The Markov–Bernstein inequality and Hermite–Fejér interpolation for exponential-type weights

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Abstract

We investigate the coefficients of Hermite–Fejér interpolation polynomials based at zeros of orthogonal polynomials with respect to exponential-type weights. First, we obtain the modified Markov–Bernstein inequalities with respect to $w \in \mathcal{F}(\text{Lip}^{1/2})$. Then using the modified Markov–Bernstein inequalities, we estimate the value of $\left| p_n^{(r)}(w_\rho^2, x)/p_n'(w_\rho^2, x) \right|$ for $r = 1, 2, \ldots$ at zeros of $p_n(w_\rho^2; x)$ and we apply this to estimate the coefficients of Hermite–Fejér interpolation polynomials. Here, $p_n(w_\rho^2, x)$ denotes the $n$th orthogonal polynomial with respect to an exponential-type weight $w_\rho(x) = |x|\rho w(x), x \in \mathbb{R}, \rho > -1/2$. © 2010 Elsevier Inc. All rights reserved.

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1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$. Let $Q \in C^2 : \mathbb{R} \rightarrow \mathbb{R}^+ = [0, \infty)$ be an even function and $w(x) = \exp(-Q(x))$ be such that $\int_{0}^{\infty} x^n w(x) \, dx < \infty$ for all $n = 0, 1, 2, \ldots$. For $\rho > -1/2$, we set

$$w_\rho(x) := |x|^\rho w(x), \quad x \in \mathbb{R}.$$
Then we can construct the orthonormal polynomials $p_{n,\rho}(x) = p_n(w_\rho^2; x)$ of degree $n$ with respect to $w_\rho^2(x)$. That is,

$$\int_{-\infty}^{\infty} p_{n,\rho}(x) p_{m,\rho}(x) w_\rho^2(x) dx = \delta_{mn} \text{(Kronecker’s delta)}$$

and

$$p_{n,\rho}(x) = \gamma_n x^n + \cdots, \quad \gamma_n = \gamma_{n,\rho} > 0.$$ 

We denote the zeros of $p_{n,\rho}(x)$ by

$$-\infty < x_{n,\rho} < x_{n-1,\rho} < \cdots < x_{2,\rho} < x_{1,\rho} < \infty.$$ 

For $f \in C(\mathbb{R})$ we define the Hermite–Fejér interpolation polynomial $L_n(v, f; x)$ based at the zeros $\{x_{k,n}\}_{k=1}^n$ as follows:

$$L_n(v, f; x_{k,n}) = f(x_{k,n,\rho}), \quad L_n^{(i)}(v, f; x_{k,n}) = 0,$$

for $i = 1, 2, \ldots, v-1$. In particular, we let $L_n(1, f; x)$ be the Lagrange interpolation polynomial based at the zeros $\{x_{k,n}\}_{k=1}^n$. Moreover, $L_n(2, f; x)$ is the ordinary Hermite–Fejér interpolation polynomial and $L_n(4, f; x)$ is the Krilov–Stayermann polynomial. Then we have by (1.1)

$$L_n(v, f; x) = \sum_{k=1}^n f(x_{k,n,\rho}) h_{k,n,\rho}(v; x)$$

where the fundamental polynomials $h_{k,n,\rho}(v; x)$ for the Hermite–Fejér interpolation polynomial $L_n(v, f; x)$:

$$h_{k,n,\rho}(v; x) = l_{k,n,\rho}^v(x) \sum_{i=0}^{v-1} e_i(v, k, n)(x - x_{k,n,\rho})^i, \quad k = 1, 2, \ldots, n$$

are unique polynomials of degree exactly $v n - 1$ satisfying that for $k, p = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, v-1$

$$h_{k,n,\rho}(v; x_{p,n,\rho}) = \delta_{k,p} \quad \text{and} \quad h_{k,n,\rho}^{(i)}(v; x_{p,n,\rho}) = 0.$$ 

Here, $l_{k,n,\rho}(x)$ is the fundamental Lagrange interpolation polynomial of degree $n - 1$ (cf. [2, p. 23]) given by

$$l_{k,n,\rho}(x) = \frac{p_n(w_\rho^2; x)}{(x - x_{k,n,\rho}) p'_n(w_\rho^2; x_{k,n,\rho})}$$

and $l_{k,n,\rho}^v(x)$ is the $v$th power of $l_{k,n,\rho}(x)$. Furthermore, we extend the operator $L_n(v, f; \cdot)$. Let $l$ be a non-negative integer and let $v - 1 \geq l$. For $f \in C^{(l)}(\mathbb{R})$ we define the $(l, v)$-order Hermite–Fejér interpolation polynomial $L_n(l, v, f; x) \in \mathcal{P}_{vn-1}$ as follows: For each $k = 1, 2, \ldots, n$,

$$L_n(l, v, f; x_{k,n,\rho}) = f(x_{k,n,\rho}),$$

$$L_n^{(j)}(l, v, f; x_{k,n,\rho}) = f^{(j)}(x_{k,n,\rho}), \quad j = 1, 2, \ldots, l,$$

$$L_n^{(j)}(l, v, f; x_{k,n,\rho}) = 0, \quad j = l + 1, l + 2, \ldots, v - 1.$$
In particular, $L_n(0, v, f; x)$ is equal to $L_n(v, f; x)$. The fundamental polynomials $h_{s,k,n,\rho}(v; x)$, $k = 1, 2, \ldots, n$, of $L_n(l, v, f; x)$:

$$h_{s,k,n,\rho}(l, v; x) = l^v_{k,n,\rho}(x) \sum_{i=s}^{v-1} e_{s,i}(l, v, k, n)(x - x_{k,n,\rho})^i$$

are unique polynomials of degree exactly $vn - 1$ satisfying that for $j, s = 0, 1, \ldots, v - 1$ and $p = 1, 2, \ldots, n$,

$$h_{s,k,n,\rho}^{(j)}(l, v; x_{p,n,\rho}) = \delta_{s,j}\delta_{k,p}.$$  

Then we have

$$L_n(l, v, f; x) = \sum_{k=1}^{n} \sum_{s=0}^{l} f^{(s)}(x_{k,n,\rho})h_{s,k,n,\rho}(l, v; x).$$

We note that for any polynomial $P$ with degree $\leq vn - 1$ we have $L_n(v - 1, v, P; x) = P(x)$ and since $L_n(l, v, f; x) = 1$ for $f(x) = 1$, we see that

$$\sum_{k=1}^{n} h_{0,k,n,\rho}(l, v; x) = 1.$$  

In this paper, our main concern is to estimate the coefficients $e_i(v, k, n)$ and $e_{s,i}(l, v, k, n)$ of $L_n(v, f; x)$ or $L_n(l, v, f; x)$. The expression for the usual Hermite–Fejér interpolation $L_n(0, 2, f; x)$ (cf. [11, p. 330]) is given by

$$L_n(0, 2, f; x) = L_n(2, f; x) = \sum_{k=1}^{n} f(x_{k,n,\rho}) \left(1 - \frac{p''_{n,\rho}(x_{k,n,\rho})}{p'_{n,\rho}(x_{k,n,\rho})}(x - x_{k,n,\rho})\right) l^2_{k,n,\rho}(x).$$

So we know that $e_{0,0}(0, 2, k, n) = 1$ and

$$e_{0,1}(0, 2, k, n) = -\frac{p''_{n,\rho}(x_{k,n,\rho})}{p'_{n,\rho}(x_{k,n,\rho})}.$$  

To estimate the coefficients for Hermite–Fejér interpolation in the general case, we will estimate the upper bound of

$$\left|\frac{p''_{n,\rho}(x_{k,n,\rho})}{p'_{n,\rho}(x_{k,n,\rho})}\right|$$

using the Markov–Bernstein inequality for restricted ranges.

