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Optimality and duality for nondifferentiable multiobjective variational problems ☆

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Abstract

The concept of efficiency is used to formulate duality for nondifferentiable multiobjective variational problems. Wolfe and Mond–Weir type vector dual problems are formulated. By using the generalized Schwarz inequality and a characterization of efficient solution, we established the weak, strong, and converse duality theorems under generalized (F, ρ)-convexity assumptions.

1. Introduction

Several authors have been interested in optimality conditions and duality theorems for multiobjective variational problems. For details, readers are advised to consult [1]. Recently, Preda [12] introduced generalized (F, ρ)-convexity, an extension of F-convexity and generalized ρ-convexity defined by Vial ([14, 15]). In [3], Egudo has used the concept of efficiency (Pareto optimum) to formulate duality for multiobjective nonlinear programs. In [9], Mishra and Mukherjee discussed duality for multiobjective variational problems involving generalized (F, ρ)-convex functions. Subsequently, Kim et al. ([4,5]) established symmetric duality for multiobjective variational problems with invexity and

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pseudo-invexity. On the other hand Lal et al. [7] derived some weak dual theorem for the nondifferentiable static multiobjective problems involving invex functions. In [6], Liu proved only some weak duality theorems for nondifferentiable static multiobjective variational problems involving generalized (F, ρ)-convex functions.

In this paper, a nondifferentiable multiobjective variational problem is considered. We formulate the Wolfe type dual and Mond–Weir type dual problems. By using the generalized Schwarz inequality, we prove the weak duality theorem under (F, ρ)-convexity assumptions. We employ a characterization of efficient solution due to Chankong and Haimes [2] in order to prove the strong duality theorems under generalized (F, ρ)-convexity assumptions. Also, we prove the converse duality theorem under generalized (F, ρ)-convexity assumptions.

2. Notations and preliminary results

Let $I = [a, b]$ be a real interval and $\Phi : I \times R^n \times R^n \mapsto R$ be a continuously differentiable function. In order to consider $\Phi(t, x, \dot{x})$, where $x : I \mapsto R^n$ is differentiable with derivative $\dot{x}$, we denote the partial derivatives of Φ by Φ_t,

$$\Phi_x = \left[\frac{\partial \Phi}{\partial x^1}, \ldots, \frac{\partial \Phi}{\partial x^n}\right], \qquad \Phi_{\dot{x}} = \left[\frac{\partial \Phi}{\partial \dot{x}^1}, \ldots, \frac{\partial \Phi}{\partial \dot{x}^n}\right].$$

The partial derivatives of other functions used will be written similarly. Let $C(I, R^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u^i = Dx^i \iff x^i(t) = \alpha + \int_a^t u^i(s)\,ds,$$

in which α is a given boundary value. Therefore, $D = \frac{d}{dt}$ except at discontinuities.

We now consider the following multiobjective continuous programming problem:

$$\text{(MP) Minimize } \left(\int_a^b \left[f^1\big(t, x(t), \dot{x}(t)\big) + \big(x(t)^T B_1(t) x(t)\big)^{1/2} \right] \mathrm{d}t, \ldots, \int_a^b \left[f^p\big(t, x(t), \dot{x}(t)\big) + \big(x(t)^T B_p(t) x(t)\big)^{1/2} \right] dt \right)$$

$$\text{subject to } x(a) = \alpha, \quad x(b) = \beta, \tag{1}$$

$$g\big(t, x(t), \dot{x}(t)\big) \leqslant 0, \tag{2}$$

$$x \in C(I, R^n),$$

where $f^i : I \times R^n \times R^n \mapsto R$, $i \in P = \{1, \dots, p\}$, $g : I \times R^n \times R^n \mapsto R^m$ are assumed to be continuously differentiable functions, and for each $t \in I$, $i \in P$, $B_i(t)$ is an $n \times n$ positive semidefinite (symmetric) matrix, with $B(\cdot)$ continuous on I. Let us now denote by X the set of feasible solutions of problem (MP).

The following generalized Schwarz inequality [13, p. 262] is required in the sequel:

$$v^T B\omega \leqslant \left(v^T Bv\right)^{\frac{1}{2}} \left(\omega^T B\omega\right)^{\frac{1}{2}} \quad \text{for all } v, \omega \in R^n.$$

Definition 1 [2]. A point $x^* \in X$ is said to be an efficient solution of (MP) if for all $x \in X$

$$\begin{aligned}
&\int_a^b \left[f^i\left(t, x^*(t), \dot{x}^*(t)\right) + \left(x^*(t)^T B_i(t) x^*(t)\right)^{1/2}\right] dt \\
&\quad \geqslant \int_a^b \left[f^i\left(t, x(t), \dot{x}(t)\right) + \left(x(t)^T B_i(t) x(t)\right)^{1/2}\right] dt \quad \text{for all } i \in P \\
&\Longrightarrow \int_a^b \left[f^i\left(t, x^*(t), \dot{x}^*(t)\right) + \left(x^*(t)^T B_i(t) x^*(t)\right)^{1/2}\right] dt \\
&\quad = \int_a^b \left[f^i\left(t, x(t), \dot{x}(t)\right) + \left(x(t)^T B_i(t) x(t)\right)^{1/2}\right] \mathrm{d}t \quad \text{for all } i \in P.
\end{aligned}$$

In order to prove the strong duality theorem we will invoke the following lemma due to Changkong and Haimes [2].

Lemma 1. *A point $x^0 \in X$ is an efficient solution for* (MP) *if and only if x^0 solves*

$$\begin{aligned}
&\left(\mathrm{P}_k(x^0)\right) \ \textit{Minimize} \ \int_a^b \left[f^k\left(t, x(t), \dot{x}(t)\right) + \left(x(t)^T B_k(t) x(t)\right)^{1/2}\right] \mathrm{d}t \\
&\textit{subject to } x(a) = \alpha, \quad x(b) = \beta, \\
&\qquad \int_a^b \left[f^j\left(t, x(t), \dot{x}(t)\right) + \left(x(t)^T B_j(t) x(t)\right)^{1/2}\right] \mathrm{d}t \\
&\qquad\quad \leqslant \int_a^b \left[f^j\left(t, x^0(t), \dot{x}^0(t)\right) + \left(x^0(t)^T B_j(t) x^0(t)\right)^{1/2}\right] dt \\
&\qquad \textit{for all } j \neq k, \\
&\qquad g\left(t, x(t), \dot{x}(t)\right) \leqslant 0.
\end{aligned}$$

Definition 2. The functional $F : I \times R^n \times R^n \times R^n \times R^n \times R^n \to R$ is sublinear if for any $x, x^0 \in R^n,\ \dot{x}, \dot{x}^0 \in R^n$,

$$F\big(t, x, \dot{x}, x^0, \dot{x}^0; a_1 + a_2\big) \leqslant F\big(t, x, \dot{x}, x^0, \dot{x}^0; a_1\big) + F\big(t, x, \dot{x}, x^0, \dot{x}^0; a_2\big), \tag{A}$$

for any $a_1, a_2 \in R^n$, and

$$F\big(t, x, \dot{x}, x^0, \dot{x}^0; \alpha a\big) = \alpha F\big(t, x, \dot{x}, x^0, \dot{x}^0; a\big), \tag{B}$$

for any $\alpha \in R, \alpha \geqslant 0$, and $a \in R^n$. From (B), $F(t, x, \dot{x}, x^0, \dot{x}^0; 0) = 0$ follows by substituting $\alpha = 0$.

Now consider the function $\Phi : I \times R^n \times R^n \to R$, and suppose that Φ is a continuously differentiable function. Let $d(t, \cdot, \cdot)$ be a pseudometric on R^n, and $\rho \in R$.

