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Proper maps of locales

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Abstract

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We investigate the basic properties of stably closed, or proper maps of locales, in a setting formally similar to that developed by A. Joyal and M. Tierney for treating the descent theory of localic open maps. We show that proper maps are precisely the compact (perfect) maps previously considered by P.T. Johnstone, and that proper surjections are stable coequalizers, effective for descent in the category of locales.

Introduction

If X is a topological space, the partially ordered set of open subsets of X is a complete lattice, in which the infinite distributive law

$$U \wedge \bigvee \mathscr{S} = \bigvee \{U \wedge S \mid S \in \mathscr{S}\}$$

holds for all open subsets U and collections of open subsets $\mathscr{S}$ in X. We recall that a *frame* is an abstract lattice with these properties; like inverse image along a continuous mapping, a frame homomorphism is taken to preserve joins and finite meets.

Locales are frames viewed as spaces, that is, the category of locales is dual to that of frames: a locale X is specified by a frame $\mathcal{O}X$, its lattice of formal opens, and a map $f: Y \to X$ between locales by a frame homomorphism $f^{-}: \mathcal{O}X \to \mathcal{O}Y$ defining formal inverse image along f. This is an instance of a more general idea [1], namely that a Grothendieck topos—a category of sheaves of sets on a site—is a space in an intuitive sense. Using arguments involving, amongst other things, the formal properties of open maps of Grothendieck toposes, it was shown in [13] that the mathematical step from locales to arbitrary Grothendieck toposes is indeed one of remarkable conceptual simplicity: it consists in allowing for the action of a localic groupoid.

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In this paper we state and prove the basic properties of *proper*, that is to say, stably (under pullback) closed maps of locales. A geometric morphism between Grothendieck toposes is proper if its localic part is (we shall not need the general definition here—we only mention that it is strictly weaker than that used in [16]). It turns out that localic proper maps behave essentially like their topological counterparts [4], also known as perfect maps. In particular, proper maps share most of the formal properties of open maps of locales crucial to the "open" picture of toposes presented in [13]; although this point will not be pursued here, the notion of properness might conceivably constitute the backbone of a corresponding "closed" structure of toposes.

The definition of properness for maps of locales and Grothendieck toposes in a sense equivalent to that understood here, seems to have made its first appearance in [8]: a map $f: Y \to X$ of locales is there called perfect if f is compact as locale in the topos of sheaves over X (with the term "proper" reserved for compact regular maps). Under mild separation conditions on the base, this reduces to one of the equivalent definitions for maps of topological spaces, that of being closed with compact fibers [4]. Here we retain internal compactness as one characterization of properness, showing that it is equivalent both to the property of being stably closed, and a simple lattice-theoretic condition on f which is to become our working definition.

A core of the results presented here (stability of compactness under localic change of base, and effectivity of proper surjections for descent) were originally obtained using the direct methods of [18, 19]. However, it transpired that these results fit neatly into an elegant algebraic formalism based on the tensor-category of *preframes* [2, 12], analogous to that used with reference to the category of sup-lattices in [13] for dealing with the descent theory of open maps. Aspects of this connection, which is closely linked with the original technical insight of P.T. Johnstone [9] leading to a choice-free, and essentially constructive [18] proof of the localic Tychonoff Theorem, were observed in [2], and spelled out more explicitly in [12].

We give a brief outline of the contents. The first section deals with preliminaries in a fairly broad sense, and is primarily intended to smooth the way somewhat for the reader unfamiliar with [13]. Section 2 contains a summary of the "linear algebra" of *$\mathscr{C}$-lattices*, the name used here for the order duals of preframes. It is based (by analogy) on [13, Chapters I and II], using the results of [12]; we thus refer to these for more details. In Section 3 we give, and prove the equivalence of, three definitions of properness; the crucial results here are Propositions 3.4 and 3.7. As far as we know, the latter is originally due to A. Tozzi and A. Pultr; however, an adaptation of their proof was necessary to make it base-independent, essential for our purposes. The final two sections treat basic preservation and exactness properties of proper maps, in the spirit of [4]. From these we shall in particular be able to conclude that proper surjections are effective descent morphisms in the category of locales [13, 17, 20]. Our treatment of proper equivalence relations was to some extent modelled on [14].

1. Notation and preliminaries

In this section we establish terminology, and recall various needed (mostly well-known) facts about the localic formalism in the topos context. Our basic reference is [13], but see also [10]. We shall mostly work with locales in terms of their "closeds", as opposed to opens as is customary; although this switch is formally trivial, it will render our arguments more geometrically intuitive.

