting the following passage of Dedekind with especial approval [Peano 1959, 264]:

> *Jedesmal nun, wenn ein Schnitt (A_1, A_2) vorliegt, welcher durch keine rationale Zahl hervorgebracht wird, so erschaffen wir eine neue, eine irrationale Zahl α, welche wir als durch diesen Schnitt (A_1, A_2) vollständig definiert ansehen.*

Peano notes: "In this 'erschaffen' (create) is precisely indicated that the real number is considered as an entity different from section, or segment."

Near the end of the article, Peano gives four statements, which include:

(a) Every class of (rational) numbers, all less than a given number, has a real number as least upper bound.

(c) Every sequence (of rationals) that satisfies the criterion of convergence, effectively converges toward a limit.

He concludes the article by commenting on these statements [Peano 1959, 267]:

> *These propositions, all true, even if one suppresses the condition 'of rationals' written between parentheses, are easily deduced one from the other. One of them may be transformed into the definition of irrationals. Our authors' opinions vary on this choice.*
>
> *In the applications, especially to analysis, proposition (a) is used more often than (c). Indeed, in most texts, to recognize the convergence of series, or of integrals, etc., almost constant use is made of (c). Hence the opinion of Cantor [*Mathematische Annalen *21 (1883), 567], is justified, that his method is 'die einfachste und natürlichste von allen' and speaking of that of Dedekind, he says that 'die Zahlen in der Analysis* niemals *in der Form von "Schnitten" darbieten, in welche sie erst mit grosser Kunst und Umständlichkeit gebracht werden müssen.'*
>
> *But in several of my works I have shown that the definition of integral depends on the sole concept of least upper bound, and not on that of the limit toward which a function converges. Similarly, in geometrical applications we define, for example, the length of an arc of a curve as the least upper bound of the lengths of the inscribed*

> *polygons, rather than as the limit toward which their lengths converge (see* Enzyclopädie der Mathematischen Wissenschaften, *p. 72). Thus the most appropriate form of the principle and the one that is applied continually in analysis appears to me to be (a), to which the definition of the irrational can therefore be directly attached.*

2.6 *Real numbers q*

Negative real numbers were not defined by Peano in *Arithmetices principia* (1889). In "Sul concetto di numero (1891) the set of real numbers q was defined directly, without first defining Q, as [Peano 1959, 108]:

$$q = (l'Kr)(-\iota\,\infty)(-\iota\,-\infty)$$

"Real numbers are the least upper bounds of the classes of rational numbers, excluding $+\infty$ and $-\infty$." This definition arises as an alternative to Dedekind's "cut," which Peano had just analysed, since, he says:

> *...instead of considering both classes A_1 and A_2, it is sufficient, as Dedekind has already observed, to consider only the first, A_1, seeing that the other, A_2, is the set of rational numbers not in A_1.*

This least upper bound; $l'A$ where A is a set of rational numbers, is to be distinguished from the set A. In this regard Peano criticized Pasch for confusing the two, remarking [Peano 1959, 107]:

> *Hence, to remove this difficulty it is necessary to make correspond to every segment A a new entity, that I shall indicate by $l'A$ (least upper bound of A); and to indicate the relation in question one writes $l'A = a$. Real numbers are therefore least upper bounds of segments.*

The difficulty that Peano is trying to avoid occurs, of course, in the case where the least upper bound of a set A is rational. Then the identification of the set A with its real (in this case rational) least upper bound a requires $a = A$, whereas the appropriate relation is $a \varepsilon A$.

Peano repeated the above definition of reals in 1899, but that same year in the *Formulaire de Mathématiques*, after having introduced the positive reals Q, he gave the definition: $q = Q \cup -Q \cup \iota\, 0$, and this definition was repeated in all later editions of the *Formulaire*.