Definition 3 [6]. The functional $\Phi(t, \cdot, \cdot)$ is said to be (F, ρ)-convex at $x^0 \in X$ if for all $x \in X$, we have

$$\int_a^b \big[\Phi\big(t, x(t), \dot{x}(t)\big) - \Phi\big(t, x^0(t), \dot{x}^0(t)\big)\big]\, dt \geqslant \int_a^b F\big(t, x(t), \dot{x}(t), x^0(t), \dot{x}^0(t); \Phi_x\big(t, x^0(t), \dot{x}^0(t)\big) - \frac{d}{dt}\big(\Phi_{\dot{x}}\big(t, x^0, \dot{x}^0\big)\big)\big)\, dt + \rho \int_a^b d^2\big(t, x(t), \dot{x}^0(t)\big)\, dt.$$

This function Φ is said to be strongly F-convex, F-convex, or weakly F-convex at x^0 according to $\rho > 0,\ \rho = 0$, or $\rho < 0$.

Definition 4. The functional $\Phi(t, \cdot, \cdot)$ is said to be (F, ρ)-quasiconvex at $x^0 \in X$ if for all $x \in X$ such that

$$\int_a^b \Phi\big(t, x(t), \dot{x}(t)\big)\, dt \leqslant \int_a^b \Phi\big(t, x^0(t), \dot{x}^0(t)\big)\, dt,$$

we have

$$\int_a^b F\Big(t, x(t), \dot{x}(t), x^0(t), \dot{x}^0(t); \Phi_x\big(t, x^0(t), \dot{x}^0(t)\big)$$
$$- \frac{d}{dt}\big(\Phi_{\dot{x}}\big(t, x^0(t), \dot{x}^0(t)\big)\big)\Big)\,dt \leqslant -\rho \int_a^b d^2\big(t, x(t), \dot{x}^0(t)\big)\,dt.$$

We say that $\Phi(t,\cdot,\cdot)$ is strongly F-quasiconvex, F-quasiconvex, or weakly F-quasiconvex at x^0 according to $\rho > 0$, $\rho = 0$, or $\rho < 0$.

Definition 5. The functional $\Phi(t,\cdot,\cdot)$ is said to be (F,ρ)-pseudoconvex at $x^0 \in X$ if for all $x \in X$ such that

$$\int_a^b F\Big(t, x(t), \dot{x}(t), x^0(t), \dot{x}^0(t); \Phi_x\big(t, x^0(t), \dot{x}^0(t)\big)$$
$$- \frac{d}{dt}\big(\Phi_{\dot{x}}\big(t, x^0(t), \dot{x}^0(t)\big)\big)\Big)\,dt \geqslant -\rho \int_a^b d^2\big(t, x(t), \dot{x}^0(t)\big)\,dt,$$

we have

$$\int_a^b \Phi\big(t, x(t), \dot{x}(t)\big)\,dt \geqslant \int_a^b \Phi\big(t, x^0(t), \dot{x}^0(t)\big)\,dt.$$

We say that $\Phi(t,\cdot,\cdot)$ is strongly F-pseudoconvex, F-pseudoconvex, or weakly F-pseudoconvex at x^0 according to $\rho > 0$, $\rho = 0$, or $\rho < 0$.

Definition 6. The function $\Phi(t,\cdot,\cdot)$ is said to be strictly (F,ρ)-pseudoconvex at $x^0 \in X$ if for all $x \in X$, $x \neq x^0$ such that

$$\int_a^b F\Big(t, x(t), \dot{x}(t), x^0(t), \dot{x}^0(t); \Phi_x\big(t, x^0(t), \dot{x}^0(t)\big)$$
$$- \frac{d}{dt}\big(\Phi_{\dot{x}}\big(t, x^0(t), \dot{x}^0(t)\big)\big)\Big)\,dt \geqslant -\rho \int_a^b d^2\big(t, x(t), x^0(t)\big)\,dt,$$

and we have

$$\int_a^b \Phi\big(t, x(t), \dot{x}(t)\big)\,dt > \int_a^b \Phi\big(t, x^0(t), \dot{x}^0(t)\big)\,dt,$$

or equivalently, if

$$\int_a^b \Phi\bigl(t, x(t), \dot{x}(t)\bigr)\,dt \leqslant \int_a^b \Phi\bigl(t, x^0(t), \dot{x}^0(t)\bigr)\,dt,$$

we have

$$\int_a^b F\Bigl(t, x(t), \dot{x}(t), x^0(t), \dot{x}^0(t); \Phi_x\bigl(t, x^0(t), \dot{x}^0(t)\bigr) - \frac{d}{dt}\bigl(\Phi_{\dot{x}}\bigl(t, x^0(t), \dot{x}^0(t)\bigr)\bigr)\Bigr)\,dt < -\rho \int_a^b d^2\bigl(t, x(t), x^0(t)\bigr)\,dt.$$

3. Optimality

In this section we give the necessary optimality theorem for $(P_k(x^0))$.

Lemma 2. *Define a function $h : R^n \mapsto R$ by $h(x(t)) = (x(t)^T B(t)x(t))^{1/2}$, where B is a symmetric and positive semidefinite $n \times n$ matrix and let $x^0 \in R^n$. Then h is convex, and*

$$\partial h\bigl(x^0(t)\bigr) = \bigl\{B(t)\omega(t)\colon\ \omega(t)^T B(t)\omega(t) \geqslant 1\bigr\},$$

where the $\partial h(x(t))$ is subgradient of h at $x(t)$.

Consider a nonlinear optimization problem:

$$\begin{aligned}&\text{(P) minimize} \quad f\bigl(t, x(t), \dot{x}(t)\bigr)\\ &\text{subject to} \quad g\bigl(t, x, \dot{x}(t)\bigr) \leqslant 0,\end{aligned}$$

where f and g^i are Lipschitz functions from R^n into R for $i = 1, 2, \ldots, m$.

Theorem 1. *Let f and g^i $(i = 1, 2, \ldots, m)$ be locally Lipschitz functions. If x^0 solves* (P)*, then there exists α and $r_i \geqslant 0$ $(i = 1, 2, \ldots, m)$, not all zero, such that*

$$0 \in \alpha\partial f\bigl(t, x^0(t), \dot{x}^0(t)\bigr) + \sum_{i=1}^m r_i\, \partial g^i\bigl(t, x^0(t), \dot{x}^0(t)\bigr) \quad \textit{and}$$

$$\sum_{i=1}^m r_i g^i\bigl(t, x^0(t), \dot{x}^0(t)\bigr) = 0.$$

Now, we have the following Fritz John type necessary optimality conditions for above minimization problem $(P_k(x^0))$.

Theorem 2. *If x^0 is optimal to $(P_k(x^0))$, then there exist $\tau_i^0 \in R,\ i \in R,\ \lambda \in R^m$ and $\omega^0 \in R^n$ such that*

$$\begin{aligned}
&\lambda(t)^T g\big(t, x^0(t), \dot{x}^0(t)\big) = 0,\\
&\tau_k^0\big[\nabla f^k\big(t, x^0(t), \dot{x}^0(t)\big) + B_k(t)\omega^0(t)\big]\\
&\quad + \sum_{i \neq k}^{p} \tau_i^0\big[\nabla f^i\big(t, x^0(t), \dot{x}^0(t)\big) + B_i(t)\omega^0(t)\big]\\
&\quad + \nabla\lambda(t)^T g\big(t, x^0(t), \dot{x}^0(t)\big) = 0,\\
&\omega^0(t) B_i(t)\omega^0(t) \leqslant 1,\\
&\big(x^0(t)^T B_i(t) x^0(t)\big)^{1/2} = x^0(t)^T B_i(t)\omega^0(t),\\
&\big(\tau^0, \lambda\big) \geqslant 0 \quad \textit{and} \quad \big(\tau^0, \lambda\big) \neq 0.
\end{aligned}$$