This paper is organized as follows. In Section 2, we introduce our class of weights and state the main results. In Section 3, we prove the Markov–Bernstein inequality for restricted ranges. We prove the results of Section 2 in Section 4. Finally the Appendix contains various estimates and known theorems from [3,4].

2. Preliminaries and theorems

A function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$ for $0 < x < y$. For any two sequences $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ of nonzero real numbers (or functions), we write $b_n \lesssim c_n$ if there exists a constant $C > 0$ independent of $n$ (or $x$).
such that \( b_n \leq Cc_n \) for \( n \) large enough. We write \( b_n \sim c_n \) if \( b_n \lesssim c_n \) and \( c_n \lesssim b_n \). We denote the class of polynomials of degree at most \( n \) by \( \mathcal{P}_n \).

Throughout, \( C, C_1, C_2, \ldots \) denote positive constants independent of \( n, x, t, \) and polynomials of degree at most \( n \). The same symbol does not necessarily denote the same constant at different occurrences.

We shall be interested in the following subclass of weights from \([6]\).

**Definition 2.1.** Let \( Q : \mathbb{R} \to \mathbb{R}^+ \) be even and satisfy the following properties:

(a) \( Q'(x) \) is continuous in \( \mathbb{R} \), with \( Q(0) = 0 \).

(b) \( Q''(x) \) exists and is positive in \( \mathbb{R} \setminus \{0\} \).

(c) \( \lim_{x \to \infty} Q(x) = \infty \).

(d) The function
\[
T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0
\]

is quasi-increasing in \((0, \infty)\) with
\[
T(x) \geq A > 1, \quad x \in \mathbb{R}^+ \setminus \{0\}.
\]

(e) There exists \( C_1 > 0 \) such that
\[
\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e.} \ x \in \mathbb{R} \setminus \{0\}.
\]

Then we write \( w \in \mathcal{F}(C^2) \). If there also exist a compact subinterval \( J(\ni 0) \) of \( \mathbb{R} \) and \( C_2 > 0 \) such that
\[
\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e.} \ x \in \mathbb{R} \setminus J,
\]

then we write \( w \in \mathcal{F}(C^2+) \).

Now we will consider some typical examples of \( \mathcal{F}(C^2+) \):

**Example 2.2.** (a) If an exponential \( Q(x) \) satisfies
\[
1 < A_1 \leq \frac{(xQ'(x))'}{Q'(x)} \leq A_2
\]

where \( A_i, i = 1, 2, \) are constants, then we call \( w(x) = \exp(-Q(x)) \) the Freud weight. The class \( \mathcal{F}(C^2+) \) contains the Freud weights.

(b) Define, for \( \alpha + m > 1, m \geq 0, l \geq 1 \) and \( \alpha \geq 0 \),
\[
Q_{l,\alpha,m}(x) := |x|^m(\exp(l|x|^\alpha) - \alpha^* \exp(0))
\]

where \( \alpha^* = 0 \) if \( \alpha = 0 \), and otherwise \( \alpha^* = 1 \).

(c) Define
\[
Q_\alpha(x) := (1 + |x|)^{l|x|^\alpha} - 1, \quad \alpha > 1.
\]

Here we let \( \exp_0(x) := x \) and for \( l \geq 1, \exp_l(x) := \exp(\exp(\ldots(\exp(x))\ldots)) \) denotes the \( l \)-th iterated exponential. In particular, \( \exp_l(x) = \exp(\exp_{l-1}(x)) \).
Kanjin and Sakai [5] obtained the following result for Freud weights $w_\alpha(x) = \exp(-x^\alpha/2)$.

**Theorem 2.3** ([5]). Let $w_\alpha(x) = \exp(-x^\alpha/2)$, $\alpha = 2, 4, \ldots$ and $r = 1, 2, \ldots$. Then

$$\left| \frac{p_n^{(r)}(w^2_\alpha, x_{kn})}{p_n'(w^2_\alpha, x_{kn})} \right| \lesssim M_n(x_{kn})^{1-(r)} q_n^{(r-2)+(\alpha-1)}$$

for $k = 1, 2, \ldots, n$, where

$$\langle s \rangle := \begin{cases} 1, & s \text{ is odd} \\ 0, & s \text{ is even} \end{cases}, \quad M_n(x) = \max\{|x| q_n^{-2}, |x|^\alpha - 1\}.$$ \quad and \quad $q_n = \left(\frac{2n}{\alpha}\right)^{1/\alpha}$.

Lubinsky and Rabinowits [7,8] proved for some admissible weight class $\mathcal{W}_1$ on $\mathbb{R}$ the following:

**Theorem 2.4** ([7,8]). For $k = 1, 2, \ldots, n$,

$$\left| \frac{p_n''(w^2_\alpha, x_{kn})}{p_n'(w^2_\alpha, x_{kn})} \right| \lesssim \left( Q'(x_{kn}) + 1 \right).$$

Theorems 2.3 and 2.4 are proved by using the differential equations for $p_n(w^2_\alpha, x)$ (cf. [1,9]). But in general, a differential equation of higher order for $p_n(w^2_\alpha, x)$ with respect to some general weight class has a complicated expression. Hence, we apply Markov–Bernstein inequalities instead of differential equations to prove the following main theorems.