3. *Mathematical Definitions*

We have just surveyed Peano's development of the real number system, starting from the natural numbers. Let us now return to the natural numbers to ask: To what extent did Peano believe the postulates *defined* the natural numbers? Peano himself raised the question of the possibility of defining the natural numbers in "Sul concetto di numero." After stating the five "primitive propositions" and some "immediate consequences," he observed [Peano 1959, 84-85]:

> *The first numbers presented, with which we form all the others, are the positive integers. And the first question is : Can we define one, number, sum of two numbers? The common, Euclidean, definition of number, 'number is the collection of serveral units', may serve as a clarification, but is not satisfactory as a definition. Indeed, very young children use the words* one, two, three, *etc. They later adopt the word* number, *and only much later does the word* collection *appear in their vocabulary. Indeed, philology teaches that these words appear in this same order in the development of the Indo-European languages. Hence, from the practical side, the question appears to me to be settled, or rather, there is no need for the teacher to give any definition of number, seeing that this idea is very clear to the pupils, and any definition would only have the effect of confusing them. The majority of authors also share this opinion.*
>
> *From the theoretical side, to decide the question of the definition of number, one should be told first what ideas he may use. Here we suppose known only the ideas represented by the signs* $\cap$ *(and),* $\cup$ *(or),* $-$ *(not),* ε *(is), etc., which have been treated in the preceding note. Therefore,* number cannot be defined, *since it is evident that however these words are combined among themselves, we can never have an expression equivalent to* number. *If number cannot be defined, however, we can still state those properties from which the many other well known properties of the numbers are derived.*
>
> *The concepts, then, that we do not define are those of* number N, *of* one 1, *and of* successor of a number a, *which we indicate for the moment by* $a+$. *These concepts may not be obtained by deduction; it is necessary to obtain them by induction (abstraction). The successor of* a *is here indicated by* $a+$, *instead of the customary* $a + 1$, *and this is done so as to indicate by a single sign,* $+$, *the fundamental operation 'successor of'. Besides, in the following sections, having defined the sum* $a + b$ *of two numbers, we shall see that* $a + 1$ *has precisely the value of* $a+$, *i.e. the successor of* a, *and thus we return to the customary notation.*

A bit further on he adds [Peano 1959, 88]:

> *Between the preceding and what Dedekind says there is an apparent contradiction that should immediately be*

pointed out. Here, number is not defined, but its principal properties are stated. Instead, Dedekind defines number as precisely that which satisfies the preceding conditions. Evidently, the two coincide.

With regard to this quotation, two remarks may be made. First, we should notice that Peano answers his questions from two viewpoints: didactic and theoretical. His "theoretical" view is of more interest here, but we must always realize that it was colored by his views on teaching, and seldom did he make the distinction so clearly as here. Indeed, in his later years he discussed the didactic question more often. Second, Peano's notion of definition in mathematics varied a great deal, and his answers must be interpreted in the light of his acceptance of the various types of definitions at the time. In 1891 his view was rather narrow. It is clear that he is asking for a nominal definition of the form

$$x = (\textit{expression composed of preceding symbols}).$$

This he has not given for the natural numbers, and he asserts its relative impossibility. Yet, his acceptance of Dedekind's use of the term "definition" ("Evidently the two coincide") is probably more than just conciliatory, for he shortly came closer to what he here described as Dedekind's position.

But what did Peano mean when he said that an idea was obtained "by abstraction"? He next discussed this in 1894 [Peano 1958, 167-168]:

There are some ideas obtained by abstraction and with which the mathematical sciences are constantly being enriched that cannot be defined in the form stated. Let u be an object; by abstraction, one deduces a new object ϕu. We cannot form an equality

$$\phi u = \textit{known expression},$$

for ϕu is an object of a nature different from all those that we have considered up to the present. Rather, we define the equality $\phi u = \phi v$ by setting

$$h_{u,v} \;.\; \supset \;:\; \phi u = \phi v \;.\; = \;.\; p_{u,v} \qquad \textit{Def.}$$

where $h_{u,v}$ is the hypothesis on the objects u and v. Thus $\phi u = \phi v$ means the same as $p_{u,v}$, which is a condition, or relation, between u and v, having a previously known meaning. This relation must satisfy the three conditions of equality that follow: [Here Peano describes the reflexive, symmetric, and transitive properties of an equivalence relation.]...