Proof. (a) If $x^0(t)^T B_i(t) x^0(t) > 0$, then $f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}}$, for $i \in P$, is differentiable in a sufficiently small neighborhood of x^0. Since x^0 is optimal to $(P_k(x^0))$, by the generalized Fritz John conditions [8], there exist $\tau_i \in R,\ i \in R$ and $\lambda \in R^m$ such that

$$\begin{aligned}
&\lambda(t)^T g\big(t, x^0(t), \dot{x}^0(t)\big) = 0,\\
&\tau_k^0\big[\nabla f^k\big(t, x^0(t), \dot{x}^0(t)\big) + B_k(t)x^0(t)\big/\big(x^0(t)^T B_k(t)x^0(t)\big)^{1/2}\big]\\
&\quad + \sum_{i \neq k}^{p} \tau_i^0\big[\nabla f^i\big(t, x^0(t), \dot{x}^0(t)\big) + B_i(t)x^0(t)\big/\big(x^0(t)^T B_i(t)x^0(t)\big)^{1/2}\big]\\
&\quad + \nabla\lambda(t)^T g\big(t, x^0(t), \dot{x}^0(t)\big) = 0,\\
&\big(\tau^0, \lambda\big) \geqslant 0 \quad \text{and} \quad \big(\tau^0, \lambda\big) \neq 0.
\end{aligned}$$

Setting $\omega^0(t) = x^0(t)/(x^0(t)^T B_i(t) x^0(t))^{1/2}$, for each $i \in P$, then

$$\begin{aligned}
&\tau_k^0\big[\nabla f^k\big(t, x^0(t), \dot{x}^0(t)\big) + B_k(t)\omega^0(t)\big]\\
&\quad + \sum_{i \neq k}^{p} \tau_i^0\big[\nabla f^i\big(t, x^0(t), \dot{x}^0(t)\big) + B_i(t)\omega^0(t)\big]\\
&\quad + \nabla\lambda(t)^T g\big(t, x^0(t), \dot{x}^0(t)\big) = 0.
\end{aligned}$$

It is clear that $\omega^0(t)^T B_i(t)\omega^0(t) = 1$ and $(x^0(t)^T B_i(t))^{1/2} = x^0(t)^T B_i(t)\omega^0(t)$.

(b) Assume $x^0(t)^T B_i(t) x^0(t) = 0$. Define a function $h^i : R^n \mapsto R$ by $h^i(x(t)) = (x(t)^T B_i(t) x(t))^{\frac{1}{2}}$, for all $x \in R^n$ and $i \in P$. Then $h^i, i \in P$, is not differentiable and, by Lemma 2, $\partial h^i(x^0(t)) = \{B_i(t)\omega(t){:}\ \omega(t)^T B_i(t)\omega(t) \leqslant 1\}$. Since f^i and g^i are continuously differentiable functions, then f^i and g^i are locally Lipschitz function and $\partial f^i(t, x^0(t), \dot{x}^0(t)) = \{\nabla f^i(t, x^0(t), \dot{x}^0(t))\}$, and

$\partial g^j(t, x^0(t), \dot{x}^0(t)) = \{\nabla g^j(t, x^0(t), \dot{x}^0(t))\}$, for $i \in P$ and $j = 1, 2, \ldots, m$, respectively. Automatically h^i for $i \in P$, is locally Lipschitz function. By Theorem 1, there exists τ_k^0 and τ_i^0, for $i (\neq k) \in P$, and $\lambda_j \geqslant 0$, $j = 1, 2, \ldots, m$, not all zero, such that

$$\begin{aligned} 0 \in{} & \tau_k^0\big[\partial f^k\big(t, x^0(t), \dot{x}^0(t)\big) + \partial\big(x(t)^T B_k(t)x(t)\big)^{1/2}\big|_{x=x^0}\big] \\ & + \sum_{i\neq k}^{p} \tau_i^0\big[\partial f^i\big(t, x^0(t), \dot{x}^0(t)\big) + \partial\big(x(t)^T B_i(t)x(t)\big)^{1/2}\big|_{x=x^0}\big] \\ & + \sum_{j=1}^{m} \lambda_j \partial g^j\big(t, x^0(t), \dot{x}^0(t)\big) \end{aligned}$$

and

$$\sum_{j=1}^{m} \lambda_j g^j\big(t, x^0(t), \dot{x}^0(t)\big) = 0.$$

Since $\partial f^i(t, x^0(t), \dot{x}^0(t)) = \{\nabla f^i(t, x^0(t), \dot{x}^0(t))\}$, $\partial g^j(t, x^0(t), \dot{x}^0(t)) = \{\nabla g^j(t, x^0(t), \dot{x}^0(t))\}$ and $\partial h^i(x^0(t)) = \{B_i(t)w(t)\colon\ w(t)^T B_i(t)w(t) \leqslant 1\}$, for $i \in P$ and $j = 1, \ldots, m$, respectively. Then

$$\begin{aligned} 0 ={} & \tau_k^0\big[\nabla f^k\big(t, x^0(t), \dot{x}^0(t)\big) + \big(B_k(t)w(t)\colon\ w(t)^T B_i(t)w(t) \leqslant 1\big)\big] \\ & + \sum_{i\neq k}^{p} \tau_i^0\big[\nabla f^i\big(t, x^0(t), \dot{x}^0(t)\big) + \big(B_i(t)w(t)\colon\ w(t)^T B_i(t)w(t) \leqslant 1\big)\big] \\ & + \sum_{j=1}^{m} \lambda_j \nabla g^j\big(t, x^0(t), \dot{x}^0(t)\big) \end{aligned}$$

and

$$\sum_{j=1}^{m} \lambda_j g^j\big(t, x^0(t), \dot{x}^0(t)\big) = 0.$$

So, there exists $w^0(t) \in R^n$ such that

$$\begin{aligned} 0 ={} & \tau_k^0\big[\nabla f^k\big(t, x^0(t), \dot{x}^0(t)\big) + B_k(t)w(t)\big] \\ & + \sum_{i\neq k}^{p} \tau_i^0\big[\nabla f^i\big(t, x^0(t), \dot{x}^0(t)\big) + B_i(t)w(t)n\big] \\ & + \sum_{j=1}^{m} \lambda_j \nabla g^j\big(t, x^0(t), \dot{x}^0(t)\big), \quad \sum_{j=1}^{m} \lambda_j g^j\big(t, x^0(t), \dot{x}^0(t)\big) = 0 \end{aligned}$$

and

$$w^0(t)^T B_i(t) w^0(t) \leqslant 1.$$

By generalized Schwarz inequality, $x^0(t)^T B_i(t) x^0(t) = 0$ implies that $B_i(t) x^0(t) = 0$. So $(x^0(t)^T B_i(t) x^0(t))^{\frac{1}{2}} = x^0(t) B_i(t) w^0(t)$. Hence, Theorem 2 follows. □

4. Wolfe vector duality

By using the generalized Schwarz inequality, we derive the following lemma in order to prove the weak duality theorem for multiobjective variational problem (MP).

Lemma 3. *Let $A(t)$ be an $n \times n$ positive semidefinite (symmetric) matrix, with $A(\cdot)$ continuous on I, and $\omega(t)^T A(t)\omega(t) \leqslant 1$. Then*

$$\int_a^b \bigl(x(t)^T A(t) x(t)\bigr)^{1/2}\, dt \geqslant \int_a^b x(t)^T A(t)\omega(t)\, dt.$$

Proof. With the generalized Schwarz inequality, we obtain

$$\int_a^b \bigl(x(t)^T A(t) x(t)\bigr)^{1/2} \bigl(\omega(t)^T A(t)\omega(t)\bigr)^{1/2}\, dt \geqslant \int_a^b x(t)^T A(t)\omega(t)\, dt.$$

Since

$$\omega(t)^T A(t)\omega(t) \leqslant 1.$$

Hence

$$\int_a^b \bigl(x(t)^T A(t) x(t)\bigr)^{1/2}\, dt \geqslant \int_a^b x(t)^T A(t)\omega(t)\, dt. \quad \square$$