Open- and closed maps

Let $f: Y \to X$ be a map of locales. Formal inverse image f^- has a right adjoint $f^*: \mathcal{O}Y \to \mathcal{O}X$. By simply reversing the order, we could also view X as being given in terms of a *coframe* (order-dual of a frame) $\mathcal{C}X \simeq \mathcal{O}X^{\mathrm{op}}$, the closeds of X, and f by a coframe homomorphism $f^-: \mathcal{C}X \to \mathcal{C}Y$ with left adjoint $f_!$:

$$\begin{array}{ccccc} Y & \quad & \mathcal{O}Y & \quad & \mathcal{C}Y \\ \downarrow f & & f_* \downarrow \vdash \uparrow f^- & \xleftrightarrow{\mathrm{op}} & f_! \downarrow \dashv \uparrow f^- \\ X & & \mathcal{O}X & & \mathcal{C}X. \end{array}$$

The category of locales (**Loc**) has limits and colimits, and any f can be factored essentially uniquely into a *surjection* $Y \twoheadrightarrow f[Y]$ (epimorphism, corresponding to a coframe inclusion), followed by a *sublocale inclusion* $f[Y] \hookrightarrow X$ (regular monomorphism, given by a coframe surjection). Let $\mathcal{S}X$ denote the complete lattice of sublocales of X (equivalence classes of inclusions, which form a set). The map f induces the pair

$$\mathcal{S}X \underset{f[-]}{\overset{f^-}{\rightleftarrows}} \mathcal{S}Y$$

of order-preserving mappings, where $f[-] \equiv$ categorical direct image, left adjoint to $f^- \equiv$ categorical inverse image (pullback). Our notation here is consistent, for we recall the following:

Proposition 1.1. *For every locale X, there is an embedding of $\mathcal{O}X$ into $\mathcal{S}X$—defining the* open *sublocales of X—which preserves finite meets and arbitrary joins, and such that for any $f: Y \to X$, the square*

$$\begin{array}{ccccc} Y & \quad & \mathcal{O}Y & \hookrightarrow & \mathcal{S}Y \\ \downarrow f & & f^- \uparrow & & \uparrow f^- \\ X & & \mathcal{O}X & \hookrightarrow & \mathcal{S}X \end{array}$$

commutes. □

Proposition 1.2. *Each open sublocale $U \hookrightarrow X$ has a complement $-U$ in $\mathscr{S}X$, preserved under inverse image; the assignment $U \mapsto -U$ thus gives an embedding of $\mathscr{C}X$ into $\mathscr{S}X$—defining the* closed *sublocales of X—which preserves finite meets and arbitrary joins, and such that for any $f: Y \to X$, the square*

$$
\begin{array}{ccccc}
Y & \qquad & \mathscr{C}Y & \hookrightarrow & \mathscr{S}Y \\
\big\downarrow f & & \big\uparrow f^- & & \big\uparrow f^- \\
X & & \mathscr{C}X & \hookrightarrow & \mathscr{S}X
\end{array}
$$

commutes. □

An open inclusion $i: U \hookrightarrow X$ and its closed complement $j: C \hookrightarrow X$ may be reconstructed from the locales U and C by "Artin glueing" [7] along the "fringe map" $p = j^- i_*: \mathscr{O}U \to \mathscr{O}C$; p preserves finite meets, and prescribes how U and C fit together in X at their boundaries. More precisely, the fact that U and C cover X means there is a surjection

$$U + C \overset{q}{\twoheadrightarrow} X. \tag{1}$$

The inclusion $\mathscr{O}X \hookrightarrow \mathscr{O}U \times \mathscr{O}C$ of $\mathscr{O}X$ as q-saturated opens of $U + C$ is determined as the fix-points of the co-closure operator $q^- q_*$; one checks that

$$q^- q_* \langle P, Q \rangle = \langle P, pP \wedge Q \rangle \quad \text{for } P \in \mathscr{O}U, Q \in \mathscr{O}C. \tag{2}$$

Thus, $P + Q \in \mathscr{O}U + \mathscr{O}C$ is of the form $q^- U$ precisely when $Q \leq pP$. The relation (2) defines an order-preserving correspondence between images of $U + C$ of the form (1)—that is, under which U remains open and disjoint from C—and finite meet-preserving mappings $p: \mathscr{O}U \to \mathscr{O}C$.