The object indicated by ϕu is therefore what one obtains by considering in u all and only those properties that it has in common with the other objects v such that $\phi u = \phi v$.

Peano gives as an example of this Euclid's definition of ratio of two magnitudes in Book V of the *Elements*. He is still unwilling to admit that objects are truly *defined* "by abstraction," but he moved a bit closer to this view in his statement near the end of this article [Peano 1958, 175]:

> *Whatever the manner of reasoning, if a science does not contain primitive ideas, as happens in every advanced theory, one can define and prove everything in it. But if the science touches its very elements, and if there are ideas that cannot be defined, one will also find propositions that cannot be proved, and from which all the others follow. We shall call these* primitive *propositions, abbreviated by Pp; they are also called axioms, postulates, and sometimes hypotheses, experimental laws, etc. These propositions determine or, if you like, define the primitive ideas that have not been given a direct definition.*

In 1897 Peano was still insisting on this form of definition [Peano 1958, 208]:

> *By symbolic definition of a new symbol x we understand the convention of calling x a group of symbols already having a known meaning; and we indicate this by*
>
> $x = a$ *Def.*

In 1898, with regard to the postulates for the natural numbers, he wrote [Peano 1959, 217]:

> *Les idées primitives sont déterminees par les 5 propositions primitives 002 desquelles découlent toutes les P[ropositions] de l'Arithmêtique.*

Peano meant that the postulates "determine" the natural numbers to the extent that the theorems of arithmetic follow. He was aware from the beginning that they do not uniquely characterize the natural numbers, as he explicitly pointed out in "Sul concetto di numero" [Peano 1959, 87]:

> *These propositions express the necessary and sufficient conditions that the entities of a system can be put into one-to-one correspondence with the series of natural numbers.*

In 1898, again with regard to the postulates for the natural numbers, he wrote [Peano 1959, 243]:

> *The analysis of the ideas of arithmetic contained in F_2§2 [i.e. the* Formulaire *of 1898] is the only one in existence today. To this group of propositions 002·1-5, which may be called a definition of the positive integers, using the word definition in a wider sense than that given in F_2§1 P7 [i.e. in the* Formulaire *of 1897, as quoted above], the nearest work is that of Dedekind of 1888....*

By 1899 Peano was willing to admit [Peano 1959, 261]: "The word 'definition', even books of mathematics, has several meanings." He went on to say:

> *Prof. C. Burali-Forti, in his text* Logica Matematica, *Milan, 1898, pp. 120-148, has classified the definitions that are met in the theories already expressed in ideographic symbols, distinguishing them by the names : nominal definition, definition by induction, by abstraction, etc.* [The "etc." seems superfluous, since Burali-Forti only discusses the three types mentioned.] *Of these various types of definitions, the nominal appears to be the most satisfactory. Many definitions of the other types contained in the early works of mathematical logic could be transformed into nominal definitions. Of definitions by abstraction, in F_2N2 (Arithmetic)* [i.e. the *Formulaire* of 1898] *use is made only once, in P210·1, to define the cardinal number, or power, of a set.*

Since Peano says that he used definition by abstraction only once in the *Formulaire* of 1898, we must understand that he is using the term "definition by abstraction" as a technical term, since the expression "by abstraction" is used in that book in referring to the postulates for the natural numbers, presumably used in a wider sense. After noting that there are an infinity of systems that satisfy the postulates, he wrote [Peano 1959, 218]:

> *Tous les systèmes qui satisfont aux 5 Pp sont en correspondence réciproque avec les nombres. Le nombre, N_0, est ce qu'on obtient par abstraction de tous ces systèmes; autrement dit, le nombre N_0 est le système qui a toutes et seules les propriêtés ênoncées par les 5 P primitives.*

This sentence, slightly altered, was repeated in the *Formulaire* of 1899, of 1901, and of 1903, but was omitted from the final edition of 1908.