Consider the following Wolfe vector dual of (MP):

$$\begin{aligned}
(\mathrm{MDP})_1 \ \text{Maximize}\ \Biggl(\int_a^b & \bigl[f^1\bigl(t, y(t), \dot y(t)\bigr) + y(t)^T B_1(t)\omega(t) \\
& + \lambda(t)^T g\bigl(t, y(t), \dot y(t)\bigr)\bigr]\, dt, \ldots, \\
\int_a^b & \bigl[f^p\bigl(t, y(t), \dot y(t)\bigr) + y(t)^T B_p(t)\omega(t) \\
& + \lambda(t)^T g\bigl(t, y(t), \dot y(t)\bigr)\bigr]\, dt \Biggr)
\end{aligned}$$

$$\begin{aligned}\text{subject to}\quad & y(a)=\alpha, \quad y(b)=\beta,\\ & \sum_{i=1}^{p}\tau_i\big[f_x^i\big(t,y(t),\dot y(t)\big)+B_i(t)\omega(t)\big]+\lambda(t)^T g_x\big(t,y(t),\dot y(t)\big)\\ &\quad = D\left\{\sum_{i=1}^{p}\tau_i f_{\dot x}^i\big(t,y(t),\dot y(t)\big)+\lambda(t)^T g_{\dot x}\big(t,y(t),\dot y(t)\big)\right\}, \qquad (3)\\ & \omega^T B_i\omega\leqslant 1, \quad i\in P, \qquad (4)\\ & \lambda(t)\geqslant 0, \quad \tau_i\geqslant 0, \quad \sum_{i=1}^{p}\tau_i=1, \qquad (5)\\ & y\in C(I,R^n), \quad \omega\in C(I,R^n), \quad \lambda\in C(I,R^m).\end{aligned}$$

Theorem 3 (Weak Duality). *Assume that for all feasible x for* (MP) *and all feasible (y,λ,ω,τ) for* $(\mathrm{MDP})_1$, *either*

(i) $\tau_i>0, \sum_{i=1}^{p}\tau_i[f^i(t,\cdot,\cdot)+(\cdot)^T B_i w]$ *is* (F,ρ_1)*-convex, for all* $i\in P$, $\lambda(t)^T g(t,\cdot,\cdot)$ *is* (F,ρ_2)*-convex, and* $\rho_1+\rho_2\geqslant 0$; *or*
(ii) $\sum_{i=1}^{p}\tau_i[f^i(t,\cdot,\cdot)+(\cdot)^T B_i w]$ *is strictly* (F,ρ_1)*-convex, for all* $i\in P$, $\lambda(t)^T g(t,\cdot,\cdot)$ *is strictly* (F,ρ_2)*-convex and* $\rho_1+\rho_2\geqslant 0$.

Then, the following cannot hold:

$$\begin{aligned}&\int_a^b\big[f^i\big(t,x(t),\dot x(t)\big)+\big(x(t)^T B_i(t)x(t)\big)^{1/2}\big]\,dt\\ &\quad\leqslant\int_a^b\big[f^i\big(t,y(t),\dot y(t)\big)+y(t)^T B_i(t)\omega(t)\\ &\qquad\quad+\lambda(t)^T g\big(t,y(t),\dot y(t)\big)\big]\,dt, \qquad (6)\end{aligned}$$

for all $i\in P$ and

$$\begin{aligned}&\int_a^b\big[f^j Lb(t,x(t),\dot x(t)\big)+\big(x(t)^T B_j(t)x(t)\big)^{1/2}\big]\,dt\\ &\quad<\int_a^b\big[f^j\big(t,y(t),\dot y(t)\big)+y(t)^T B_j(t)\omega(t)\\ &\qquad\quad+\lambda(t)^T g\big(t,y(t),\dot y(t)\big)\big]\,dt \qquad (7)\end{aligned}$$

for some $j\in P$.

Proof. Suppose, contrary to the result, that (6) and (7) hold. With Lemma 3 and $\lambda(t) \geqslant 0$, we have

$$\int_a^b \left[f^i\big(t, x(t), \dot{x}(t)\big) + x(t)^T B_i(t)\omega(t) + \lambda(t)^T g\big(t, x(t), \dot{x}(t)\big)\right] \mathrm{d}t$$
$$\leqslant \int_a^b \left[f^i\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_i(t)\omega(t) + \lambda(t)^T g\big(t, y(t), \dot{y}(t)\big)\right] dt, \tag{8}$$

for all $i \in P$ and

$$\int_a^b \left[f^j\big(t, x(t), \dot{x}(t)\big) + x(t)^T B_j(t)\omega(t) + \lambda(t)^T g\big(t, x(t), \dot{x}(t)\big)\right] dt$$
$$< \int_a^b \left[f^j\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_j(t)\omega(t) + \lambda(t)^T g\big(t, y(t), \dot{y}(t)\big)\right] dt, \tag{9}$$

for some $j \in P$, respectively. Now assumption (i) $\tau_i > 0$ and $\sum_{i=1}^p \tau_i = 1$, (8) and (9) imply

$$\sum_{i=1}^p \tau_i \int_a^b \left[f^i\big(t, x(t), \dot{x}^t\big) + x(t)^T B_i(t)\omega(t) + \lambda(t)^T g\big(t, x(t), \dot{x}(t)\big)\right] dt$$
$$< \sum_{i=1}^p \tau_i \int_a^b \left[f^i\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_i(t)\omega(t) + \lambda(t)^T g\big(t, y(t), \dot{y}(t)\big)\right] dt. \tag{10}$$

Under assumption (i) $\sum_{i=1}^p \tau_i[f^i(t, \cdot, \cdot) + (\cdot)^T B_i\omega]$ is (F, ρ_1)-convex, for all $i \in P$, and $\lambda(t)^T g(t, \cdot, \cdot)$ is (F, ρ_2)-convex.

$$\sum_{i=1}^p \tau_i \int_a^b \left\{\left[f^i\big(t, x(t), \dot{x}(t)\big) + x(t)^T B_i(t)\omega(t)\right] - \left[f^i\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_i(t)\omega(t)\right]\right\} dt$$
$$\geqslant \int_a^b F\left(t, x(t), \dot{x}(t), y(t), \dot{y}(t);\ \sum_{i=1}^p \tau_i\left[f_x^i\big(t, y(t), \dot{y}(t)\big) + B_i(t)\omega(t)\right] - \frac{d}{dt}\sum_{i=1}^p \tau_i f_{\dot{x}}^i\big(t, y(t), \dot{y}(t)\big)\right) dt + \rho_1 \int_a^b d^2\big(t, x(t), y(t)\big)\,\mathrm{d}t. \tag{11}$$

$$\begin{aligned}
&\int_a^b \left[\lambda(t)^T g\big(t, x(t), \dot{x}(t)\big) - \lambda(t)^T g\big(t, y(t), \dot{y}(t)\big)\right] dt \\
&\quad \geqslant \int_a^b F\bigg(t, x(t), \dot{x}(t), y(t), \dot{y}(t); \lambda(t)^T g_x\big(t, y(t), \dot{y}(t)\big) \\
&\qquad - \frac{d}{dt}\lambda(t)^T g_{\dot{x}}\big(t, y(t), \dot{y}(t)\big)\bigg) dt + \rho_2 \int_a^b d^2\big(t, x(t), y(t)\big)\, dt. \qquad (12)
\end{aligned}$$

By (10), (11) and (12), we have

$$\begin{aligned}
&\int_a^b F\bigg(t, x(t), \dot{x}(t), y(t), \dot{y}(t); \sum_{i=1}^p \tau_i\big[f_x^i\big(t, y(t), \dot{y}(t)\big) + B_i(t)\omega(t)\big] \\
&\quad - \frac{d}{dt}\sum_{i=1}^p \tau_i f_{\dot{x}}^i\big(t, y(t), \dot{y}(t)\big)\bigg) dt + \rho_1 \int_a^b d^2\big(t, x(t), y(t)\big)\, dt \\
&\quad + \int_a^b F\bigg(t, x(t), \dot{x}(t), y(t), \dot{y}(t); \lambda(t)^t g_x\big(t, y(t), \dot{y}(t)\big) \\
&\quad - \frac{d}{dt}\lambda(t)^T g_{\dot{x}}\big(t, y(t), \dot{y}(t)\big) \\
&\quad + \rho_2 \int_a^b d^2\big(t, x(t), y(t)\big)\bigg) dt < 0.
\end{aligned}$$