By Propositions 1.1 and 1.2, open sublocales (with their joins) and closed sublocales are preserved under inverse image. Given a closed embedding $i: C \hookrightarrow X$, the (images of) closed sublocales of C in $\mathscr{S}X$ are of the form $C \wedge D$ for some $D \in \mathscr{O}X$, hence closed. Thus $i[-]$ restricts to closed sublocales, or equivalently, closed inclusions are preserved under composition:

$$\mathscr{C}X \underset{i[-]}{\overset{i^-}{\rightleftarrows}} \mathscr{C}C \;\simeq\; \mathscr{C}X \underset{\text{incl}}{\overset{C \wedge -}{\rightleftarrows}} \downarrow C.$$

More generally, a map $f: Y \to X$ of locales is *closed* if direct image along f, $f[-]: \mathscr{S}Y \to \mathscr{S}X$, restricts to closed sublocales. It is immediate from the definition that the following maps are closed: inclusions of closed sublocales, in particular homeomorphisms (isomorphisms); the composite of two closed maps; the result of cancelling from a closed map an inclusion on the right or an epimorphism on the left.

Open maps are defined similarly; in addition to satisfying the stability properties just mentioned for closed maps, open surjections, hence arbitrary open maps are like open inclusions stable under pullback. As in topology, this is not true for arbitrary localic closed maps, which is the reason for introducing proper maps.

A map $f\colon Y \to X$ factors through a sublocale $S \hookrightarrow X$ if and only if for all $C, D \in \mathscr{C}X$, $S \wedge D \leq C$ implies $f^{-}D \leq f^{-}C$; this is easily seen to imply that

$$S = \bigwedge\{U \vee C \mid U \in \mathcal{O}X,\ V \in \mathscr{C}X \text{ and } S \leq U \vee C\}.$$

This last fact forms the backbone of the following proposition:

Proposition 1.3. *$\mathscr{S}X$ is a coframe $\mathscr{C}X'$, and the coframe inclusion $\mathscr{C}X \hookrightarrow \mathscr{C}X'$ inverse image for a stable, monomorphic surjection $s\colon X' \twoheadrightarrow X$ of locales. Moreover, for each map $f\colon Y \to X$, inverse image $f^{-}\colon \mathscr{S}X \to \mathscr{S}Y$ is a coframe homomorphism, defining a map of locales $f'\colon Y' \to X'$. The surjection $s\colon X' \twoheadrightarrow X$ (to which we shall refer as the* splitting cover *of X) is universal amongst maps for which the inverse images of opens—and hence of all sublocales of X—are closed.* □

The fact that inverse image preserves finite joins of sublocales means the complement $-S$ (if it exists) of any $S \in \mathscr{S}X$ is stable. As a consequence, localic surjections are preserved on pulling back (restricting to) complemented sublocales. This leads to a "Frobenius reciprocity"-law:

Proposition 1.4. *For a map $f\colon Y \to X$, complemented $A \in \mathscr{S}X$ and any $B \in \mathscr{S}Y$,*

$$f[f^{-}A \wedge B] = A \wedge f[B].$$

Proof. The sublocale $A \wedge f[B] \hookrightarrow f[B]$ is complemented in $f[B]$, and the diagram

$$\begin{array}{ccc}
f^{-}A \wedge B & \hookrightarrow & B \\
{\scriptstyle f\restriction}\big\downarrow & & \big\downarrow{\scriptstyle f\restriction} \\
A \wedge f[B] & \hookrightarrow & f[B]
\end{array}$$

a pullback. □

Each sublocale S of X has a *closure* $\bar{S}$, the least closed sublocale of X containing S. In particular, $f_! D = \overline{f[D]}$ for $D \in \mathscr{C}Y$. Thus, for $U \in \mathcal{O}X$ and $C \in \mathscr{C}X$ we have $f_* - C = -f_! C$ and $-f_* U = f_! - U$.

Lemma 1.5. *The following are equivalent for* $f\colon Y \to X$:
(i) f *is closed;*
(ii) $f[-]$ *restricts to* $f_!$ on $\mathscr{C}Y$;
(iii) $f_!(f^- C \wedge D) = C \wedge f_! D$ *for all* $C \in \mathscr{C}X$, $D \in \mathscr{C}Y$.