By 1900 the problem of what constitutes a definition in mathematics had become of greater interest to Peano, and he made this the subject of his talk at the International Congress of Philosophy in Paris that summer. There he said [Peano 1958, 362]:

> *Une définition est réductible à une égalité, dont un membre (le premier) est le nom qu'on définit, et l'autre en exprime la valeur. Example :*
> *(dérivé d'une fonction) = (limite du rapport des accroissement de la fonction et de la variable.)*
> *En conséquence, un proposition qui n'est pas une égalité ne pourra pas être une définition.*

The problem posed by Peano does not concern the *form* that a definition must have -- this he consistently affirms must be an equality -- but rather is the question of which equalities are (or could be) taken as definitions. Of special interest, because of its immediate influence, is the example $0 = a - a$, as an equality that may not be taken as a definition of zero. First of all, according to Peano, it is incomplete, since we are not told what value to give to the letter a. Second, even if we say: "Let a be a number; then $0 = a - a$," the equality is still not homogeneous, for the first member is the constant symbol 0, while the second is a function of the variable letter a. Peano then shows how to properly phrase the definition [Peano 1959, 366]:

> *La proposition:*
>
> *0 = (la valeur constante de l'expression a - a, quel que soit le nombre a)*
>
> *est une égalité homogène, car, bien que dans le second membre figure la lettre a, elle n'y figure qu'en apparence, puisque la valeur de ce second membre n'est pas une fonction de a. Cette proposition est une définition possible.*

After the talk, Schröder objected that Peano's condition of homogeneity was too restrictive, but Peano (backed up by his disciple Alessandro Padoa) defended it. Ivor Grattan-Guinness has discovered from manuscript sources (to be described in detail in an article in preparation by him) that is was their discussion on this topic that convinced Bertrand Russell of Peano's superiority over Schröder.

In 1901, as part of a projected dictionary of mathematics, Peano wrote a brief dictionary of mathematical logic. There he again insisted on the "equality" form of a definition, but we find under the heading "abstraction" the following [Peano 1958, 373]:

> *In mathematical logic, what is called "definition by abstraction" is the definition of a function ϕx, having the form:*
>
> $\phi x = \phi y$. = . *(expression composed of the preceding symbols),*
>
> *that is, the isolated symbol ϕx is not defined, but only the equality $\phi x = \phi y$.*

At this time, although Peano recognized definitions by abstraction as valid, he felt that their use should be avoided where possible. This is shown in a letter of May 1902 to L. Gérard, editor of the *Bulletin des sciences mathématiques et physiques*, and published in that journal the following month. After recalling that Burali-Forti had classified the definitions used in mathematics and had called "definition of the first type" those of the form [Peano 1959, 371]:

> $x = a$ *Def.*
>
> *où x est le signe simple qu'on définit, et a est un*

groupement des signes connus. Le Signe = accompagné de Df signifie « est égal par définition » ou « nous nommons »,

he gives several examples of definitions by abstraction, concluding with [Peano 1959, 372]:

Plusieurs analystes introduisent les nombres rationnels par abstraction, en posant

$a,b,c,d \,\varepsilon$ (nombres naturels) . $\supset$: $a/b = c/d$. = . $ad = bc$, mais cela n'est pas nécessaire, car on peut en donner une Df de première espèce.

D'une façon analogue, notre Df 1 est une Df par abstraction, car elle définit une égalité.

Or, il est bon de remarquer qu'on peut définir les nombres imaginaires par des Df de première espèce, et qu'il n'est pas nécessaire de recourir à des Df par abstraction.