**Theorem 2.5.** Let us have $w \in \mathcal{F}(C^2_+)$ and $r = 1, 2, \ldots$. Then uniformly for $1 \leq k \leq n$,

$$\left| \frac{l_{k,n,\rho}^{(r)}(x_{k,n,\rho})}{l_{k,n,\rho}'(x_{k,n,\rho})} \right| \lesssim \left( \frac{n}{\sqrt{a_{2n}^2 - x_{k,n,\rho}^2}} \right)^{r-1}.$$

**Theorem 2.6.** Let us have $w \in \mathcal{F}(C^2_+)$ and $n, \nu \geq 1$. Then uniformly for $1 \leq k \leq n$ and $0 < r \leq \nu - 1$,

$$\left| e_{s,i}(l, \nu, k, n) \right| \lesssim \left( \frac{n}{\sqrt{a_{2n}^2 - x_{k,n,\rho}^2}} \right)^{i-s}.$$

(2.1)

and for $0 \leq s \leq \nu - 1$ and $s \leq i \leq \nu - 1$,

$$\left| e_{s,i}(l, \nu, k, n) \right| \lesssim \left( \frac{n}{\sqrt{a_{2n}^2 - x_{k,n,\rho}^2}} \right)^{i-s}.$$

(2.2)

Here, $a_t$ for $t > 0$ is the Mhaskar–Rahmanov–Saff number (MRS) which is defined as the positive root of the following equation:

$$t = \frac{2}{\pi} \int_0^1 \frac{a_t u Q'(a_t u)}{(1 - u^2)^{1/2}} \, du.$$

(2.3)
3. The Markov–Bernstein inequality for restricted ranges

In this section, to derive the Markov–Bernstein inequalities for restricted ranges we introduce some definitions, notation and lemmas from [6]. Let

\[ I = (c, d) \]

where

\[-\infty \leq c < 0 < d \leq \infty.\]

**Definition 3.1** ([6, Definitions 1.2 and 1.4]). Let \( w = \exp(-Q) \) where \( Q : I \to [0, \infty) \) satisfy the following properties:

(a) \( Q' \) is continuous in \( I \) and \( Q(0) = 0. \)

(b) \( Q' \) is non-decreasing in \( I. \)

(c) \[ \lim_{t \to c^+} Q(t) = \lim_{t \to d^-} Q(t) = \infty. \]

(d) The function

\[ T(t) := \frac{tQ'(t)}{Q(t)}, \quad t \neq 0 \]

is quasi-increasing in \((0, d)\), and quasi-decreasing in \((c, 0)\), with

\[ T(t) \geq \Gamma > 1, \quad t \in I \setminus \{0\}. \]

(e) There exists \( \varepsilon_0 \in (0, 1) \) such that for \( y \in I \setminus \{0\}, \)

\[ T(y) \sim T \left( y \left[ 1 - \frac{\varepsilon_0}{T(y)} \right] \right). \]

Then we write \( w \in \mathcal{F}. \) Moreover, let \( w \in \mathcal{F} \) and assume that there exist \( C, \varepsilon_1 > 0 \) such that for all \( x \in I \setminus \{0\}, \)

\[ \int_{x - \varepsilon_1 |x| / T(x)}^{x} \frac{|Q'(s) - Q'(x)|}{|x - s|^{3/2}} ds \leq C|Q'(x)| \sqrt{T(x) / |x|}. \]

Then we write \( w \in \mathcal{F}(\text{Lip}^{1/2}). \)

**Notation 3.2** ([6]).

(a) The numbers \( a_{-t} < 0 < a_t, t > 0, \) are uniquely defined if \( w \in \mathcal{F} \) by the equations

\[ t = \frac{1}{\pi} \int_{a_{-t}}^{a_t} \frac{xQ'(x)}{\sqrt{(x - a_{-t})(a_t - x)}} dx; \]

\[ 0 = \frac{1}{\pi} \int_{a_{-t}}^{a_t} \frac{Q'(x)}{\sqrt{(x - a_{-t})(a_t - x)}} dx. \]

(b) Define

\[ \delta_t := \frac{1}{2} (a_t + |a_{-t}|), \quad t > 0. \]
and
\[ \eta_{\pm t} := \left[ r T(a_{\pm t}) \sqrt{|a_{\pm t}|} \right]^{-2/3}, \quad t > 0. \]

(c) Let \( \Delta_t := [a_{-t}, a_t] \) and \( |\Delta_t| := a_t - a_{-t} \). Then for \( w = \exp(-Q) \), where \( Q(x) \) is convex, the density function \( \sigma_t(x) \) is defined by
\[ \sigma_t(x) := \frac{\sqrt{(x - a_{-t})(a_t - x)}}{\pi^2} \int_{a_{-t}}^{a_t} \frac{Q'(s) - Q'(x)}{\sqrt{(s - a_{-t})(a_t - s)}} ds, \quad x \in \Delta_t. \]

(d) The potential \( V^{\sigma_t}(z) \) is defined by
\[ V^{\sigma_t}(z) := -\int \log |z - s| \sigma_t(s) ds. \]

(e) Define \( g_{\Delta_t}(z) \) for \( z \in \mathbb{C} \) and the constant \( c_t \) by
\[ g_{\Delta_t}(z) := \frac{1}{\pi} \int_{\Delta_t} \log |z - s| \frac{1}{\sqrt{(s - a_{-t})(a_t - s)}} ds + \log \frac{4}{|\Delta_t|} \]
and
\[ c_t := \int_0^t \log \frac{4}{|\Delta_t|} dt. \]

(f) Then we know from [6, p. 36, Example 2] that for \( z \in \mathbb{C} \),
\[ g_{\Delta_t}(z) = -V^{\gamma_{\Delta_t}}(z) + \log \frac{4}{|\Delta_t|} \] (3.1)
where
\[ \gamma_{\Delta_t}(s) := \frac{1}{\pi} \frac{1}{\sqrt{(s - a_{-t})(a_t - s)}}, \quad s \in \Delta_t. \] (3.2)

(g) The function \( \varphi_t(x) \) is defined as follows:
\[ \varphi_t(x) = \begin{cases} \frac{|x - a_{-2t}|}{t \sqrt{|x - a_{-t}| + |a_{-t}| \eta_{-t}|}}, & x \in [a_{-t}, a_t]; \\
\varphi_t(a_{-t}), & x \in (a_t, d); \\
\varphi_t(a_{-1}), & x \in (c, a_{-1}). \end{cases} \]

The following lemma for the potential \( V^{\sigma_t}(z) \) is proved in [6, Theorem 5.5].

**Lemma 3.3** ([6, Theorem 5.5]). Let us have \( w \in \mathcal{F}(\text{Lip}_{1/2}^2) \). Then
\[ V^{\sigma_t}(x + iy) - V^{\sigma_t}(x) = O(1) \]
uniformly for \( t > 0 \), for \( x \in \mathbb{R} \) and for \( y \) in the range \( |y| \leq \varphi_t(x) \).

The following lemma is valid under the mild assumptions on \( Q \):

**Lemma 3.4** ([6, Lemma 9.6]). Let \( w := \exp(-Q) \) where \( Q : I \rightarrow [0, \infty) \) is convex with \( Q(c+) = \infty = Q(d-) \) and \( Q(x) > 0 = Q(0), x \in I \setminus \{0\} \). Let \( s, t > 0 \) and \( P \in \mathcal{P}_3 \). Then we have for \( z \in \mathbb{C} \setminus \Delta_t \),
\[ |P(z)| \leq \exp \left( -V^{\sigma_t}(z) + c_t + (s - t)g_{\Delta_t}(z) \right) \|Pw\|_{L_{\infty}(\Delta_t)}. \]
For the proof of our Markov–Bernstein inequality for restricted ranges, we base consideration on the following lemma.