By the sublinearity of F and $\rho_1 + \rho_2 \geqslant 0,$ we have

$$\begin{aligned}
&\int_a^b F\bigg(t, x(t), \dot{x}(t), y(t), \dot{y}(t); \sum_{i=1}^p \tau_i\big[f_x^i\big(t, y(t), \dot{y}(t)\big) + B_i(t)\omega(t)\big] \\
&\quad + \lambda(t)^T g_x\big(t, y(t), \dot{y}(t)\big) \\
&\quad - \frac{d}{dt}\left[\sum_{i=1}^p \tau_i f_{\dot{x}}^i\big(t, y(t), \dot{y}(t)\big) + \lambda(t)^T g_{\dot{x}}\big(t, y(t), \dot{y}(t)\big)\right]\bigg) dt \\
&\quad + (\rho_1 + \rho_2) \int_a^b d^2\big(t, x(t), y(t)\big)\, dt < 0.
\end{aligned}$$

Hence

$$\int_a^b F\biggl(t, x(t), \dot{x}(t), y(t), \dot{y}(t); \sum_{i=1}^p \tau_i\bigl[f_x^i\bigl(t, y(t), \dot{y}(t)\bigr) + B_i(t)\omega(t)\bigr] \\ + \lambda(t)^T g_x\bigl(t, y(t), \dot{y}(t)\bigr) \\ - \frac{d}{dt}\biggl[\sum_{i=1}^p \tau_i f_{\dot{x}}^i\bigl(t, y(t), \dot{y}(t) + \lambda(t)^T g_{\dot{x}}\bigl(t, y(t), \dot{y}(t)\bigr)\biggr]\biggr)\, dt < 0, \tag{13}$$

which contradicts (3), because $\int_a^b F(t, x(t), \dot{x}(t), y(t), \dot{y}(t); 0)\, dt = 0$. Hence, the result follows.

If the assumption (ii) holds, since $\tau_i \geqslant 0$, for all $i \in P$, and $\sum_{i=1}^p \tau_i = 1$, (8) and (9) imply that

$$\sum_{i=1}^p \tau_i \int_a^b \bigl[f^i\bigl(t, x(t), \dot{x}(t)\bigr) + x(t)^T B_i(t)x(t) + \lambda(t)^T g\bigl(t, x(t), \dot{x}(t)\bigr)\bigr]\, dt \\ \leqslant \sum_{i=1}^p \tau_i \int_a^b \bigl[f^i\bigl(t, y(t), \dot{y}(t)\bigr) + y(t)^T B_i(t)\omega(t) \\ + \lambda(t)^T g\bigl(t, y(t), \dot{y}(t)\bigr)\bigr]\, dt,$$

and then again we reach (13). Hence, the proof is complete. □

Corollary 1. *Let* $(y^0, \lambda^0, \omega^0, \tau^0)$ *be a feasible solution for* $(\mathrm{MDP})_1$ *such that*

$$\int_a^b \lambda^0(t)^T g(t, y^0(t), \dot{y}^0(t)\, dt = 0$$

and

$$\int_a^b \bigl(y^0(t)^T B_i(t) y^0(t)\bigr)^{1/2}\, dt = \int_a^b y^0(t)^T Bi(t)\omega^0(t)\, dt,$$

for each $i \in P$ *and assume that* y^0 *is feasible for* (MP)*. If weak duality holds between* (MP) *and* $(\mathrm{MDP})_1$*, then* y^0 *is efficient for* (MP) *and* $(y^0, \lambda^0, \omega^0, \tau^0)$ *is efficient for* $(\mathrm{MDP})_1$.

Proof. Suppose that y^0 is not efficient for (MP). Since

$$\int_a^b \bigl(y^0(t)^T B_i(t) y^0(t)\bigr)^{1/2}\, dt = \int_a^b y^0(t)^T B_i(t)\omega^0(t)\, dt,$$

and

$$\int_a^b \lambda^0(t)^T g\big(t, y^0(t), \dot y^0(t)\big)\, dt = 0,$$

we obtain

$$\begin{aligned}
&\int_a^b \big[f^i\big(t, x(t), \dot x(t)\big) + \big(x(t)^T B_i(t) x(t)\big)^{1/2}\big]\, dt \\
&\quad < \int_a^b \big[f^i\big(t, y^0(t), \dot y^0(t)\big) + y^0(t)^T B_i(t)\omega^0(t) \\
&\qquad\qquad + \lambda^0(t)^T g\big(t, y^0(t), \dot y^0(t)\big)\big]\, dt,
\end{aligned}$$

for some $i \in P$ and

$$\begin{aligned}
&\int_a^b \big[f^j\big(t, x(t), \dot x(t)\big) + \big(x(t)^T B_j(t) x(t)\big)^{1/2}\big]\, dt \\
&\quad \leqslant \int_a^b \big[f^j\big(t, y^0(t), \dot y^0(t)\big) + y^0(t)^T B_j(t)\omega^0(t) \\
&\qquad + \lambda^0(t)^T g\big(t, y^0(t), \dot y^0(t)\big)\big]\, dt,
\end{aligned}$$

for all $j \in P$.

Since $(y^0, \lambda^0, \omega^0, \tau^0)$ is feasible for $(\mathrm{MDP})_1$ and x is feasible for (MP), these inequalities contradict weak duality (Theorem 3).

Also, suppose that $(y^0, \lambda^0, \omega^0, \tau^0)$ is not efficient for $(\mathrm{MDP})_1$. Since

$$\int_a^b \big(y^0(t)^T B_i(t) y^0(t)\big)^{1/2}\, dt = \int_a^b y^0(t)^T B_i(t)\omega^0(t)\, dt,$$

and

$$\int_a^b \lambda^0(t)^T g\big(t, y^0(t), \dot y^0(t)\big)\, dt = 0,$$

we obtain

$$\begin{aligned}
&\int_a^b \big[f^i\big(t, y(t), \dot y(t)\big) + y(t)^T B_i(t)\omega(t) + \lambda(t)^T g\big(t, y(t), \dot y(t)\big)\big]\, dt \\
&\quad > \int_a^b \big[f^i\big(t, y^0(t), \dot y^0(t)\big) + \big(y^0(t)^T B_i(t) y^0(t)\big)^{1/2}\big]\, dt,
\end{aligned}$$

for some $i \in P$ and

$$\int_a^b \left[f^j\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_j(t)\omega(t) + \lambda(t)^T g\big(t, y(t), \dot{y}(t)\big)\right] dt$$

$$\geqslant \int_a^b \left[f^j\big(t, y^0(t), \dot{y}^0(t)\big) + \big(y^0(t)^T B_j(t) y^0(t)\big)^{1/2}\right] dt,$$

for all $j \in P$, respectively. Since y^0 is feasible for (MP), these inequalities contradict weak duality.

Therefore y^0 and $(y^0, \lambda^0, \omega^0, \tau^0)$ are efficient for their respective programs. □

Theorem 4 (Strong Duality). *Let x^0 be a feasible solution for* (MP) *and assume that*

(i) x^0 *is an efficient solution*;
(ii) *for at least one i, $i \in P$, x^0 satisfies a constraint qualification* [11] *for problem* $(\mathrm{P}_i(x^0))$.

Then there exists $\tau^0 \in R^p$, $\lambda^0 \in R^m$, such that $(x^0, \lambda^0, \omega^0, \tau^0)$ is feasible for $(\mathrm{MDP})_1$ *and* $\int_a^b \lambda^0(t)^T g(t, x^0(t), \dot{x}^0(t))\, dt = 0$.