Peano continued to be interested in the question of definitions in mathematics (and in the last year of his life assigned this as a thesis topic to the young Ludovico Geymonat, now Professor of the Philosophy of Science at the University of Milan, Italy), but his next publication on this topic was not until 1911 and by that time he had ceased to be the innovator in this field, and although he was still capable of incisive comments, he was usually content to describe the views of others. This resulted from two major events, both dating from 1903. One was the introduction of *Latino sine flexione* as an international auxiliary language. The following years saw his almost complete dedication to the international auxiliary language movement, confirmed by his election in 1908 to the directorship of the Academia pro Interlingua, a position he held until his death in 1932.

The second event of 1903 that influenced Peano was the publication of Bertrand Russell's *Principles of Mathematics*, which, as Peano wrote to Russell [Kennedy 1974, 31], "marked an epoch in the field of philosophy of mathematics." He had already written Russell in 1901 [Kennedy 1974?]:

Permettez-moi de me féliciter avec vous de la facilité et de la précision avec lequelles vous maniez les symboles de Logique.

Now he was content to let Russell take over leadership in this field, and in 1910 could only write him [Kennedy 1974?]:

Je vous remercie du livre Principia Mathematica, *que je me propose de lire avec attention. Maintenant, mes heures libres de l'école sont occupées dans la question de l'Interlingua,....*

Peano did not, however, accept everything that Russell was doing. In particular, he rejected Russell's "class of classes" definition of cardinal number, published by Russell in Peano's journal in July 1901 [Russell 1901, 121]. Curiously, Peano's rejection was published several months earlier in *Formulaire de mathématiques*, vol. 3, dated 1 January 1901. There Peano defined the cardinal number of a set *a*, symbolized by *Num a*, "by abstraction," adding [Peano 1901, 70]: "mais on ne peut pas identifier Num a avec la Cls de Cls considérée, car ces objets ont des propriétés différentes." Although this publication preceded that of Russell, Peano would have had Russell's manuscript since October 1900 and so was probably prompted by it to consider the "class of classes" definition. At any rate, Russell did not alter his manuscript, but commented in *Principles of Mathematics* [Russel 1903, section 111]: "He does not tell us what these properties are, and for my part I am unable to discover them." Significantly, however, Peano does identify the finite cardinal numbers with the natural numbers. Indeed, he had already done this in "Sul concetto di numero" where, although Peano was aware that the postulates do not characterize the concept of natural numbers, nevertheless, as Ugo Cassina has noted [letter to the author dated 4 June 1961], "the way in which he attacks the problem of numeration [in Peano 1959, 100], i.e., his inductive definition of 'number of objects of a [finite] class' results in giving to the primitive entities 0, N_0, + (or 'suc') the intuitive meaning."

What, then, was Peano's objection to the "class of classes" definition? Most probably it was the artificiality of this concept as opposed to what Peano saw as the "natural" concept of number. Consider, for example, the passage of 1906 [Peano 1957, 343]:

> *That is, we have deduced theorems identical to the postulates of arithmetic. Therefore, for the sumbols of arithmetic 0, N_0, +, there exists an interpretation that satisfies the system of postulates. Thus it has been proved (if proof were necessary), that the postulates of arithmetic, which the collaborators of the* Formulaire *have shown to be necessary and sufficient, do not involve a self-contradiction.*
>
> *Other examples of entities that satisfy the system of postulates have been given by Burali-Forti and by Russell. But a proof that a system of postulates of arithmetic, or of geometry, does not involve a self-contradiction is not, I think, necessary. For we do not create postulates at will, but we assume as postulates the simplest statements that, either written in an explicit way or implicitly, are in every treatise of arithmetic or of geometry. Our analysis of the principles of these sciences is the reduction of the ordinary statements to a necessary and sufficient minimum. Systems of postulates of arithmetic and of geometry are satisfied by the ideas of number and point that every writer of arithmetic and geometry has.*

