Linear maps on $M_n(\mathbb{C})$ preserving the local spectrum

Manuel González a, Mostafa Mbekhta b, *

a Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, E-39071 Santander, Spain
b Université de Lille I, UFR de Mathématiques, 59655 Villeneuve d’Ascq Cedex, France

Received 24 April 2007; accepted 5 July 2007
Available online 22 August 2007
Submitted by P. Šemrl

Abstract

Let $x_0 \in \mathbb{C}^n$ be a nonzero vector. We prove that if a linear map $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ preserves the local spectrum at $x_0$; i.e., $\sigma_T(x_0) = \sigma_{\varphi(T)}(x_0)$ for all $T \in M_n(\mathbb{C})$, then there exists an invertible matrix $A$ such that $A(x_0) = x_0$ and $\varphi(T) = A T A^{-1}$ for every $T$.

© 2007 Elsevier Inc. All rights reserved.

AMS classification: Primary 47B49; Secondary 47A11

Keywords: Linear maps preserving the local spectrum

1. Introduction

In 1897, Frobenius [8] showed that a linear mapping $\varphi$ from $M_n(\mathbb{C})$ into $M_n(\mathbb{C})$ which preserves the determinant is the composition of an automorphism or an anti-automorphism with a left multiplication by a matrix of determinant 1. From this result, it is not difficult to derive that if $\varphi$ preserves the spectrum ($\sigma(\varphi(x)) = \sigma(x)$ for all $x \in M_n(\mathbb{C})$), then it is an automorphism or an anti-automorphism; i.e., it has one of the forms $\varphi(x) = axa^{-1}$ or $\varphi(x) = a^T x a^{-1}$, for some invertible matrix $a$. Dieudonné [7] and Marcus and Purves [14] obtained the same representation, assuming $\varphi$ onto and $\sigma(\varphi(x)) \subset \sigma(x)$ for all $x$.

Over the last years there has been a considerable interest in the so called linear preserver problems (see the survey papers [3,6,12,13,15]). The goal is to describe the general form of linear

* The research was partially supported by DGI (Spain). Proyecto MTM2007-67994.
* Corresponding author. Tel.: +33 3 2043 4238; fax: +33 3 2043 6758.
E-mail addresses: gonzalem@unican.es (M. González), mostafa.mbekhta@math.univ-lille1.fr (M. Mbekhta).

0024-3795/$ - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2007.07.005
maps between two Banach algebras that preserve a certain property, or a certain class of elements, or a certain relation. One of the most famous problems in this direction is Kaplansky’s problem [10] asking whether every surjective unital invertibility preserving linear map between two semi-simple Banach algebras is a Jordan homomorphism. In the commutative case the well-known Gleason–Kahane–Zelazko theorem provides the affirmative answer. In the non-commutative case the best known results so far are due to Aupetit and Sourour. They showed that the answer to the Kaplansky question is in the affirmative for von Neumann algebras [4] and for bijective unital linear invertibility preserving maps acting on the algebra of all bounded operators on a Banach space [17].

Recently, Bourhim and Ransford [5] studied additive maps on \( L(X) \), the bounded operators acting on a complex Banach space \( X \), preserving the local spectrum \( \sigma_T(x) \) of \( T \) at \( x \in X \). They showed that if \( \varphi : L(X) \to L(X) \) is an additive map such that \( \sigma_{\varphi(T)}(x) = \sigma_T(x) \) for all \( T \) and all \( x \), then \( \varphi(T) = T \) for all \( T \).

In this paper, we consider a more general situation where a linear map preserves the local spectrum at only a fixed nonzero vector.

**Problem.** Let \( x_0 \in X \) be a fixed nonzero vector. Characterize those linear maps \( \varphi \) on \( L(X) \) which preserve the local spectrum at \( x_0 \) (i.e \( \sigma_{\varphi(T)}(x_0) = \sigma_T(x_0) \) for all \( T \)).

In this paper we answer the problem in the case of a finite dimensional space \( X \). Note that, when \( \dim X = n \), the space of operators \( L(X) \) can be identified with the space of \( n \times n \) square matrices \( M_n(\mathbb{C}) \). We show that if \( \varphi : L(X) \to L(X) \) is a linear map, then \( \sigma_{\varphi(T)}(x_0) = \sigma_T(x_0) \) for all \( T \in L(X) \) if and only if \( \varphi(T) = A T A^{-1} \) for all \( T \), with \( A \) an invertible operator on \( X \) satisfying \( A(x_0) = x_0 \).

Throughout this paper, \( X \) is a complex Banach space and \( L(X) \) denotes the algebra of all bounded operators acting on \( X \). Given \( T \in L(X) \), we denote by \( N(T) \) and \( R(T) \) the kernel and the range of \( T \).