**Lemma 3.5 (Cf. [6, Lemma 10.4]).** Let us have \( w \in \mathcal{F}(\text{Lip}_2) \). Let us have \( \varepsilon \in (0, 1) \) and \( 0 < \alpha \leq 1 \). Then for \( n \geq 1 \), \( P \in P_n \), \( x \in I \) and for some constant \( C(\varepsilon) > 0 \),

\[
\left| (Pw)'(x) \right| \leq C(\varepsilon)\varphi_{an}^{-1}(x) \| Pw \|_{L_\infty(\Delta_{an})} \times \max_{|z-x|=\varepsilon\varphi_{an}(x)} \exp(U_{an}(Re(z)) + n(1 - \alpha)g_{\Delta_{an}}(z) + H(Re(z), x))
\]

where

\[
H(y, x) := Q(y) - Q(x) - Q'(x)(y - x)
\]

and

\[
U_t(x) := -[V^{\sigma_t} + Q - c_t](x).
\]

**Proof.** We define \( \hat{w}(t) := \exp(-\hat{Q}(t)) \) and

\[
\hat{Q}(t) := Q(x) + Q'(x)(t - x), \quad t \in \mathbb{C}.
\]

Then \( \hat{w}(t) \) is an entire function and

\[
\hat{w}^{(j)}(x) = w^{(j)}(x), \quad j = 0, 1.
\]

Then for \( P \in P_n \), by Cauchy’s integral formula,

\[
(Pw)'(x) = (P\hat{w})'(x) = \frac{1}{2\pi i} \int_{\Gamma} \frac{P\hat{w}(z)}{(z-x)^2} dz
\]

where \( \Gamma := \{ z \in \mathbb{C} \mid |z-x|=r \} \). Then by Lemma 3.4, (3.4) and (3.5)

\[
\left| (Pw)'(x) \right| \leq \frac{1}{r} \max_{z \in \Gamma} \left| P\hat{w}(z) \right|
\]

\[
\leq \frac{1}{r} \| Pw \|_{L_\infty(\Delta_{an})} \max_{|z-x|=r} \exp(-V^{\sigma_{an}}(z) + c_{an} + n(1 - \alpha)g_{\Delta_{an}}(z) - Q(x) - Q'(x)(Re(z) - x))
\]

\[
\leq \frac{1}{r} \| Pw \|_{L_\infty(\Delta_{an})} \max_{|z-x|=r} \exp(-[V^{\sigma_{an}}(z) - V^{\sigma_{an}}(Re(z))]
\]

\[
+ U_{an}(Re(z)) + n(1 - \alpha)g_{\Delta_{an}}(z) + H(Re(z), x)).
\]

If we choose \( r = \varepsilon\varphi_{an}(x) \), then Lemma 3.3 shows that the term \( [\cdot] \) is \( O(1) \) uniformly in \( n \), \( z \), \( x \in I \). Then (3.3) follows. \( \square \)

**Lemma 3.6 ([6, Theorem 1.11]).** Let us have \( w \in \mathcal{F}(\text{Lip}_2) \) and \( 0 < \alpha < 1 \). Then

\[
\sigma_t(x) \sim \frac{t}{\sqrt{(x - a_t)(a_t - x)}}, \quad x \in \Delta_{at}.
\]

The following are our Markov–Bernstein inequalities for restricted ranges:

**Theorem 3.7.** Let us have \( w \in \mathcal{F}(\text{Lip}_2) \) and \( 0 < \beta < \alpha < 1 \). Then for \( n \geq 1 \), \( P \in P_n \), and for \( x \in \Delta_{\beta n} \), there exists some constant \( C \neq C(x, n, P) \) such that

\[
\left| (Pw)'(x) \right| \leq C\varphi_{an}^{-1}(x) \| Pw \|_{L_\infty(\Delta_{an})}.
\]
Proof. Using Lemma 3.5 we will prove that
\[ \max_{|z-x|=\varepsilon\psi_{an}(x)} \exp(U_{an}(Re(z)) + n(1-\alpha)g_{\Delta_{an}}(z) + H(Re(z), x)) \leq C. \]
First, we know that \( U_{an} = 0 \) in \( \Delta_{an} \) (see [6, (2.14)]) and \( U_{an} \leq 0 \) in \( I \setminus \Delta_{an} \). Secondly, we know that from [6, p. 296, (10.11)]
\[ H(Re(z), x) \leq C, \quad x \in \Delta_{2an}, \quad |Re(z) - x| \leq \psi_{an}(x). \]
Next, we will show that
\[ n(1-\alpha)g_{\Delta_{an}}(z) \leq C. \]
Since by (A.3)
\[ \delta_{t}T(a_{t})/a_{t} \leq Ct^{2-\eta}, \]
we can choose \( 0 < \varepsilon < 1 \) satisfying that for \( x \in \Delta_{\beta n} \).
\[ x \pm \varepsilon \psi_{an}(x) \in \Delta_{an}. \]
Then since \( g_{\Delta_{i}} = 0 \) in \( \Delta_{i} \) (see [6, p. 36, Example 2]), and \( Re(z) \in \Delta_{an} \) for \( |z-x| = \varepsilon\psi_{an}(x) \),
\[ g_{\Delta_{an}}(z) = g_{\Delta_{an}}(z) - g_{\Delta_{an}}(Re(z)). \]
Then we can prove, by (3.1), (3.2) and Lemma 3.6,
\[ 0 \leq g_{\Delta_{an}}(z) - g_{\Delta_{an}}(Re(z)) = -(V^{\prime}g_{\Delta_{an}}(z) - V^{\prime}g_{\Delta_{an}}(Re(z))) \]
\[ \leq C \int_{\Delta_{an}} \log \left[ 1 + \left( \frac{y}{Re(z) - s} \right)^{2} \right] \frac{ds}{\sqrt{(s-a_{an})(a_{an}-s)}} \]
\[ \leq C \frac{|y|}{\sqrt{(Re(z) - a_{an})(a_{an} - Re(z))}} = O(1/n) \]
where \( iy := z - Re(z) \) and here we used that for \( x \in \Delta_{\beta n} \),
\[ |y| \leq \varepsilon \psi_{an}(x) \sim \sqrt{(Re(z) - a_{an})(a_{an} - Re(z))} / n, \]
(see [6, p. 130, the proof of Lemma 5.10(a)]). Therefore, (3.6) is proved. \( \square \)