With the publication of Whitehead and Russell's *Principia Mathematica*, Peano seems willing to concede to them the "class of classes" definition of number. He wrote in 1913 in his review of Vol. 1 [Peano 1958, 397-398]:

> *Page 363 begins the treatment of cardinal numbers. The authors eliminate definitions by abstraction. In many cases mathematicians introduce a new entity ϕx, not by a definition of the form*
>
> ϕx = *expression composed of x and known symbols,*
>
> *but they define only the equality :*
>
> $\phi x = \phi y$. = . *relation $p_{x,y}$ composed of x,y, and known elements*
>
> *The authors prove that definitions by abstraction can be reduced to nominal definitions; it suffices to set*
>
> $x = y \ni (p_{x,y})$.

Peano's acceptance of all this, however, is not complete, and in 1915 he again definds definitions by abstraction in an article with the title "Le definizioni per astrazione" [Peano 1958, 402-416]. Perhaps he felt, in some way, the necessity of defending what was his, for although the recognition of the importance of the procedure that leads to it goes back at least to H. Grassmann's *Ausdehnungslehre* of 1844, nevertheless it was Peano who introduced the name "definition by abstraction" in 1894 [Peano 1958, 167]. But his objections are on practical grounds; he seems to accept the logical validity of replacing definitions by abstraction by nominal definitions. In this same article there is an interesting comment apropos definitions by abstraction and the cardinal numbers (of Cantor). After noting that one says the cardinal numbers of two sets are the same if the sets can be put into a one-to-one correspondence, he concludes [Peano 1958, 404]:

> *One thus defines the equality of two numbers, and not number itself; and this because this definition may be placed before arithmetic, and also because the number that results is not the finite number of arithmetic.*

It it not clear what distinction he intends here, since in the *Formulaire* he identified the finite cardinals with the natural numbers -- but perhaps he only means that some cardinals are infinite. Peano next describes Padoa's theory of equivalence relations (relations that are reflexive, symmetric, and transitive -- Padoa called them "egualiforme") and concludes [Peano 1958, 413]; "Between the two theories, definition by abstraction and equivalence, I see only a difference of language."

Finally, Peano comments on Russell's practice of replacing definitions by abstraction by nominal definitions. But by 1915 he does not want to oppose the authority of Russell, and he definds his neutrality. He wrote [Peano 1958, 414]:

The question of abstraction pertains to pure logic, and we can give non-mathematical examples. Are the following equations true or not?

whiteness = white things,
sickness = sick people,
youth = young people,
Italy = the Italians,
justice = judges, police, jail.

The theory of Russell answers in the affirmative. I, investing myself with the authority of Euclid (like a live ass covered with the hide of a dead lion), neither affirm nor deny. This identity is denied by the doctor who says 'there are no sicknesses, but only sick people', as well as by the opposite theory that says 'I conquered the sickness and killed the sick person'.

The question of definitions in mathematics continued to interest Peano, and he returned to this topic in 1921 in an article discussing the various types of definitions. He even included a discussion of "primitive ideas" and their use in setting up an axiom system [Peano 1958, 433; 1973, 244]:

The fundamental properties of the primitive ideas are determined by the "primitive propositions", or the propositions that are not proven, and from which are deduced all the other properties of the entities considered. The primitive propositions function in a certain fashion as definitions of the primitive ideas.

With this last remark, we apparently have Peano's complete acceptance of the Postulates for the Natural Numbers as a definition, something he appeared reluctant to accept in 1891. Having in the meantime searched for another definition, and not finding it, he ends by adopting a naive axiomatic viewpoint, while still equivocating with the restriction "in a certain fashion."