Proof. That (i) is equivalent to (ii) is immediate. Given (ii), (iii) is just the Frobenius-identity (Proposition 1.4), hence is necessary for f to be closed; it is also sufficient, since it implies (for arbitrary $E \in \mathscr{C}X$) that $f[C] \wedge D \leq E$ only if $f_! C \wedge D \leq E$, that is, that $f[-] \leq f_!$. $\square$

Locales over a base

Given a locale X, the frame $\mathcal{O}X$ as a category may be completed under colimits, freely up to the preservation of existing covers (i.e. joins) in $\mathcal{O}X$, to yield the topos $\mathscr{E}X$ of "generalized opens" or *sheaves* on X; it is constructed in the standard way as the category of contravariant functors $\mathcal{O}X \to$ Sets satisfying a patching condition on open covers. This extends 2-functorially to maps $f\colon Y \to X$ to yield a 2-full embedding (the correspondence between morphisms is an equivalence of categories rather than a bijection of sets) of the 2-category of locales into that of Grothendieck toposes:

$$
\begin{array}{c}
Y\\
\Big\downarrow f\\
X
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{O}Y & \hookrightarrow & \mathscr{E}Y\\
f_*\Big\downarrow\dashv\Big\uparrow f^- & & f_*\Big\downarrow\dashv\Big\uparrow f^*\\
\mathcal{O}X & \hookrightarrow & \mathscr{E}X
\end{array}
$$

Here the (Yoneda-)embedding $\mathcal{O}X \hookrightarrow \mathscr{E}X$ preserves finite limits, as does the colimit-extension $f^*\colon \mathscr{E}X \to \mathscr{E}Y$ of f^-. f^* has a right adjoint, restricting to that of f^-. One may therefore identify such a *localic topos* with the corresponding locale, that is, X is determined by its "topos of opens" $\mathscr{E}X$, and conversely.

A map $p\colon E \to X$ of locales is said to be *etale* (or a local homeomorphism) if p, as well as the inclusion $\Delta\colon Y \hookrightarrow Y \times_X Y$ of the diagonal into the fibered product are open. Given a locale X, we write $\mathscr{L}X$ for the comma-category **Loc**/X of locales over X. A map $f\colon Y \to X$ induces the functor $f^*\colon \mathscr{L}X \to \mathscr{L}Y$ (pullback or "change of base") which preserves etale maps. Proposition 1.1 now extends to generalized opens, to read:

Proposition 1.6. *For every locale* X, *there is a full embedding of* $\mathscr{E}X$ *into* $\mathscr{L}X$*—with image the etale maps into* X*—which preserves finite limits and arbitrary colimits, and* *such that for each map $f: Y \to X$, the square*

$$
\begin{array}{ccccc}
Y & \quad & \mathscr{E}Y & \hookrightarrow & \mathscr{L}Y \\
\Big\downarrow{\scriptstyle f} & & \Big\uparrow{\scriptstyle f^*} & & \Big\uparrow{\scriptstyle f^*} \\
X & & \mathscr{E}X & \hookrightarrow & \mathscr{L}X
\end{array}
$$

commutes up to isomorphism. □

The data in Proposition 1.1 is recovered from the above by restricting to (regular) subojects of the terminal object 1 in the respective categories.

The category $\mathscr{L}X$ is the fiber over X of a fibration of categories

$$\mathbf{Loc}^2 \xrightarrow{\text{cod}} \mathbf{Loc},$$

a natural structure for describing the internal (categorical) logic of locales as *indexing objects* (cf. [3, 15]). Briefly, a map $p: E \to X$ represents a family $\{E[x] \mid x \in X\}$ of locales, continuously parametrized by X. Given any $f: Y \to X$, the family $\langle E, p\rangle$ is re-indexed along f by applying f^*, that is, $f^*\langle E, p\rangle \equiv \{E[f(y)] \mid y \in Y\}$; re-indexing along a (constant) point $p: 1 \to X$ of X produces a (constant) locale, the fiber above p of the corresponding map. A map in $\mathscr{L}X$ is similarly thought of as a family of localic maps indexed by X. A basic idea now is that a truly "internal" class of locales is completely defined only in terms of the collection of all (formal) locale-indexed families of its members; to make sense, it is clearly necessary that such a collection of families (as maps) be stable under re-indexing. An internal class is "small" if all the corresponding families may be obtained by re-indexing a single, generic family, of which the indexing locale is said to "classify" the (members of the) class.