### 2. Preliminaries

Here we describe the basic concepts of local spectral theory and give some results that are relevant for our paper.

Let \( T \in L(X) \) and let \( x \in X \). A complex number \( \lambda \in \mathbb{C} \) belongs to the local resolvent set of \( T \) at \( x \), denoted \( \lambda \in \rho_T(x) \), if there exists an open neighborhood \( U \) of \( \lambda \) and an analytic function \( \hat{x}_T : U \to X \) such that \( (T - \mu)\hat{x}_T(\mu) = x \) for all \( \mu \in U \). The local spectrum of \( T \) at \( x \) is \( \sigma_T(x) := \mathbb{C} \setminus \rho_T(x) \).

Let us state some basic properties of the local spectrum for future reference.

**Proposition 1** [1,11]. Let \( T \in L(X) \).

1. \( \sigma_T([0]) = \emptyset \);
2. \( \sigma_T(x) \subseteq \sigma_{\text{sur}}(T) := \{ \lambda \in \mathbb{C} : R(T - \lambda) \neq X \}, \text{ for all } x \in X \);
3. there exists \( x \in X \) such that \( \sigma_T(x) = \sigma_{\text{sur}}(T) \);
4. if \( S \in L(X) \) commutes with \( T \), then \( \sigma_T(Sx) \subseteq \sigma_T(x) \), for all \( x \in X \);
5. if \( x = y + z \), then \( \sigma_T(x) \subseteq \sigma_T(y) \cup \sigma_T(z) \).

In the case \( X \) is a finite dimensional space, we have a good description of the concepts involved in local spectral theory. Indeed, if the spectrum of \( T \) is
\[ \sigma(T) = \{\lambda_1, \ldots, \lambda_k\}, \quad \text{with } \lambda_i \neq \lambda_j \quad \text{for } i \neq j, \]
then we can write
\[ X = N(T - \lambda_1)^{n_1} \oplus \cdots \oplus N(T - \lambda_k)^{n_k}. \] (1)

Moreover,
\[ R(T - \lambda_i)^{n_i} = \bigoplus_{j=1; j \neq i}^k N(T - \lambda_j)^{n_j} \quad \text{for every } i = 1, \ldots, k. \]

We denote by \( P_i \) the projection on \( X \) with \( R(P_i) = N(T - \lambda_i)^{n_i} \) and \( N(P_i) = R(T - \lambda_i)^{n_i} \), for \( i = 1, \ldots, k \). Note that \( P_i P_j = 0 \) for \( i \neq j \), and \( P_1 + \cdots + P_k = I \).

The operators \( Q_i := (\lambda_i - T) P_i \) are nilpotent; indeed, \( Q_i^{n_i} = 0 \) for \( i = 1, \ldots, k \), and for every \( \lambda \in \mathbb{C} \setminus \sigma(T) \) we can write
\[ (T - \lambda)^{-1} P_i = (\lambda_i - \lambda - Q_i)^{-1} P_i = (\lambda_i - \lambda)^{-1} (I - (\lambda_i - \lambda)^{-1} Q_i)^{-1} P_i. \]

The resolvent operator \( R_T(\lambda) := (T - \lambda)^{-1}, \lambda \in \mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_k\} \), is given by
\[ R_T(\lambda) = \sum_{i=1}^k \left( \sum_{j=1}^{n_i} (\lambda_i - \lambda)^{-j} Q_i^{j-1} \right) P_i. \]

Using this expression for \( R_T(\lambda) \) it is not difficult to show that, for every \( x \in X \),
\[ \sigma_T(x) = \{\lambda_i : 1 \leq i \leq k, P_i(x) \neq 0\}. \]

The previous description of the local spectrum for operators acting on finite dimensional spaces is well-known. See, for example, [18]. We include it for the convenience of the reader. For additional information on local spectral theory we refer to [1] or [11].

Let \( \mathcal{K}(\mathbb{C}) \) denote the set of all nonempty compact subsets of \( \mathbb{C} \), endowed with the Hausdorff metric.

**Proposition 2.** Let \( X \) be a finite dimensional normed space. Then the map
\[ \sigma : T \in \mathcal{L}(X) \longrightarrow \sigma(T) \in \mathcal{K}(\mathbb{C}) \]
is continuous.

**Proof.** It is a consequence of [2, Corollary 3.4.5]. \( \square \)

3. Main results

Let \( x_0 \in X \) be a fixed nonzero vector. We say that a linear map \( \varphi : \mathcal{L}(X) \rightarrow \mathcal{L}(X) \) preserves the local spectrum at \( x_0 \) if \( \sigma_{\varphi(T)}(x_0) = \sigma_T(x_0) \) for all \( T \in \mathcal{L}(X) \).