Further, if the assumptions of Theorem 3 are satisfied, then $(x^0, \lambda^0, \omega^0, \tau^0)$ is efficient for $(\mathrm{MDP})_1$.

$$\begin{aligned}
&\tau_k^0\left[\nabla f^k\big(t, x^0(t), \dot{x}^0(t)\big) + B_k(t) w^0(t)\right] \\
&\quad + \sum_{i \neq k} \tau_i^0\left[\nabla f^i\big(t, x^0(t), \dot{x}^0(t)\big) + B_i(t) w^0(t)\right] \\
&\quad + \nabla \lambda(t)^T g\big(t, x^0(t), \dot{x}^0(t)\big) = 0, \qquad (14)\\
&\lambda(t)^T g\big(t, x^0(t), \dot{x}^0(t)\big) = 0, \qquad (15)\\
&w^0(t) B_i(t) w^0(t) \leqslant 1,\\
&\big(x^0(t)^T B_i(t) x^0(t)\big)^{1/2} = x^0(t)^T B_i(t) w^0(t),\\
&\big(\tau^0, \lambda\big) \geqslant 0 \quad \text{and} \quad \big(\tau^0, \lambda\big) \neq 0.
\end{aligned}$$

By (14), we have

$$\sum_{i=1}^p \tau_i^0\left[f_x^i\big(t, x^0(t), \dot{x}^0(t)\big) + B_i(t) w^0(t)\right] + \lambda^0(t)^T g_x\big(t, x^0(t), \dot{x}^0(t)\big)$$

$$= D\left[\sum_{i=1}^p \tau_i^0 f_x^i\big(t, x^0(t), \dot{x}^0(t)\big) + \lambda^0(t)^T g_{\dot{x}}\big(t, x^0(t), \dot{x}^0(t)\big)\right].$$

From (15), we have

$$\int_a^b \lambda^0(t)^T g\big(t, x^0(t), \dot{x}^0(t)\big)\, dt = 0.$$

Since

$$x^0(t)^T B_i(t) w^0(t) = \big(x^0(t)^T B_i(t) x^0(t)\big)^{1/2},$$

we have

$$\begin{aligned}
&\int_a^b \Big[f^i\big(t, x^0(t), \dot{x}^0(t)\big) + \big(x^0(t)^T B_i(t) x^0(t)\big)^{1/2}\Big]\, dt \\
&\quad = \int_a^b \Big[f^i\big(t, x^0(t), \dot{x}^0(t)\big) + x^0(t)^T B_i(t) w^0(t) \\
&\qquad\quad + \lambda^0(t)^T g\big(t, x^0(t), \dot{x}^0(t)\big)\Big]\, dt,
\end{aligned}$$

and $w^0(t)^T B_i(t) w^0(t) \leqslant 1$, we conclude that $(x^0, \lambda^0, w^0, \tau^0)$ is feasible for $(\mathrm{MDP})_1$. Efficiency of $(x^0, \lambda^0, w^0, \tau^0)$ for $(\mathrm{MDP})_1$ now follows from Corollary 1.

For the converse duality, we make the assumption that Z denotes the space of the piecewise differentiable function $x : I \mapsto R^n$ for which $x(a) = 0 = x(b)$ equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty$.

$(\mathrm{MDP})_1$ may be rewritten in the following form:

$$\begin{aligned}
\text{Minimize } \Bigg(&-\int_a^b \Big[f^1\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_1(t)\omega(t) \\
&\qquad + \lambda(t)^T g\big(t, y(t), \dot{y}(t)\big)\Big]\, dt, \ldots, \\
&-\int_a^b \Big[f^p\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_p(t)\omega(t) \\
&\qquad + \lambda(t)^T g\big(t, y(t), \dot{y}(t)\big)\Big]\, dt\Bigg) \\
\text{subject to } \ & y(a) = \alpha, \quad y(b) = \beta, \\
& \theta\big(t, y(t), \dot{y}(t), \ddot{y}(t), \lambda(t), \tau\big) = 0, \\
& \omega^T B_i \omega \leqslant 1, \quad i \in P, \\
& \lambda(t) \geqslant 0, \quad \tau_i \geqslant 0, \quad \sum_{i=1}^p \tau_i = 1,
\end{aligned}$$

where

$$\begin{aligned}\theta &= \theta\big(t, y(t), \dot y(t), \ddot y(t), \lambda(t), \tau\big)\\ &= \sum_{i=1}^{p} \tau_i\big[f_x^i\big(t, y(t), \dot y(t)\big) + B_i(t)\omega(t)\big] + \lambda(t)^T g_x\big(t, y(t), \dot y(t)\big)\\ &\quad - D\left\{\sum_{i=1}^{p} \tau_i f_{\dot x}^i\big(t, y(t), \dot y(t)\big) + \lambda(t)^T g_{\dot x}\big(t, y(t), \dot y(t)\big)\right\},\\ &\quad \text{with } \ddot y = D^2 y(t).\end{aligned}$$

Consider $\theta(\cdot, y(\cdot), \dot y(\cdot), \ddot y(\cdot), \lambda(\cdot), \tau)$ as defining a map $\phi : Z \times W \times R^p \mapsto A$, where W is the space of piecewise differentiable function $\lambda : I \mapsto R^m$ and A is Banach space.

Theorem 5 (Converse Duality). *Let* $(y^0, \lambda^0, \omega^0, \tau^0)$ *be a efficient solution for* $(\text{MDP})_1$. *Assume that*

(i) *the Frechet derivative* ϕ' *have a* (*weak**) *closed range,*
(ii) f *and* g *be twice continuously differentiable,*
(iii) $f_x^i + B_i\omega - Df_{\dot x}^i$, $i \in P$, *is linearly independent, and*
(iv) $(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot x} + D^2\beta(t)^T\theta_{\ddot x})\beta(t) = 0 \Rightarrow \beta(t) = 0, t \in I$.

Further, if the assumptions of Theorem 3 *are satisfied, then* y^0 *is an efficient solution of* (MP).

Proof. Since $(y^0, \lambda^0, \omega^0, \tau^0)$, with $y^0 \in Z$ and ϕ' having a (weak*) closed range, is an efficient solution, there exist $\xi \in R$, $\gamma \in R$, $\delta \in R$, $\varepsilon \in R^p$ and piecewise smooth functions $\beta : I \mapsto R^n$ and $\mu : I \mapsto R^m$, satisfying the following Fritz John conditions.

$$\begin{aligned}&\big(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot x} + D^2\beta(t)^T\theta_{\ddot x}\big) + \delta\big[\lambda^0(t)^T g_x\big(t, y^0(t), \dot y^0(t)\big)\\ &\quad - D\lambda^0(t)^T g_{\dot x}\big(t, y^0(t), \dot y^0(t)\big)\big] + \gamma\sum_{i=1}^{p}\tau_i\big\{\big[f_x^i\big(t, y^0(t), \dot y^0(t)\big)\\ &\quad + B_i(t)\omega^0(t)\big] - Df_{\dot x}^i\big(t, y^0(t), \dot y^0(t)\big)\big\} = 0, \qquad (16)\end{aligned}$$

$$\begin{aligned}&\beta(t)^T\big\{\big[f_x^i\big(t, y^0(t), \dot y^0(t)\big) + B_i(t)\omega^0(t)\big]\\ &\quad - Df_{\dot x}^i\big(t, y^0(t), \dot y^0(t)\big)\big\} + \varepsilon = 0, \qquad (17)\end{aligned}$$

$$\begin{aligned}&\beta(t)^T\big[g_x\big(t, y^0(t), \dot y^0(t)\big) - Dg_{\dot x}\big(t, y^0(t), \dot y^0(t)\big)\big]\\ &\quad + \delta g\big(t, y^0(t), \dot y^0(t)\big) + \mu = 0, \qquad (18)\end{aligned}$$

$$\delta\big(B_i(t)y^0(t)\big) - \beta(t)^T B_i(t) - 2\xi\big(B_i(t)\omega^0(t)\big) = 0, \qquad (19)$$

$$\delta\lambda^0(t)^T g\big(t, y^0(t), \dot y^0(t)\big) = 0, \tag{20}$$

$$\xi\big(\omega^0(t)^T B_i(t)\omega^0(t) - 1\big) = 0, \tag{21}$$

$$\varepsilon \sum_{i=1}^{p} \tau_i = 0, \tag{22}$$

$$\mu^T \lambda^0(t) = 0, \tag{23}$$

$$(\beta, \gamma, \delta, \varepsilon, \xi, \mu) \geqslant 0 \quad \text{and} \quad (\beta, \gamma, \delta, \varepsilon, \xi, \mu) \neq 0. \tag{24}$$