4. *Conclusion*

Peano made no pretense of being a philosopher and, indeed, denied competence in this field. Nevertheless several things stand out in his development of, and comment on, the real number system, as it has been traced above. Perhaps the most important aspect of his work in the foundations of mathematics is that he made no attempt to *found* mathematics on any prior discipline. In particular, he denied the validity of Russell's reduction of mathematics to pure logic. Peano was concerned with reducing arithmetic, say, to the minimum number of undefined terms and axioms, from which all the rest could be defined and proved. For him the goal of the axiomatic method was to make the theory clearer, more precise, and *easier to learn*.

Peano saw mathematical logic as contributing to this goal. Already in 1894 he wrote Felix Klein [Peano 1894]:

> *Mathematical logic, with a very limited number of signs (actually seven, but further reducible), has succeeded in expressing all the logical relations imaginable among classes and propositions; or rather, the analysis of these relations has led to the use of these signs, with which everything can be expressed, even the most complicated relations that are expressed in ordinary language with fatigue and difficulty. But the advantage is not limited to simplifying the writing; its usefulness lies especially in the analysis of the ideas and reasoning that make up mathematics.*

As an example of this usefulness, he continues:

> *I could cite many so-called theories that* evaporate *when translated into symbols; they exist only in appearance, by exchanging a new name for an old idea. I limit myself to mentioning that several parts of the theory of fields and moduli of Dedekind are merely propositions of logic, and as such are included in Part I of the* Formulaire.

It seems clear, too, that Peano was not a formalist, in the sense of Hilbert. This point has already been emphasized by his ardent disciple Ugo Cassina (1894-1964) [see Cassina 1961]. Mathematics, then, has content and is not merely a game. In 1923 Peano wrote [Peano 1923, 383]:

> *Mathematics has a place between logic and the experimental sciences. It is pure logic; all its propositions are of the form: 'If one supposes A, then B is true.' But these logical constructions must not be made for the mere pleasure of reasoning about them. The object studied by them is given by the experimental sciences; they must have a practical goal.*

Thus Peano believed that mathematics can, and must, be based on experience, as we have already seen above in his comment on consistency proofs of an axiom system.

Many of Peano's views on the concept of number were echoed in 1939, in an article that has recently been reprinted, by A.Ya. Khinchin (Александр Яковлевич Хинчин 1894-1959). For example [Khinchin 1968, 3]:

> *The concept of number is distinguished from many other concepts of the school course by its* primarity. *This means that in the overwhelming majority of ways in which mathematics can be developed as a logical system, the idea of number belongs to the set of those concepts which are not defined in terms of other concepts, but together with*

the axioms enter into the ranks of the initial data. It means that mathematics does not contain within itself an answer to the question 'what is a number?' -- an answer, that is, which would consist of a definition of this concept in terms of concepts that had been introduced at an earlier stage; mathematics gives this answer in a different form, by listing the properties of a number as axioms.

Further, I believe that Peano, with his great concern for the teaching of mathematics, would have approved Khinchin's advice to the teacher whose pupil asked the question 'what is a number?' [Khinchin 1968, 4]:

Tell him that the question he has posed is one of the most difficult questions in the philosophy of science, one to which we are still far from having a complete answer; that a number, in the same way as any mathematical concept, is the reflection in our own consciousness of certain relations in the real world; but that the question, precisely which relations of the world find their reflection in the concept of number, which relations are quantitative, *is a deep and difficult philosophical problem; mathematics itself can only point out to those who study it what types of numbers there are, what are their properties, and how they can and should be manipulated.*

That Peano himself recognized the seriousness of this problem and was not misled by the facile "philosophical" solutions current in his lifetime (e.g. logicism and formalism) is confirmed by the conclusion of L. Geymonat with regard to the difficulty in determining Peano's views. He wrote [Geymonat 1955, 62-63]:

If we must recognize this hesitation of Peano, it would nevertheless be superficial to attribute it to his incapacity to detect the philosophical problem underlying the mathematical one. It seems to me more exact to recognize, on the contrary, the critical depth *of his position, inspired by a somewhat excessive caution. The slightly, but constantly, ironic tone with which he talked about philosophical discussions -- was it not perhaps the trench behind which he wished to defend himself from the temptations of the rash theories of the 'philosophers'?*

NOTES

1. Since for most readers the volumes of the *Opere scelte* [Peano 1957, 1958, 1959] will be the most available source of Peano's writings, I have given page references to them wherever possible.
2. I have translated into English all quotations that were originally in Italian.