Now, we consider \( w \in F(C^{2}) \). Then we see the following properties:
(a) \( I = (-\infty, \infty) \).
(b) \( a_{-t} = -a_{t}, t > 0 \), and \( a_{t} \) are the Mhaskar–Rahmanov–Saff numbers (MRS) which are the positive roots of (2.3).
(c) The function \( \varphi_{t}(x), t > 0 \), is
\[ \varphi_{t}(x) = \begin{cases} \frac{a_{t}^{2} - x^{2}}{t[(a_{t} + x)(a_{t} - a_{t}\eta_{t})]^{1/2}}, & |x| \leq a_{t}; \\ \varphi_{t}(a_{t}), & a_{t} < |x|. \end{cases} \] (3.7)
(d) The list of inclusions between the various classes of weights may be summarized as follows:
\[ F(C^{2}+) \subset F(C^{2}) \subset F\left(Lip \frac{1}{2}\right) \subset F \]
(see [6, (1.50)]).
\textbf{Theorem 3.8} (Cf. [3] and [6, Theorem 1.15 and Corollary 1.16]). Let us have \( w \in \mathcal{F}(C^2) \). Let us also have \( \rho \in \mathbb{R} \) and \( 0 < \beta < \alpha < 1 \). Then for \( n \geq 1 \), \( P \in \mathcal{P}_n \), and for \( x \in \Delta_{\beta n} \), there exists some constant \( C \neq C(x, n, P) \) such that

\[
\left| (Pw)'(x) \left( |x| + \frac{a_n}{n} \right)^\rho \varphi_{an}(x) \right| \leq \left\| (Pw)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L^\infty(\Delta_{an})}
\]

and

\[
\left| P'(x) \left( |x| + \frac{a_n}{n} \right)^\rho \varphi_{an}(x) \right| \leq \left\| (Pw)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L^\infty(\Delta_{an})}.
\]

\textbf{Proof.} Let \( S_{n, \rho}(x) \in \mathcal{P}_n \, (n \geq 2) \) be a polynomial satisfying that

\[
S_{n, \rho}(x) \sim \left( |x| + \frac{a_n}{n} \right)^\rho, \quad x \in (-a_n, a_n)
\]

and

\[
|S'_{n, \rho}(x)| \lesssim \left( |x| + \frac{a_n}{n} \right)^{\rho - 1}, \quad |x| \leq a_n
\]

(see [3, Lemma 3.1]). Then we have for \( |x| \leq a_{\beta n} \),

\[
\left| (Pw)'(x) \left( |x| + \frac{a_n}{n} \right)^\rho \varphi_{an}(x) \right| \lesssim \left| (PS_{n, \rho}w)'(x)\varphi_{an}(x) \right| + \left| (wS'_{n, \rho})(x)\varphi_{an}(x) \right|.
\]

Then since \( 0 < \beta/2 < \alpha/2 < 1 \) and \( PS_{n, \rho} \in \mathcal{P}_2n \), we have from \textbf{Theorem 3.7} for \( |x| \leq a_{\beta n} \),

\[
\left| (PS_{n, \rho}w)'(x)\varphi_{an}(x) \right| \lesssim \left\| PS_{n, \rho}w \right\|_{L^\infty(\Delta_{an})} \sim \left\| (Pw)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L^\infty(\Delta_{an})}.
\]

Moreover, we have, by \textbf{(A.6)},

\[
\left| (PwS'_{n, \rho})(x)\varphi_{an}(x) \right| \leq \left| (Pw)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right|.
\]

Therefore, we have, for \( |x| \leq a_{\beta n} \),

\[
\left| (Pw)'(x) \left( |x| + \frac{a_n}{n} \right)^\rho \varphi_{an}(x) \right| \lesssim \left\| (Pw)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L^\infty(\Delta_{an})}.
\]

On the other hand, since, from [6, Lemma 3.8(a)] and \textbf{(3.7)} for \( |x| \leq a_{\beta n} \),

\[
|Q'(x)| \leq C \frac{n}{a_n \sqrt{1 - |x|/a_n}} \quad \text{and} \quad \varphi_{an}(x) \lesssim \frac{a_n}{n} \left( 1 - \frac{x}{a_n} \right)^{1/2},
\]

we see that for \( |x| \leq a_{\beta n} \),

\[
|Q'(x)| \varphi_{an}(x) \lesssim 1.
\]

Therefore, we have for \( |x| \leq a_{\beta n} \),

\[
\left| P'w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \varphi_{an}(x) \right| \leq \left| (Pw)'(x) \left( |x| + \frac{a_n}{n} \right)^\rho \varphi_{an}(x) \right|
\]

\[
+ \left| Q'(x)(Pw)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \varphi_{an}(x) \right|
\]

\[
\lesssim \left\| (Pw)(x) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L^\infty(\Delta_{an})}.
\]
Lemma 3.9 ([10]). Let us have \( w \in \mathcal{F}(C^2) \). Fix \( \alpha = \gamma + j \) where \( 0 \leq \gamma < 1 \) and \( j = 0, 1, 2, \ldots \). Let

\[
u(x) := (1 - x^2)^{-\gamma}, \quad x \in [-1, 1]
\]

and

\[
R_n(x) := \frac{1}{n} \lambda_n^{-1} (u, x/a_{5n}) \left( 1 - (x/a_{5n})^2 \right)^j
\]

where \( \lambda_n(u, x) \) is the Christoffel function for \( u(x) \) and is represented in

\[
\lambda_n(u; x) := \inf_{p \in \mathcal{P}_{n-1}} \int_{-1}^{1} P^2(t) u(t) dt / P^2(x) = \frac{1}{\sum_{j=0}^{n-1} P_j^2(u; x)}.
\]

Then for large enough \( n \) and uniformly for \(|x| \leq a_{5n}(1 - n^{-2})\),

\[
R_n(x) \sim \left( 1 - (x/a_{5n})^2 \right)^{\alpha-1/2}
\]

and

\[
|R'_n(x)| \lesssim \frac{1}{a_{5n}} \left| 1 - (x/a_{5n})^2 \right|^{\alpha-3/2}.
\]

Theorem 3.10. Let us have \( w \in \mathcal{F}(C^2) \) and \( \alpha \geq 0 \) any fixed number. Let \( 0 < 4\beta_1 < 4\beta_2 < 1 \). Then for \(|x| \leq a_{4\beta_1n} \) and \( P \in \mathcal{P}_n \),

\[
\left| P'(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^\alpha \right| \\
\lesssim \frac{n}{a_n} \max_{[-a_{4\beta_2n}, a_{4\beta_2n}]} \left| P(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{\alpha-1/2} \right|.
\]

Proof. If we let \( R_n \) as in Lemma 3.9, then for \( P \in \mathcal{P}_n \), we have \( PR_n \in \mathcal{P}_{4n} \) for \( n \) large enough, because \( R_n \in \mathcal{P}_{2n+2j} \). Since \( 0 < 4\beta_1 < 4\beta_2 < 1 \), we have by (3.8) for \(|x| \leq a_{4\beta_1n} \),

\[
\left| P'(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^\alpha \right| \\
\sim \left| P'(x) R_n(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{1/2} \right| \\
\lesssim \left| (PR_n)' w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{1/2} \right| \\
+ \left| R'_n(x) \right| \left| P(x) w(x) \left( |x| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{1/2} \right|.
\]