By feasibility of $(y^0, \lambda^0, \omega^0, \tau^0)$, from (16), we get

$$(\gamma - \delta)\sum_{i=1}^{p} \tau_i\big\{\big[f_x^i\big(t, y^0(t), \dot y^0(t)\big) + B_i(t)\omega^0(t)\big] - Df_{\dot x}^i\big(t, y^0(t), \dot y^0(t)\big)\big\} + \big(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot x} + D^2\beta(t)^T\theta_{\ddot x}\big) = 0. \tag{25}$$

Multiplying (17) by τ_i, $i \in P$, and using (22) we have

$$\sum_{i=1}^{p} \tau_i\big\{\big[f_x^i\big(t, y^0(t), \dot y^0(t)\big) + B_i(t)\omega^0(t)\big] - Df_{\dot x}^i\big(t, y^0(t), \dot y^0(t)\big)\big\}\beta(t) = 0.$$

Multiplying (25) by $\beta(t)$ and using the above equation, (25) becomes

$$\big(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot x} + D^2\beta(t)^T\theta_{\ddot x}\big)\beta(t) = 0,$$

which along with assumption (iv) gives

$$\beta(t) = 0. \tag{26}$$

Eqs. (25) and (26) now yield

$$(\gamma - \delta)\sum_{i=1}^{p} \tau_i\big\{\big[f_x^i\big(t, y^0(t), \dot y^0(t)\big) + B_i(t)\omega^0(t)\big] - Df_{\dot x}^i\big(t, y^0(t), \dot y^0(t)\big)\big\} = 0,$$

which along with assumption (iii) and $\tau_i \geqslant 0$, $\sum_{i=1}^{p} \tau_i = 1$, yields

$$\gamma = \delta.$$

We claim that $\gamma = \delta > 0$. If $\gamma = \delta = 0$, the from (17) and (18) we have $\varepsilon = \mu = 0$, and $\xi = 0$ from (18). Thus $(\beta, \gamma, \delta, \varepsilon, \xi, \mu) = 0$, which contradicts (24). Therefore from (19), y^0 is feasible for (MP).

From (20) and $\delta > 0$, we have

$$\lambda^0(t)^T g\big(t, y^0(t), \dot y^0(t)\big) = 0. \tag{27}$$

Also, $\beta(t) = 0$, $\delta > 0$ and (19) give

$$B_i(t)y^0(t) = (2\xi/\delta)\big(B_i(t)\omega^0(t)\big). \tag{28}$$

Hence

$$\big(y^0(t)^T B_i(t)\omega^0(t)\big) = \big(y^0(t)^T B_i(t) y^0(t)\big)^{1/2}\big(\omega^0(t)^T B_i(t)\omega^0(t)\big)^{1/2}. \quad (29)$$

If $\xi > 0$, then (21) gives $\omega^0(t)^T B_i(t)\omega^0(t) = 1$ and so (29) yields

$$\big(y^0(t)^T B_i(t)\omega^0(t)\big) = \big(y^0(t)^T B_i(t) y^0(t)\big)^{1/2}.$$

If $\xi = 0$, then (28) gives $B_i(t)y^0(t) = 0$. So we still get

$$\big(y^0(t)^T B_i(t)\omega^0(t)\big) = \big(y^0(t)^T B_i(t) y^0(t)\big)^{1/2}.$$

Thus in either case, we obtain

$$\big(y^0(t)^T B_i(t)\omega^0(t)\big) = \big(y^0(t)^T B_i(t) y^0(t)\big)^{1/2}. \quad (30)$$

Therefore from (27) and (30), we have

$$\begin{aligned}
&\int_a^b \big[f^i\big(t, y^0(t), \dot{y}^0(t)\big) + \big(y^0(t)^T B_i(t) y^0(t)\big)^{1/2}\big]\,dt \\
&\quad = \int_a^b \big[f^i\big(t, y^0(t), \dot{y}^0(t)\big) + y^0(t)^T B_i(t)\omega^0(t) \\
&\qquad + \lambda^0(t)^T g\big(t, y^0(t), \dot{y}^0(t)\big)\big]\,dt,
\end{aligned}$$

and, by Corollary 1, $y^0(t)$ is efficient for (MP). □

5. Mond–Weir vector duality

In this section, we establish various duality theorems for the following Mond–Weir [10] vector dual problem:

$$\begin{aligned}
(\mathrm{MDP})_2\ \text{Maximize}\ &\left(\int_a^b \big[f^1\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_1(t)\omega(t)\big]\,dt, \ldots,\right. \\
&\left.\int_a^b \big[f^p\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_p(t)\omega(t)\big]\,dt\right) \\
\text{subject to}\ \ & y(a) = \alpha, \quad y(b) = \beta \\
&\sum_{i=1}^p \tau_i\big[f_x^i\big(t, y(t), \dot{y}(t)\big) + B_i(t)\omega(t)\big] + \lambda(t)^T g_x\big(t, y(t), \dot{y}(t)\big) \\
&\quad = D\left\{\sum_{i=1}^p \tau_i f_{\dot{x}}^i\big(t, y(t), \dot{y}(t)\big) + \lambda(t)^T g_{\dot{x}}\big(t, y(t), \dot{y}(t)\big)\right\}, \quad (31)
\end{aligned}$$

$$\lambda(t)^T g\big(t, y(t), \dot{y}(t)\big) \geqslant 0, \tag{32}$$

$$\omega^T B_i \omega \leqslant 1, \quad i \in P, \tag{33}$$

$$\lambda(t) \geqslant 0, \quad \tau_i \geqslant 0, \quad \sum_{i=1}^{p} \tau_i = 1, \tag{34}$$

$$y \in C(I, R^n), \quad \omega \in C(I, R^n), \quad \lambda \in C(I, R^m).$$

Theorem 6 (Weak Duality). *Assume that for all feasible x for* (MP) *and all feasible $(y, \lambda, \omega, \tau)$ for* $(\mathrm{MDP})_2$*, either*

(i) $\tau_i > 0$, $\sum_{i=1}^{p} \tau_i [f^i(t, \cdot, \cdot) + (\cdot)^T B_i \omega]$ *is (F, ρ_1)-pseudoconvex, for all $i \in P$, $\lambda(t)^T g(t, \cdot, \cdot)$ is (F, ρ_2)-quasiconvex, and $\rho_1 + \rho_2 \geqslant 0$; or*

(ii) $\sum_{i=1}^{p} \tau_i [f^i(t, \cdot, \cdot) + (\cdot)^T B_i \omega]$ *is strictly (F, ρ_1)-pseudoconvex, for all $i \in P$, $\lambda(t)^T g(t, \cdot, \cdot)$ is (F, ρ_2)-quasiconvex, and $\rho_1 + \rho_2 \geqslant 0$.*

Then the following cannot hold:

$$\int_a^b \big[f^i\big(t, x(t), \dot{x}(t)\big) + \big(x(t)^T B_i(t) x(t)\big)^{1/2}\big]\, dt \leqslant \int_a^b \big[f^i\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_i(t) \omega(t)\big]\, dt, \tag{35}$$

for all $i \in P$ and

$$\int_a^b \big[f^j\big(t, x(t), \dot{x}(t)\big) + \big(x(t)^T B_j(t) x(t)\big)^{1/2}\big]\, dt < \int_a^b \big[f^j\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_j(t) \omega(t)\big]\, dt, \tag{36}$$

for some $j \in P$.