Next we state our main result, which characterizes the linear maps \( \varphi \) that preserve the local spectrum at a fixed point \( x_0 \) in the case of a finite dimensional space \( X \).

**Theorem 3.** Let \( X \) be a finite dimensional complex vector space. Then a linear map \( \varphi : \mathcal{L}(X) \rightarrow \mathcal{L}(X) \) preserves the local spectrum at \( x_0 \) if and only if there exists an invertible \( A \in \mathcal{L}(X) \) such that \( A(x_0) = x_0 \) and \( \varphi(T) = ATA^{-1} \) for every \( T \in \mathcal{L}(X) \).

Before presenting the proof of Theorem 3, we give some auxiliary results.
Lemma 4. Let $A \in \mathcal{L}(X)$ be an invertible operator and let $x_0 \in X$. Then the automorphism $\varphi_A : \mathcal{L}(X) \to \mathcal{L}(X)$ defined by $\varphi_A(T) := AT A^{-1}$ preserves the local spectrum at $x_0 \in X$ if and only if

$$x_0 \in \bigcup_{\lambda \in \sigma_p(A)} N(A - \lambda),$$

where $\sigma_p(T)$ is the point spectrum of $T$.

Proof. It follows from the definition of the local spectrum that

$$\sigma_T(x_0) = \sigma_{AT A^{-1}}(Ax_0) \quad \text{for every } T \in \mathcal{L}(X).$$

Thus, if $Ax_0 = \lambda x_0$ for some nonzero $\lambda \in \mathbb{C}$, then we have

$$\sigma_T(x_0) = \sigma_{AT A^{-1}}(\lambda x_0) = \sigma_{AT A^{-1}}(x_0) \quad \text{for every } T \in \mathcal{L}(X).$$

Conversely, suppose that $\sigma_T(x_0) = \sigma_{AT A^{-1}}(x_0)$ for every $T \in \mathcal{L}(X)$, but $x_0$ and $Ax_0$ are linearly independent. Then, there exists $T \in \mathcal{L}(X)$ such that $Tx_0 = x_0$ and $T(A^{-1}x_0) = 0$; thus $\sigma_T(x_0) = \{0\}$ and $\sigma_{\varphi_A(T)}(x_0) = \{0\}$, and we get a contradiction. \qed

Lemma 5. Suppose $\dim(X) > 1$. Let $x_0 \in X$ be a nonzero vector, let $B \in \mathcal{L}(X^*, X)$ be a bijective operator and let $\varphi_B : \mathcal{L}(X) \to \mathcal{L}(X)$ be the map defined by $\varphi_B(T) := BT^* B^{-1}$, where $T^* \in \mathcal{L}(X^*)$ is the conjugate operator of $T$. Then there exists an operator $T \in \mathcal{L}(X)$ such that

$$\sigma_T(x_0) = \{0\} \neq \sigma_{\varphi_B(T)}(x_0).$$

Hence $\varphi_B$ does not preserve the local spectrum at $x_0$.

Proof. Note that for every $T \in \mathcal{L}(X)$,

$$\sigma_{T^*}(B^{-1}(x_0)) = \sigma_{\varphi_B(T)}(x_0).$$

We take $w \in X$ such that $w$ and $x_0$ are linearly independent and $(B^{-1}(x_0))(w) \neq 0$, and choose $f \in X^*$ such that $f(x_0) = 0$ and $f(w) = 1$. Then the operator $T = w \otimes f \in \mathcal{L}(X)$ satisfies $T(x_0) = 0$ and $T(w) = w$. Thus $\sigma_T(x_0) = \{0\}$. Moreover, for every $p \in \mathbb{N}$

$$T^p(B^{-1}(x_0))(w) = B^{-1}(x_0)(T^p(w)) = B^{-1}(x_0)(w) \neq 0.$$ 

Then $\sigma_{\varphi_B(T)}(x_0) \neq \{0\}$; hence that $\varphi_B$ does not preserves the local spectrum at $x_0$. \qed

For a finite set $A$ we denote by $|A|$ the number of elements of $A$, and given a $n$-dimensional vector space $X$ and a nonzero vector $x_0 \in X$, we define

$$\mathcal{G}(X; x_0) := \{T \in \mathcal{L}(X) : \{Tx_0, \ldots, T^n x_0\} \text{ is a basis of } X \text{ and } |\sigma(T)| = n\}.$$ 

The following result will be the key in the proof of Theorem 3.