Since we see from (3.7) for \(|x| \leq a_{4\beta_1n} \),

\[
\varphi_{4\beta_2n}(x) \sim \frac{a_n}{n} (1 - (x/a_{5n})^2)^{1/2},
\]
we have by Theorem 3.8, for $|x| \leq a_{4\beta_1n}$,
\[
\left| (PR_n)'(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\theta \left| 1 - (x/a_{5n})^2 \right|^{1/2} \right| 
\lesssim \frac{n}{a_n} \left| (PR_n)'(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\theta \varphi_{4\beta_2n}(x) \right| 
\lesssim \frac{n}{a_n} \max_{x \in [-a_{4\beta_2n}, a_{4\beta_2n}]} \left| P(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\rho \left( 1 - (x/a_{5n})^2 \right)^{\alpha - 1/2} \right|.
\]

On the other hand, since we see by Lemma A.1 for $|x| \leq a_{4\beta_1n}$,
\[
\left| 1 - (x/a_{5n})^2 \right|^{1/2} \lesssim T^{1/2}(a_n) \lesssim n,
\]
we have by (3.9),
\[
\left| R_n(x) \right| \left| P(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{1/2} \right| 
\lesssim \frac{1}{a_{5n}} \left| 1 - (x/a_{5n})^2 \right|^\alpha \left| P(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\rho \right| 
\lesssim \frac{n}{a_n} \max_{x \in [-a_{4\beta_2n}, a_{4\beta_2n}]} \left| P(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\rho \left( 1 - (x/a_{5n})^2 \right)^{\alpha - 1/2} \right|. \quad \square
\]

4. Proofs of Theorems 2.5 and 2.6

Proof of Theorem 2.5. Now, let $0 < 4\beta_1 < 4\beta_2 < \cdots < 4\beta_{r+1} < 1$. Then by Theorem 3.10,
\[
\max_{|x| \leq a_{4\beta_1n}} \left| p_{n,\rho}^{(r)}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{r/2+1/4} \right| 
\lesssim \frac{n}{a_n} \max_{|x| \leq a_{4\beta_2n}} \left| p_{n,\rho}^{(r-1)}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{(r-1)/2+1/4} \right|.
\]

Repeating the same processes $r$ times, we have by Theorem A.6,
\[
\max_{|x| \leq a_{4\beta_1n}} \left| p_{n,\rho}^{(r)}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{r/2+1/4} \right| 
\lesssim \left( \frac{n}{a_n} \right)^r \max_{|x| \leq a_{4\beta_{r+1}n}} \left| p_{n,\rho}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{1/4} \right| 
\lesssim \left( \frac{n}{a_n} \right)^r \max_{|x| \leq a_{4\beta_{r+1}n}} \left| p_{n,\rho}(x)w(x) \left( \left| x \right| + \frac{a_n}{n} \right)^\rho \left| 1 - (x/a_{5n})^2 \right|^{1/4} \right| 
\lesssim \left( \frac{n}{a_n} \right)^r a_n^{-1/2}
\]
since $1 - |x|/a_{5n} \sim 1 - |x|/a_{n}$ for $|x| \leq a_{4\beta_{r+1}n}$. Then for $|x_{k,n,\rho}| \leq a_{4\beta_1n}$,
\[
\left| p_{n,\rho}(x_{k,n,\rho})w(x_{k,n,\rho}) \left( \left| x_{k,n,\rho} \right| + \frac{a_n}{n} \right)^\rho \right| \lesssim \left( \frac{n}{a_n} \right)^r a_n^{-1/2} \left| 1 - \left( x_{k,n,\rho}/a_{5n} \right)^2 \right|^{-r/2-1/4} 
\lesssim \left( \frac{n}{a_n} \right)^r a_n^{-1/2} \left| 1 - \left( x_{k,n,\rho}/a_{5n} \right)^2 \right|^{-r/2-1/4}.
\]
Then since
\[ \varphi_n(x_{k,n,\rho}) \sim \frac{a_n}{n} (1 - (x_{k,n,\rho}/a_n)^2)^{1/2}, \]
we have by Theorem A.7 and Lemma A.1 that for \(|x_{k,n,\rho}| \leq a_4 \beta_1 n\),
\[
\begin{aligned}
\left| \frac{P_n^{(r)}(x_{k,n,\rho})}{P_n^{(r)}(x_{k,n,\rho})} \right| & = \frac{P_n^{(r)}(x_{k,n,\rho}) w(x_{k,n,\rho}) \left( |x_{k,n,\rho}| + \frac{a_n}{n} \right)^{\rho}}{P_n^{(r)}(x_{k,n,\rho}) w(x_{k,n,\rho}) \left( |x_{k,n,\rho}| + \frac{a_n}{n} \right)^{\rho}} \\
& \lesssim \left( \frac{n}{a_n} \right)^{r-1} \left( 1 - (x_{k,n,\rho}/a_n)^2 \right)^{-(r-1)/2} \left( \sqrt{\frac{n}{a_n^2} - \frac{x_{k,n,\rho}^2}{a_n^2}} \right)^{r-1}.
\end{aligned}
\]