Proof. Suppose, contrary to the result, that (35) and (36) hold. With Lemma 3, we have

$$\int_a^b \big[f^i\big(t, x(t), \dot{x}(t)\big) + x(t)^T B_i(t) \omega(t)\big]\, dt \leqslant \int_a^b \big[f^i\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_i(t) \omega(t)\big]\, dt, \tag{37}$$

for all $i \in P$ and

$$\int_a^b \left[f^j\left(t, x(t), \dot{x}(t)\right) + x(t)^T B_j(t)\omega(t)\right] dt$$
$$\leqslant \int_a^b \left[f^j\left(t, y(t), \dot{y}(t)\right) + y(t)^T B_j(t)\omega(t)\right] dt, \tag{38}$$

for some $j \in P$, respectively. Now assumption (i) $\tau_i \geqslant 0$ and $\sum_{i=1}^p \tau_i = 1$, (37) and (38) imply

$$\sum_{i=1}^p \tau_i \int_a^b \left[f^i\left(t, x(t), \dot{x}(t)\right) + x(t)^T B_i(t)\omega(t)\right] dt$$
$$< \sum_{i=1}^p \tau_i \int_a^b \left[f^i\left(t, y(t), \dot{y}(t)\right) + y(t)^T B_i(t)\omega(t)\right] dt. \tag{39}$$

Since (i) $\sum_{i=1}^p \tau_i[f^i(t,\cdot,\cdot) + (\cdot)^T B_i\omega]$ is (F, ρ_1)-pseudoconvex, for all $i \in P$, we get from (39)

$$\int_a^b F\Biggl(t, x(t), \dot{x}(t), y(t), \dot{y}(t); \sum_{i=1}^p \tau_i\left[f_x^i\left(t, y(t), \dot{y}(t)\right) + B_i(t)w(t)\right]$$
$$- \frac{d}{dt}\sum_{i=1}^p \tau_i f_{\dot{x}}^i\left(t, y(t), \dot{y}(t)\right)\Biggr) dt < -\rho_1 \int_a^b d^2\left(t, x(t), y(t)\right) dt. \tag{40}$$

As x is feasible for (MP) and $(y, \lambda, \omega, \tau)$ is feasible for $(\mathrm{MDP})_2$. Then in view of $\lambda \geqslant 0$, we have that

$$\int_a^b \lambda(t)^T g\left(t, x(t), \dot{x}(t)\right) dt \leqslant \int_a^b \lambda(t)^T g\left(t, y(t), \dot{y}(t)\right) dt.$$

Since $\lambda(t)^T g(t,\cdot,\cdot)$ is (F, ρ_2)-quasiconvex, this implies

$$\int_a^b F\Biggl(t, x(t), \dot{x}(t), y(t), \dot{y}(t); \lambda(t)^T g_x\left(t, y(t), \dot{y}(t)\right)$$
$$- \frac{d}{dt}\lambda(t)^T g_{\dot{x}}\left(t, y(t), \dot{y}(t)\right)\Biggr) dt \leqslant -\rho_2 \int_a^b d^2\left(t, x(t), y(t)\right) dt. \tag{41}$$

From (31), (41) and $\rho_1 + \rho_2 \geqslant 0$, we have

$$\int_a^b F\Bigg(t, x(t), \dot{x}(t), y(t), \dot{y}(t); \sum_{i=1}^p \tau_i \big[f_x^i\big(t, y(t), \dot{y}(t)\big) + B_i(t)w(t)\big] - \frac{d}{dt}\sum_{i=1}^p \tau_i f_{\dot{x}}^i\big(t, y(t), \dot{y}(t)\big)\Bigg)\,dt \geqslant -\rho_1 \int_a^b d^2\big(t, x(t), y(t)\big)\,dt, \tag{42}$$

which is a contradiction to (40). Hence, the result follows.

If the assumption (ii) holds, since $\tau_i \geqslant 0$, for all $i \in P$, and $\sum_{i=1}^p \tau_i = 1$, (37) and (38) imply that

$$\sum_{i=1}^p \tau_i \int_a^b \big[f^i\big(t, x(t), \dot{x}(t)\big) + \big(x(t)^T B_i(t) x(t)\big)\big]\,dt \leqslant \sum_{i=1}^p \tau_i \int_a^b \big[f^i\big(t, y(t), \dot{y}(t)\big) + y(t)^T B_i(t)\omega(t)\big]\,dt,$$

and, then we have a contradiction to (42). Hence, the proof is complete. □

By the similar method of Corollary 1, the following corollary can be proved.

Corollary 2. *Let* $(y^0, \lambda^0, \omega^0, \tau^0)$ *be a feasible solution for* $(\mathrm{MDP})_2$ *such that*

$$\int_a^b \big(y^0(t)^T B_i(t) y^0(t)\big)^{1/2}\,dt = \int_a^b y^0(t)^T B_i(t)\omega^0(t)\,dt,$$

for each $i \in P$ *and assume that* y^0 *is feasible for (MP). If weak duality holds between* (MP) *and* $(\mathrm{MDP})_2$ *then* y^0 *is efficient for* (MP) *and* $(y^0, \lambda^0, \omega^0, \tau^0)$ *is efficient for* $(\mathrm{MDP})_2$.

The following duality theorem can be proved along the lines of Theorem 4.

Theorem 7 (Strong Duality). *Let* x^0 *be a feasible solution for* (MP) *and assume that*

(i) x^0 *is an efficient solution*;
(ii) *for at least one* i, $i \in P$, x^0 *satisfies a constraint qualification* [11] *for problem* $(P_i(x^0))$.

Then there exists $\tau^0 \in R^p$, $\lambda^0 \in R^m$, $(x^0, \lambda^0, \omega^0, \tau^0)$ *is feasible for* $(\mathrm{MDP})_2$.

Further, if the assumption of Theorem 6 are satisfied, then $(x^0, \lambda^0, \omega^0, \tau^0)$ is efficient for $(\mathrm{MDP})_2$.

$(\mathrm{MDP})_2$ may be rewritten in the following form:

$$\text{Minimize}\ \left(-\int_a^b \left[f^1\big(t, y(t), \dot y(t)\big) + y(t)^T B_1(t)\omega(t)\right] dt, \ldots, \right.$$
$$\left. -\int_a^b \left[f^p\big(t, y(t), \dot y(t)\big) + y(t)^T B_p(t)\omega(t)\right] dt\right)$$
$$\begin{aligned}\text{subject to}\ & y(a) = \alpha, \quad y(b) = \beta,\\ & \theta\big(t, y(t), \dot y(t), \ddot y(t), \lambda(t), \tau\big) = 0,\\ & \lambda(t)^T g\big(t, y(t), \dot y(t)\big) \geqslant 0,\\ & \omega^T B_i \omega \leqslant 1, \quad i \in P,\\ & \lambda(t) \geqslant 0, \quad \tau_i \geqslant 0, \quad \sum_{i=1}^p \tau_i = 1,\end{aligned}$$

where

$$\begin{aligned}\theta &= \theta\big(t, y(t), \dot y(t), \ddot y(t), \lambda(t), \tau\big)\\ &= \sum_{i=1}^p \tau_i \left[f_x^i\big(t, y(t), \dot y(t)\big) + B_i(t)\omega(t)\right] + \lambda(t)^T g_x\big(t, y(t), \dot y(t)\big)\\ &\quad - D\left\{\sum_{i=1}^p \tau_i f_{\dot x}^i\big(t, y(t), \dot y(t)\big) + \lambda(t)^T g_{\dot x}\big(t, y(t), \dot y(t)\big)\right\},\\ &\qquad \text{with } \ddot y = D^2 y(t).\end{aligned}$$

Consider $\theta(\cdot, y(\cdot), \dot y(\cdot), \ddot y(\cdot), \lambda(\cdot), \tau)$ as defining a map $\phi : Z \times W \times R^p \mapsto A$, where W is the space of piecewise differentiable function $\lambda : I \mapsto R^m$ and A is Banach space. A converse duality theorem may be stated: the proof would be analogous to that of Theorem 5.

Theorem 8 (Converse Duality). *Let* $(y^0, \lambda^0, \omega^0, \tau^0)$ *be a efficient solution for* $(\mathrm{MDP})_2$. *Assume that*

(i) *the Frechet derivative* ϕ' *have a* (*weak**) *closed range,*
(ii) f *and* g *be twice continuously differentiable,*
(iii) $f_x^i + B_i\omega - Df_{\dot x}^i$, $i \in P$, *is linearly independent, and*
(iv) $(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot x} + D^2\beta(t)^T\theta_{\ddot x})\beta(t) = 0 \Rightarrow \beta(t) = 0$, $t \in I$.

Further, if the assumptions of Theorem 6 *are satisfied then* y^0 *is an efficient solution of* (MP).

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