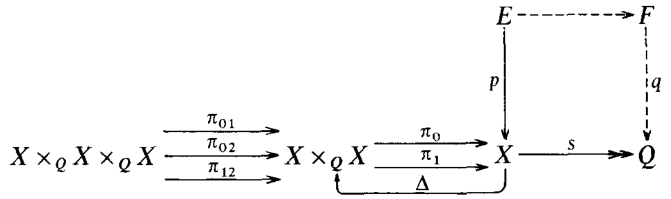
Now, as is well-known, the internal logic of a topos can support set-theoretic arguments provided these are constructive in the appropriate sense (see [7]). In particular, the notion of locale can be interpeted in any topos, as can the facts and arguments expressed in set-theoretic language about them given in this paper. For example, one shows that the unique map from a locale X to the terminal locale is etale precisely when X is discrete, that is, has for opens all subsets of a set; the resulting full embedding $\mathbf{Set} \hookrightarrow \mathbf{Loc}$ is just that of Proposition 1.6 in the case $X \simeq 1$. This is constructive, and can be interpreted with the topos $\mathscr{E}X$ in the place of $\mathbf{Set} \equiv \mathscr{E}1$. It can be shown that the result is exactly the embedding of Proposition 1.6 for general X; more precisely, we have the following:

Proposition 1.7. *There is an equivalence of categories $\mathscr{L}X \simeq \mathbf{Loc}_{\mathscr{E}X}$, identifying the inclusion of $\mathscr{E}X$ as etale maps with its inclusion as discrete locales.* □

A locale continuously varying over X is therefore a locale in *a* category of sets, namely that of (constant) sets varying continously over X. In this way any constructive property of a locale which on interpretation in localic toposes yields a stable class of maps, gives rise to an internal class of locales. Thus, etale maps represent the class of discrete locales, inclusions that of sublocales of 1, and open inclusions the class of open sublocales of 1 or truth values, which is small (classified by the Sierpinski-locale $\mathbb{S}$). More generally, there is an internal class of *open* locales, locales with open support; whereas the class of inhabited locales (locales with global support) is not internal, that of open inhabited locales is, since open surjections are stable.

Effective descent

Given a localic surjection $s: X \twoheadrightarrow Q$, consider the diagram



For any X-indexed family $\langle E, p\rangle$ to be the pullback of a family $\langle F, q\rangle$ indexed by Q, it is necessary that re-indexing along the projections π_0 and π_1 leads to isomorphic results, $\phi: \pi_0^*\langle E, p\rangle \simeq \pi_1^*\langle E, p\rangle$; moreover, the $X \times_Q X$-indexed family of isomorphisms ϕ needs to be compatible, in the sense that

$$\Delta^*\phi = \mathrm{id}_{\langle E, p\rangle} \quad \text{and} \quad \pi_{12}^*\phi \circ \pi_{01}^*\phi = \pi_{02}^*\phi. \tag{3}$$

If, conversely, an arbitrary $\langle E, p\rangle$ and ϕ satisfying the *cocycle-conditions* (3) (called *descent data*) determine such $\langle F, q\rangle$ uniquely up to isomorphism, s is said to be an *effective descent morphism of locales.* We shall apply the following criterion for effectivity of descent [17]:

Proposition 1.8. *The members of a stable class $\mathscr{S}$ of localic surjections which contains the homeomorphisms and is closed under composition, are effective descent morphims provided that*

(i) *each $s \in \mathscr{S}$ is a coequalizer ("identification map"),*

(ii) *any equivalence relation*

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X$$

for which α (or equivalently β) is in $\mathscr{S}$ has a stable coequalizer. □

"Stable" in (ii) means when changing base, the pullback of the coequalizer of α and β remains the coequalizer of their pullbacks.

2. Coframes as ring objects

By a *$\mathscr{C}$-lattice* we shall mean a partially ordered set P which has finite joins and meets of filtered (i.e. down-directed) subsets, satisfying the distributive law

$$x \vee \bigwedge F = \bigwedge \{x \vee f \mid f \in F\} \quad \text{for all } x \in P \text{ and filtered } F \subseteq P.$$

$\mathscr{C}$-lattice homomorphisms preserve finite joins and filtered meets. Recall that the category of partially ordered sets and monotone mappings (**Pos**) has limits and colimits, and is cartesian closed under the pointwise order on its hom-sets.

The next result is essentially contained in [12]:

Theorem 2.1. *The category* **Cl** *of $\mathscr{C}$-lattices has colimits, and is commutative monadic (in the strong sense) over the category of partially ordered sets. It is therefore a complete and cocomplete, symmetric closed* **Pos**-*category, in which coframes are the meet-semi-lattices.* □