Now consider \(|x_{k,n,\rho}| \geq a_4 \beta_1 n\). Since \(l_{k,n,\rho}^{(r-1)}(t) \in P_{2n-2}^{(r-1)} \subset P_{2n-1}\), we have by the Gauss quadrature formula
\[
\begin{aligned}
\int l_{k,n,\rho}^{(r-1)}(t) l_{k,n,\rho}(t) w_{\rho}^2(t) \, dt &= \sum_{j=1}^{n} \lambda_{j,n,\rho} l_{k,n,\rho}^{(r-1)}(x_j) l_{k,n,\rho}(x_j) \\
& = \lambda_{k,n,\rho} l_{k,n,\rho}^{(r-1)}(x_{k,n,\rho}) l_{k,n,\rho}(x_{k,n,\rho}) \\
& = \lambda_{k,n,\rho} l_{k,n,\rho}^{(r-1)}(x_{k,n,\rho})
\end{aligned}
\]
where the Christoffel numbers are \(\lambda_{k,n,\rho} := \lambda_n(w_{\rho}^2; x_{k,n,\rho})\). On the other hand, since
\[
p_n^{(r)}(x) = p_n^{(r)}(x_{k,n,\rho}) l_{k,n,\rho}(x) (x - x_{k,n,\rho}),
\]
by differentiating \(r + 1\) times we have
\[
p_n^{(r+1)}(x) = p_n^{(r)}(x_{k,n,\rho}) l_{k,n,\rho}^{(r+1)}(x) (x - x_{k,n,\rho}) + (r + 1) p_n^{(r)}(x_{k,n,\rho}) l_{k,n,\rho}^{(r)}(x).
\]
So we have by substituting \(x_{k,n,\rho}\) into \(x\),
\[
l_{k,n,\rho}^{(r)}(x_{k,n,\rho}) = \frac{p_n^{(r+1)}(x_{k,n,\rho})}{(r + 1) p_n^{(r)}(x_{k,n,\rho})}.
\]
Therefore, we have
\[
p_n^{(r)}(x_{k,n,\rho}) = \frac{rp_n^{(r)}(x_{k,n,\rho})}{\lambda_{k,n,\rho}} \int l_{k,n,\rho}^{(r-1)}(t) l_{k,n,\rho}(t) w_{\rho}^2(t) \, dt.
\]
We can see from Theorems A.4 and A.5 and (A.7) that for \(\rho > -1/2\),
\[
\begin{aligned}
\left\| l_{k,n,\rho}^{(r-1)}(t) w_{\rho}(t) \right\|_{L_2(\mathbb{R})} & \lesssim \left\| l_{k,n,\rho}^{(r-1)}(t) w_{\rho}(t) \right\|_{L_2(L_{\frac{a_n}{n}} \leq |x| \leq a_0(1-L \eta_n))} \\
& \sim \left\| l_{k,n,\rho}^{(r-1)}(t) w(t) \left( |x| + \frac{a_n}{n} \right)^{\rho} \right\|_{L_2(L_{\frac{a_n}{n}} \leq |x| \leq a_0(1-L \eta_n))} \\
& \lesssim \frac{n T(a_n)^{1/2}}{a_n} \left\| l_{k,n,\rho}^{(r-2)}(t) w(t) \left( |x| + \frac{a_n}{n} \right)^{\rho} \right\|_{L_2(\mathbb{R})}
\end{aligned}
\]
\[
\frac{nT(a_n)^{1/2}}{a_n} \left\| l_{k,n,\rho}^{(r-2)}(t)w(t) \left( |x| + \frac{a_n}{n} \right)^\rho \right\|_{L_2(L_\infty)} \leq |x| \leq a_n(1-L\eta_n)).
\]

Repeating in the same manner, we conclude that
\[
\left\| l_{k,n,\rho}^{(r-1)}(t)w(t) \right\|_{L_2(\mathbb{R})} \lesssim \left( \frac{nT(a_n)^{1/2}}{a_n} \right)^{r-1} \left\| l_{k,n,\rho}(t)w(t) \right\|_{L_2(\mathbb{R})},
\]

and here by Gauss quadrature formula,
\[
\left\| l_{k,n,\rho}(t)w(t) \right\|_{L_2(\mathbb{R})}^2 = \lambda_{k,n,\rho}.
\]

Therefore, by the Hölder inequality,
\[
|p_{n,\rho}(x_{k,n,\rho})| \lesssim \frac{|p_{n,\rho}(x_{k,n,\rho})|}{\lambda_{k,n,\rho}} \left( \int \left( l_{k,n,\rho}^{(r-1)}(t)w(t) \right)^2 \, dt \right)^{1/2} \left( \int \left( l_{k,n,\rho}(t)w(t) \right)^2 \, dt \right)^{1/2}
\]
\[
= \frac{|p_{n,\rho}(x_{k,n,\rho})|}{\lambda_{k,n,\rho}} \left\| l_{k,n,\rho}^{(r-1)}(t)w(t) \right\|_{L_2(\mathbb{R})} \left\| l_{k,n,\rho}(t)w(t) \right\|_{L_2(\mathbb{R})}
\]
\[
\lesssim \left( \frac{nT(a_n)^{1/2}}{a_n} \right)^{r-1} \frac{|p_{n,\rho}(x_{k,n,\rho})|}{\lambda_{k,n,\rho}} \left\| l_{k,n,\rho}(t)w(t) \right\|_{L_2(\mathbb{R})}^2
\]
\[
\lesssim \left( \frac{nT(a_n)^{1/2}}{a_n} \right)^{r-1} |p_{n,\rho}'(x_{k,n,\rho})|.
\]

Moreover, we see from (A.2) and (A.4) for \(|x_{k,n,\rho}| \geq a_4\beta_1n,\)
\[
T(a_n) \sim T(x_{k,n,\rho}) \sim \left( 1 - \frac{x_{k,n,\rho}}{a_{2n}} \right)^{-1}.
\]

Then we have from (A.1) for \(k = 1, 2, \ldots, n,\)
\[
\left| \frac{p_{n,\rho}^{(r)}(x_{k,n,\rho})}{p_{n,\rho}(x_{k,n,\rho})} \right| \lesssim \left( \frac{n}{\sqrt{a_{2n}^2 - x_{k,n,\rho}^2}} \right)^{r-1} \cdot \square
\]

**Proof of Theorem 2.6.** We will prove (2.1) by induction on \(v.\) (2.1) holds for \(v = 1\) by (4.1) and Theorem 2.5. Now assume that (2.1) holds for \(t \geq 2\) and \(v = 1, 2, \ldots, t - 1.\) Then using Leibniz’s rule for differentiation we obtain
\[
\left| \left[ l_{k,n,\rho}^{(r)}(x_{k,n,\rho}) \right] \right| \lesssim \sum_{i=0}^{r} \binom{r}{i} \left| l_{k,n,\rho}^{(i)}(x_{k,n,\rho}) \right| \left| l_{k,n,\rho}^{(r-i)}(x_{k,n,\rho}) \right|
\]
\[
\lesssim \sum_{i=0}^{r} \binom{r}{i} \left( \frac{n}{\sqrt{a_{2n}^2 - x_{k,n,\rho}^2}} \right)^i \left( \frac{n}{\sqrt{a_{2n}^2 - x_{k,n,\rho}^2}} \right)^{r-i}
\]
\[
\lesssim \left( \frac{n}{\sqrt{a_{2n}^2 - x_{k,n,\rho}^2}} \right)^r.
\]
This completes the proof of (2.1). To prove the next result, we proceed by induction on \(i\). From the fact that \(h_{s,k,n,\rho}(x_{k,n,\rho}) = 1\) and the expression for \(h_{s,k,n,\rho}(x)\), we have \(e_{s,s}(l, v, k, n) = 1/s!\), and by the fact that \(h_{s,k,n,\rho}(x_{k,n,\rho}) = 0, s + 1 \leq j \leq v - 1\), we obtain easily the following recurrence relation:

\[
e_{s,j}(l, v, k, n) = -\sum_{p=s}^{j-1} \frac{1}{(j-p)!} e_{s,p}(l, v, k, n) \left[ l_{k,n,\rho}^{v} \right]^{(j-p)}(x_{k,n,\rho}),
\]

\(s + 1 \leq j \leq v - 1\). \(\)(4.2)