Proposition 6. Let $X$ be a $n$-dimensional complex vector space and let $x_0 \in X$ be a nonzero vector. Then the set $\mathcal{G}(X; x_0)$ is dense in $\mathcal{L}(X)$.

Proof. Let us say that $T \in \mathcal{L}(X)$ satisfies property $b$-x whenever $\{Tx_0, \ldots, T^n x_0\}$ is a basis of $X$. First we observe the following facts:
(1) $T$ satisfies property b-$x_0$ if and only if $Tx_0 \neq 0$ and
\[
\min\{\text{dist}(T^{k+1}x_0, \langle Tx_0, \ldots, T^kx_0 \rangle) : k = 1, \ldots, n - 1\} > 0;
\]
where $\langle y_1, \ldots, y_k \rangle$ is the subspace generated by the vectors $y_1, \ldots, y_k$.

(2) As a consequence of (1), property b-$x_0$ is stable under small norm perturbations; i.e., if $T$ satisfies property b-$x_0$, then there exists $\varepsilon > 0$ such that $S \in \mathcal{L}(X)$ satisfies it also, whenever $\|S-T\| < \varepsilon$.

(3) The set of all operators that satisfy property b-$x_0$ is dense.

Indeed, given $T \in \mathcal{L}(X)$, by making arbitrarily small modifications we can get $Tx_0 \neq 0$ and $T^{k+1} \notin \langle Tx_0, \ldots, T^kx_0 \rangle$ for $k = 1, \ldots, n - 1$.

Thus $\{T \in \mathcal{L}(X) : T$ satisfies property b-$x_0\}$ is an open dense subset of $\mathcal{L}(X)$.

It is enough to show that $\Sigma_n(X) := \{T \in \mathcal{L}(X) : |\sigma(T)| = n\}$ is an open dense subset of $\mathcal{L}(X)$. Indeed, once we have proved that, since the intersection of two open dense subsets is again open and dense, we conclude that $\mathcal{B}(X; x_0)$ is dense (and open).

On the one hand, the fact that $\Sigma_n(X)$ is open is a consequence of the continuity of the spectrum (Proposition 2). On the other hand, the fact that $\Sigma_n(X)$ is dense in $\mathcal{L}(X)$ is probably a folklore result; however, we will give a proof for the convenience of the reader. We will do it in several steps:

Step 1: First we consider $Q_0 \in \mathcal{L}(X)$ for which there exists a basis $e_1, \ldots, e_n$ in $X$ so that $Q_0(e_i) = e_{i+1}$ for $1 \leq i < n$ and $Q_0(e_n) = 0$; hence $Q_0$ is nilpotent. If we consider $n$ different complex numbers $\varepsilon_1, \ldots, \varepsilon_n$ and define $A_{\varepsilon} \in \mathcal{L}(X)$ by $A_{\varepsilon}(e_i) := \varepsilon e_i$; $i = 1, \ldots, n$, then $Q_0 + A_{\varepsilon} \in \Sigma_n(X)$. Indeed, if we also represent by $Q_0$ and $A_{\varepsilon}$ the corresponding matrices with respect to the basis $e_1, \ldots, e_n$, then
\[
\det(Q_0 + A_{\varepsilon} - \lambda) = (\varepsilon_1 - \lambda)(\varepsilon_2 - \lambda) \cdots (\varepsilon_n - \lambda).
\]

Moreover, choosing the numbers $\varepsilon_i$ small enough, we can make $\|A_{\varepsilon}\|$ arbitrarily small. Thus $Q_0$ is in the closure of $\Sigma_n(X)$.

Step 2: Next we consider $Q \in \mathcal{L}(X)$ nilpotent. In this case we derive that $Q$ is in the closure of $\Sigma_n(X)$ from Step 1 and the fact that there exists a basis of $X$ so that the matrix of $Q$ with respect to this basis is formed by zeros and some diagonal square blocks with the same form as the matrix of $Q_0$.

Step 3: Clearly, it follows from Step 2 that, for every $\lambda \in \mathbb{C}$, $\lambda - Q$ is in the closure of $\Sigma_n(X)$.

Step 4: Finally, we consider an arbitrary operator $T \in \mathcal{L}(X)$. In this case, using the decomposition of $X$ described in Formula (1), we get that
\[
T = (\lambda_1 - Q_1) \oplus \cdots \oplus (\lambda_k - Q_k),
\]

where $Q_1, \ldots, Q_k$ are nilpotent operators. Hence, this case is a consequence of the result obtained in the previous step. \qed

Let us prove our main result.