We briefly explain the terms, and then the way we intend to use Theorem 2.1. *Strong* refers to the fact that the monad $\mathbb{T} \equiv \langle T, \eta, \mu \rangle$ involved—and thus various derived constructions—respects the order on maps (T is said to be a **Pos**-functor). To describe $\mathbb{T}$, recall that the functor which assigns to a partially ordered set P its downsets $\mathscr{D}P$ ordered by subset-inclusion, and to a monotone mapping $f: P \to Q$ the function which takes a downset of P to the down-closure of its image in Q, defines a monad. The unit is the embedding $\downarrow(-): P \hookrightarrow \mathscr{D}P$ associating with an element of P its downsegment; the counit is union: $\mathscr{D}^2 P \to \mathscr{D}P$. Submonads are obtained by restricting to *ideals* (up-directed downsets) and *finitely generated* (finite unions of principal) downsets respectively; the latter may be followed (composed) with the order-dual of the former (filters replacing ideals, ordered by reverse inclusion) to give a monad having $\mathscr{C}$-lattices as algebras. Thus, limits of $\mathscr{C}$-lattices are calculated in **Pos**. Colimits are most easily constructed by "pulling down" those from the category of coframes, cf. [12].

For partially ordered sets P and Q, the exponential transposes of the identity $\mathrm{id}: P \times Q \to P \times Q$ and $T: \mathbf{Pos}\langle Q, P \times Q\rangle \to \mathbf{Pos}\langle TQ, T(P \times Q)\rangle$ induce the map

$$P \times TQ \xrightarrow{\widehat{\mathrm{id}} \times \mathrm{id}} \mathbf{Pos}\langle Q, P \times Q\rangle \times TQ \xrightarrow{\hat{T}} T(P \times Q),$$

natural in P and Q. The term *commutative in* Theorem 2.1 refers to the fact that the diagram (with the obvious maps)

$$\begin{array}{ccc} TP\times TQ & \longrightarrow & T(P\times TQ) \\ \downarrow & & \downarrow \\ T(TP\times Q) & \longrightarrow & T^2(P\times Q) \end{array}$$

commutes. Explicitly, it says all $\mathscr{C}$-lattice operations commute, that is, filtered meets and finite joins with themselves and each other. Given that **Cl** has equalizers and coequalizers, it is a purely formal consequence of commutativity that the symmetric closed structure of **Pos** lifts to $\mathscr{C}$-lattices. That is, for $\mathscr{C}$-lattices M and N, $\mathbf{Cl}\langle M, N\rangle$ becomes an object of **Cl** and the cartesian product of partially ordered sets can be modified to a symmetric tensor product ($\otimes$), with unit (I), such that there is a natural, order-preserving bijection

$$\mathbf{Cl}\langle M\otimes N, L\rangle \simeq \mathbf{Cl}\langle M, \mathbf{Cl}\langle N, L\rangle\rangle. \tag{4}$$

Explicitly, elements of the right-hand side in (4) correspond bijectively to monotone mappings $M\times N\to L$ which preserve the $\mathscr{C}$-lattice operations in each variable separately; these may be represented by a universal such bi-homomorphism $M\times N\to M\otimes N$. The unit is the free object on one generator. It follows that there is a forgetful functor

Commutative monoids in **Cl** $\to$ Commutative monoids in **Pos**,

with the commutative monoids of **Cl**—or *$\mathscr{C}$-lattice rings*—appearing in **Pos** as those having a multiplication which preserves the $\mathscr{C}$-lattice operations in each variable separately. The description of coframes in Theorem 2.1 follows by considering meet-semilattices as commutative monoids of partially ordered sets.

Limits and filtered colimits of rings are calculated in **Cl**. The initial ring is the tensor-unit I, and the coproduct of two rings R and S their tensor product:

$$R\simeq R\otimes I\xrightarrow{\mathrm{id}\otimes !}R\otimes S\xleftarrow{!\otimes\mathrm{id}}I\otimes S\simeq S. \tag{5}$$

These constructions restrict to coframes; in particular, the unit, and the tensor product of two coframes are coframes. We shall not here need to consider coequalizers of general $\mathscr{C}$-lattice rings.

The generalities of "linear algebra" can be developed in **Cl**, similar to the way it is done in the category of abelian groups (see [13] for details of the corresponding theory for sup-lattices). Thus, any ring R in **Cl** defines a strong commutative monad $C\mapsto R\otimes C$ on **Cl**; commutativity is just that of the monoid R. The algebras are actions of R in **Cl**, or *R-modules*; we refer to the morphisms as *R-linear mappings*.