When \(i = s\), \(e_{s,s}(l, v, k, n) = 1/s!\), so (2.2) is satisfied for \(i = s\). From (4.2), (2.1) and the assumption of induction on \(i\), for \(s + 1 \leq i \leq v - 1\), we obtain easily

\[
\left| e_{s,i}(l, v, k, n) \right| \lesssim \sum_{p=s}^{i-1} \left| e_{s,p}(l, v, k, n) \right| \left[ l_{k,n,\rho}^{v} \right]^{(i-p)}(x_{k,n,\rho})
\]

\[
\lesssim \sum_{p=s}^{i-1} \left( \frac{n}{\sqrt{d_{2n}^2 - x_{k,n,\rho}^2}} \right)^{p-s} \left( \frac{n}{\sqrt{d_{2n}^2 - x_{k,n,\rho}^2}} \right)^{i-p}
\]

\[
\lesssim \left( \frac{n}{\sqrt{d_{2n}^2 - x_{k,n,\rho}^2}} \right)^{i-s}.
\]

Therefore, we have (2.2). \(\square\)

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Appendix

**Lemma A.1** \([6]\). Let us have \(w \in \mathcal{F}\).

(a) Fix \(L > 1\). Then uniformly for \(t > 0\),

\[
a_{Lt} \sim a_t.
\] \(\)(A.1)

(b) Fix \(L > 0\). Then uniformly for \(t > 0\),

\[
T(a_{Lt}) \sim T(a_t).
\] \(\)(A.2)

(c) For some \(\varepsilon > 0\), and for large enough \(t\),

\[
\delta_t T(a_t)/a_t \leq Ct^{2-\varepsilon}.
\] \(\)(A.3)

(d) Fix \(L > 1\). Then uniformly for \(t > 0\),

\[
1 - a_t/a_{Lt} \sim T(a_t).
\] \(\)(A.4)

**Proof.** (a) It is from \([6, Lemma 3.5(a)]\). (b) It is from \([6, Lemma 3.5(b)]\). (c) It is from \([6, Lemma 3.7]\). (d) It is from \([6, p. 76]\). \(\square\)
Lemma A.2. Let us have \( w \in \mathcal{F}(\mathbb{C}^2) \).

(a) Uniformly for \( |x| \leq \varepsilon a_n, 0 < \varepsilon < 1 \), we have
\[
\varphi_{2n}(x) \sim \varphi_n(x) \sim \frac{a_n}{n}.
\] (A.5)

(b) For any \( x \in \mathbb{R} \),
\[
\varphi_n(x) \sim \frac{a_n}{n} \quad \text{and} \quad \left( |x| + \frac{a_n}{n} \right) \gtrsim \frac{a_n}{n}.
\] (A.6)

Proof. (a) This is easily proved by the definition of \( \varphi_n(x) \).

(b) For \( |x| \leq \frac{1}{2} a_n \), we have by (A.5),
\[
\varphi_n(x) \sim \frac{a_n}{n} \quad \text{and} \quad \left( |x| + \frac{a_n}{n} \right) \gtrsim \frac{a_n}{n}.
\]

For \( |x| \geq a_n \), we have from the definitions of \( \varphi_n(x) \) and \( \eta_n \),
\[
\varphi_n(x) = \varphi_n(a_n) \lesssim a_n \quad \text{and} \quad \left( |x| + \frac{a_n}{n} \right) \gtrsim a_n.
\]

Moreover, for \( \frac{1}{2} a_n \leq |x| \leq a_n \), we have by (A.3) and the definitions of \( \varphi_n(x) \) and \( \eta_n \),
\[
\varphi_n(x) \lesssim \frac{a_n^2}{\eta_n^2} \frac{T(a_n)^{1/2}}{a_n} \sim \frac{a_n(T(a_n))^{1/2}}{n^{2/3}} \lesssim a_n
\]
and \( \left( |x| + \frac{a_n}{n} \right) \sim a_n \). Therefore, (b) is proved. \( \square \)

Theorem A.3 ([3, Theorem 2.1]). Let us have \( w \in \mathcal{F}(\mathbb{C}^2) \), \( 0 < p \leq \infty \), and \( \beta \in \mathbb{R} \). Then for \( n \geq 1 \) and \( P \in \mathcal{P}_n \), we have
\[
\left\| (P'w)(x) \right\|_{L_p(\mathbb{R})} \lesssim \left( \frac{nT(a_n)^{1/2}}{a_n} \right) \left\| (Pw)(x) \right\|_{L_p(\mathbb{R})}.
\] (A.7)

Theorem A.4 ([3, Theorem 2.3]). Let us have \( w \in \mathcal{F}(\mathbb{C}^2) \), \( 0 < p \leq \infty \), and \( \beta \in \mathbb{R} \). Then there exists a positive constant \( L \) such that for any polynomial \( P \in \mathcal{P}_n \),
\[
\left\| (Pw)(x) \right\|_{L_p(\mathbb{R})} \lesssim \left( \frac{nT(a_n)^{1/2}}{a_n} \right) \left\| (Pw)(x) \right\|_{L_p(\mathbb{R})}.
\]

Theorem A.5 ([3, Theorem 2.6]). Let us have \( w \in \mathcal{F}(\mathbb{C}^2) \) and \( 0 < p \leq \infty \). Let us have \( \rho > -\frac{1}{p} \) if \( p < \infty \) and \( \rho \geq 0 \) if \( p = \infty \). Then there exists a positive constant \( L \) such that for any polynomial \( P \in \mathcal{P}_n \),
\[
\left\| (Pw_\rho)(x) \right\|_{L_p(\mathbb{R})} \lesssim \left( \frac{nT(a_n)^{1/2}}{a_n} \right) \left\| (Pw_\rho)(x) \right\|_{L_p(\mathbb{R})}.
\]

Theorem A.6 ([4, Theorem 2.3]). Let us have \( \rho > -\frac{1}{2} \) and \( w(x) \in \mathcal{F}(\mathbb{C}^2) \). Then uniformly for \( n \geq 1 \),
\[
\sup_{x \in \mathbb{R}} |p_{n,\rho}(x)w(x)| \left( |x| + \frac{a_n}{n} \right)^{\rho} |x^2 - a_n^2|^{1/4} \sim 1.
\]
Theorem A.7 ([4, Theorem 2.5(a)]). Let us have $w(x) \in \mathcal{F}(C^2+) \text{ and } \rho > -\frac{1}{2}$. Then there exists $n_0$ such that uniformly for $n \geq n_0$ and $1 \leq j \leq n$,

$$
|p_{n,\rho} w(x_j, n, \rho) \left(|x_j, n, \rho| + \frac{a_n}{n}\right)^\rho \sim \varphi_n(x_j, n, \rho)^{-1}\left[a_n^2 - x_j^2, n, \rho\right]^{-1/4}.
$$

References