REFERENCES

Birkhoff, Garrett 1973 "Current trends in algebra," *American Mathematical Monthly* 80, 760-782.

Cassina, Ugo 1961 *Critica dei principî della matematica e questioni di logica.* Cremonese, Rome.

Dedekind, Richard 1888 *Was sind und was sollen die Zahlen?* Braunschweig.

Geymonat, Ludovico 1955 "I fondamenti dell'aritmetica secondo Peano e le obiezioni 'filosofiche' di B. Russell," in *In Memoria di Giuseppe Peano*, ed. A. Terracini, Cuneo.

Heijenoort, Jean van 1967 *From Frege to Gödel: a Source Book in Mathematical Logic, 1879-1931.* Ed Jean van Heijenoort. Harvard University Press, Cambridge, Mass.

Kennedy, Hubert C. 1974 *Giuseppe Peano.* German translation by Ruth Amsler. Birkhäuser Verlag, Basel.

——— 1974? "Nine letters from Giuseppe Peano to Bertrand Russell," *Journal of the History of Philosophy* [forthcoming].

Khinchin, A. Ya. 1968 *The Teaching of Mathematics.* Essays by A. Ya. Khinchin, edited by B.V. Gnedenko, translated from the Russian by W. Cochrane and D. Vere-Jones. American Elsevier Publishing Company, New York.

Peano, Giuseppe 1894 [Unpublished] letter to Felix Klein, dated 29 August 1894, in the Niedersächsische Staats- und Universitätsbibliothek Göttingen Handschriftenabteilung.

——— 1898 *Formulaire de mathématiques.* Vol. 2, section 2. Bocca, Turin.

——— 1899 *Formulaire de mathématiques.* Vol. 2, section 3. Bocca, Turin.

——— 1901 *Formulaire de mathématiques.* Vol. 3. Carré & Naud, Paris.

——— 1902 *Aritmetica generale e algebra elementare.* Paravia, Turin.

——— 1903 *Formulaire mathématique.* Vol. 4. Bocca, Turin.

——— 1908 *Formulario mathematico.* Vol. 5. Bocca, Turin. Facsimile reprint, ed. U. Cassina, Cremonese, Rome, 1960.

——— 1923 Review of A. Nantucci, *Il concetto di numero, e le sue estensioni, Archeion. Archivio di storia della scienza* 4, 382-383.

——— 1957 *Opere scelte di Giuseppe Peano.* Vol. 1. Ed. U. Cassina. Cremonese, Rome.

——— 1958 *Opere scelte di Giuseppe Peano.* Vol. 2. Ed. U. Cassina. Cremonese, Rome.

——— 1959 *Opere scelte di Giuseppe Peano.* Vol. 3. Ed. U. Cassina. Cremonese, Rome.

——— 1969 *Su no Gainen ni Tsuite.* Trans. Toshio Umezawa. Kyoritsu Publishing Company, Japan.

——— 1973 *Selected Works of Giuseppe Peano.* Ed. and trans. by Hubert C. Kennedy. University of Toronto Press.

Quine, Willard van Orman 1963 *Set Theory and Its Logic.* Harvard University Press, Cambridge, Mass.

Russell, Bertrand 1901 "Sur la logique des relations avec des applications à la théorie des séries," *Rivista di matematica* 7, 115-148.

——— 1903 *The Principles of Mathematics.* Vol. 1. Cambridge University. 2nd ed. (1938) reprinted by Norton, New York, 1964.

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