R-modules are also the coalgebras for the corresponding comonad $C \mapsto \mathbf{Cl}\langle R, -\rangle$; thus, both limits and colimits of R-modules exist, and are calculated in **Cl**. As before, the closed structure of **Cl** lifts to R-modules: the relative tensor product $(\otimes_R)$ has R for its unit, with an R-linear map $M \otimes_R N \to L$ corresponding to a monotone mapping $M \times N \to L$ which is R-linear in each variable separately. We refer to rings of R-modules as *R-algebras*; these correspond to ring homomorphisms $h: R \to S$, each inducing a functor

$$S\text{-modules} \xrightarrow{h_*} R\text{-modules}$$

("restriction of base" along h). h_* has a left adjoint $M \mapsto S \otimes_R M$ ("extension of base" along h); the unit of the adjunction at (an R-module) M is

$$M \simeq h \otimes_R \mathrm{id} : R \otimes_R M \to S \otimes_R M.$$

The construction of limits and coproducts of rings "relativize" to R-algebras. We shall have frequent use for the following lemma:

Lemma 2.2. *A commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{f} & M \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ S & \xrightarrow{g} & N \end{array}$$

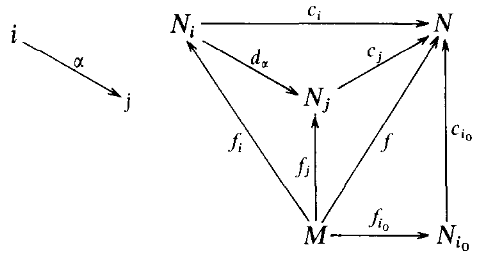
of coframes is a pushout if and only if g is the extension of f along h, with h and k units for the adjunction.

Proof. The diagram above is a pushout of coframes iff a coproduct in the category of rings under R, that is, of R-algebras; the latter is given by the (relative) tensor product as in (5). □

We note that the constructions above are all internal to the category of partially ordered sets, that is, the functors and adjunctions mentioned respect the pointwise order of maps.

Proposition 2.3. *Let R be a $\mathscr{C}$-lattice ring, M an R-module, and $D: I \to \mathbf{Cl}\backslash M$ a (small) filtered diagram in the category of R-modules under M. Then, if each of the vertices of D (R-linear maps from M) has an R-linear left adjoint, then so has the colimit of D; moreover, the latter is an injective mapping given that all the vertices are.*

Proof. Let i_0 be a fixed object of the filtered category I, and consider the commutative diagram



where α varies over the morphisms of I, where $d_\alpha : \langle f_i, N_i \rangle \to \langle f_j, N_j \rangle$ stands for $D\alpha : Di \to Dj$, and f is the colimit of D, with $c_i : \langle f_i, N_i \rangle \to \langle f, N \rangle$ the canonical maps. The top triangles define a colimit of R-modules, from which f is obtained as $f = c_{i_0} f_{i_0}$ (connected colimits of R-modules under M are calculated in the category of R-modules). For $i \in \operatorname{Ob} I$, let $f_{i!} : N_i \to M$ be the left adjoint of f_i, and put $\phi_i = \bigwedge \{ f_{k!} d_\gamma \mid \gamma : i \to k \text{ in } I \}$. Then ϕ_i is R-linear, being a filtered meet of such; moreover, given any $\alpha : i \to j$ in I,

$$\phi_i = \bigwedge \{ f_{k!} d_{\delta\alpha} \mid i \xrightarrow{\alpha} j \xrightarrow{\delta} k \text{ in } I \} \quad \text{(using the filteredness of } I)$$

$$= \bigwedge \{ f_{k!} d_\delta \mid j \xrightarrow{\delta} k \text{ in } I \} \circ d_\alpha$$

$$= \phi_j d_\alpha. \tag{6}$$

By (6) and the colimit property of the maps c_i, there exists a unique R-linear map $f_! : N \to M$ such that $\phi_i = f_! c_i$ for all $i \in \operatorname{Ob} I$. Also,

$$f_! f = f_! c_{i_0} f_{i_0}$$

$$= \bigwedge \{ f_{k!} d_\gamma f_{i_0} \mid i_0 \xrightarrow{\gamma} k \text{ in } I \}$$

$$\leq f_{i_0 !} f_{i_0} \leq \mathrm{id}, \tag{7}$$

while for all $i \in \operatorname{Ob} I$,

$$f f_! c_i = \bigwedge \{ c_k f_k f_{k!} d_\gamma \mid i_0 \xrightarrow{\gamma} k \text{ in } I \}$$

$$\geq \bigwedge \{ c_k d_\gamma \mid i_0 \xrightarrow{\gamma} k \text{ in } I \}$$

$$= c_i = \mathrm{id} \circ c_i,$$

giving $\mathrm{id} \leq f f_!$. It follows that $f_! \dashv f$, that is, f has an R-linear left adjoint. Finally, if each f_i, $i \in \operatorname{Ob} I$ is injective, we have equality in (7), so that f is injective. □