**Proof of Theorem 3.** Suppose that $A \in \mathcal{L}(X)$ is invertible and satisfies $A(x_0) = x_0$, and let us define $\varphi(T) := AT A^{-1}$; $T \in \mathcal{L}(X)$.

Let $T \in \mathcal{L}(X)$ and let $\lambda \in \mathbb{C} \setminus \sigma_T(x_0)$. Then there exists an open neighborhood $\Omega$ of $\lambda$ in $\mathbb{C}$ and an analytic function $\hat{x}_0 : \Omega \to X$ so that $(T - \mu)\hat{x}_0(\mu) = x_0$ for all $\mu \in \Omega$. Then
\[
(ATA^{-1} - \mu)A\hat{x}_0(\mu) = A(x_0) = x_0;
\]
for all $\mu \in \Omega$. 


Hence \( \lambda \in \mathbb{C} \setminus \sigma(T)(x_0) \). By symmetry, we conclude that \( \sigma(T)(x_0) = \sigma(T)(x_0) \), for every \( T \in \mathcal{L}(X) \).

Conversely, suppose that \( \sigma(T)(x_0) = \sigma(T)(x_0) \) for all \( T \in \mathcal{L}(X) \). First we will show that \( \sigma(T) = \sigma(T) \) for all \( T \in \mathcal{L}(X) \).

Let \( T \in \mathcal{G}(X; x_0) \). Since there exists \( y \in X \) such that \( \sigma(T)(y) = \sigma(T)(x_0) \) and \( y = c_1 T x_0 + \cdots + T^n x_0 \) for some \( c_i \in \mathbb{C} \), it follows from Proposition 1 that
\[
\sigma(T)(y) = \sigma(T)(T x_0) \cup \cdots \cup \sigma(T)(T^n x_0) \subset \sigma(T)(x_0);
\]

hence \( \sigma(T) = \sigma(T)(x_0) \).

Let \( T \in \mathcal{L}(X) \). By Proposition 6, there exists a sequence \( (T_p) \) in \( \mathcal{G}(X; x_0) \) such that \( T_p \to T \).

Therefore, from the continuity of the spectrum (Proposition 2) we get
\[
\sigma(T) = \lim_{p \to \infty} \sigma(T_p) = \lim_{p \to \infty} \sigma(T_p)(x_0) = \lim_{p \to \infty} \sigma(T)(x_0).
\]

Since \( |\sigma(T)(x_0)| = n \), we have \( \sigma(T_p)(x_0) = \sigma(\varphi(T_p)) \); hence
\[
\sigma(T) = \lim_{p \to \infty} \sigma(\varphi(T_p)) = \sigma(\varphi(T)).
\]

Thus \( \varphi \) is linear and spectrum preserving. Since \( \mathcal{L}(X) \) is finite dimensional, it follows from [4, Proposition 2.1] that \( \varphi \) is also bijective and unital. Therefore, it follows from [16, Theorem 2] that \( \varphi \) is of one of the following forms:
\[
\varphi(T) = AT A^{-1} \quad \text{or} \quad BT^* B^{-1}
\]

where \( T^* \in \mathcal{L}(X^*) \) is the conjugate operator of \( T \), and \( A \in \mathcal{L}(X) \) and \( B \in \mathcal{L}(X^*, X) \) are bijective operators.

By Lemma 5, the second form is not possible. Thus \( \varphi(T) = AT A^{-1} \) with \( A \in \mathcal{L}(X) \) invertible. Moreover, by Lemma 4, \( A(x_0) = \lambda x_0 \) for some \( \lambda \neq 0 \). And clearly, we can take \( \lambda = 1 \). \( \square \)

From Theorem 3, we can derive the main result of [5] in the case of a finite dimensional space.

**Corollary 7.** If a linear map \( \varphi : \mathcal{L}(X) \to \mathcal{L}(X) \) satisfies
\[
\sigma(T)(x) = \sigma(T)(x), \quad \text{for every} \ T \in \mathcal{L}(X) \text{ and every} \ x \in X,
\]
then \( \varphi(T) = T \) for every \( T \in \mathcal{L}(X) \).

**Proof.** It is a direct consequence of Theorem 3. \( \square \)

We finish our paper with the following:

**Conjecture.** Let \( X \) be a complex Banach space and let \( x_0 \in X \) be a nonzero vector. Then a linear map \( \varphi : \mathcal{L}(X) \to \mathcal{L}(X) \) preserves the local spectrum at \( x_0 \) if and only if there exists an invertible \( A \in \mathcal{L}(X) \) such that \( A(x_0) = x_0 \) and \( \varphi(T) = AT A^{-1} \) for every \( T \in \mathcal{L}(X) \).

**References**