3. Definitions of properness

Recall that a locale Y is compact if any open cover of Y contains a finite subcover; equivalently, the left adjoint to inverse image of closeds along the unique $p: Y \to 1$ preserves filtered infima. We say $f: Y \to X$ is *compact* if it is compact as a locale in the topos $\mathscr{E}X$ of sets over X. This section will be devoted to showing the following theorem:

Theorem 3.1. *The following are equivalent for* $f: Y \to X$:

(i) *f is compact;*
(ii) *f is a closed map, and $f_!$ preserves filtered infima;*
(iii) *f is stably closed.*

A map $f: Y \to X$ will be called *proper* if it satisfies any of these equivalent conditions. Stable closedness is trivially preserved under pullback. It will therefore follow from Theorem 3.1 that compactness is stable under change of base, and that proper maps have compact fibers. Thus, compactness is an "internal" notion. For brevity we refer to property (ii) as *lattice*-compactness of f; it will yield a purely lattice-theoretic characterization of properness (see Lemma 3.3). Theorem 3.1 will be proved by showing that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Compact maps are lattice-compact

We need an explicit description of compactness in a localic topos.

Lemma 3.2. *$f: Y \to X$ is compact iff for each mapping $j: \mathcal{O}Y \to \mathcal{O}X$ which satisfies*

(J0) $\quad f_*(P \to Q) \le j(Q) \to j(P)$,

(J1) $\quad X \le j(0)$,

(J2) $\quad j(P) \wedge j(Q) \le j(P \vee Q)$,

for all $P, Q \in \mathcal{O}Y$, the relation

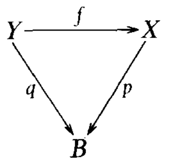
$$f_* \bigvee \{P \wedge f^- j(P) \mid P \in \mathcal{O}Y\} \le j(Y) \tag{8}$$

holds.

Proof. We use the representation of sheaves as $\mathcal{O}X$-sets, see [5]; recall that the frame-sheaf of opens of $f: Y \to X$ considered as locale in $\mathscr{E}X$ is determined by its global elements, which is $\mathcal{O}Y$. $f: Y \to X$ is compact in $\mathscr{E}X$ precisely when f_* preserves suprema indexed by (global) internal ideals of $\mathcal{O}Y$. Such an ideal $\mathcal{J}$ is described by a mapping j (the characteristic map of $\mathcal{J}$ on global opens of Y over X) as above, in terms of which (8) expresses the internal inclusion $f_* \bigvee \mathcal{J} \leq \bigvee f_*[\mathcal{J}]$. $\square$

Now, suppose $f: Y \to X$ is compact. Given any $U \in \mathcal{O}X$ and $V \in \mathcal{O}Y$, define $j: \mathcal{O}Y \to \mathcal{O}X$ by $P \mapsto U \vee f_*(P \to V)$. Then it is straightforward to check that j satisfies (J0)–(J2), while (8) reduces to $f_*(f^- U \vee V) \leq U \vee f_* V$; this shows that f is closed. Next, suppose $\mathcal{J} \subseteq \mathcal{O}Y$ is an (ordinary) ideal of $\mathcal{O}Y$, and now define $j: \mathcal{O}Y \to \mathcal{O}X$ by $P \mapsto \bigvee \{ f_*(P \to V) \mid V \in \mathcal{J} \}$. Then it is again easy to see that j satisfies (J0)–(J2), while (8) reduces to $f_* \bigvee \mathcal{J} \leq \bigvee f_*[\mathcal{J}]$; this shows that f_* preserves suprema over ideals of $\mathcal{O}Y$, or, equivalently, that $f_!$ preserves infima over filters of $\mathscr{C}Y$. Thus, f is lattice-compact.

In particular, if Y is compact, the unique $p: Y \to 1$ is closed. We may show this directly: given $c \in \mathscr{C}1$ and $D \in \mathscr{C}Y$, $p_!(p^- c \wedge D) \equiv p_!(\bigwedge \mathscr{F})$ and $c \wedge p_! D \equiv \bigwedge p_![\mathscr{F}]$, where $\mathscr{F}$ is the filtered set $\{F \in \mathscr{C}Y \mid (* \in -c \text{ and } F = 0) \text{ or } F = D\}$. We could alternatively have deduced the general case from this by observing that closedness of a map is independent of a reference base, i.e. if



commutes, f is (stably) closed iff $f: \langle Y, q \rangle \to \langle X, p \rangle$ is (stably) closed in $\mathscr{E}B$.

Lattice-compact maps are stable

Our arguments involving lattice-compactness will lean heavily on the following observation:

Lemma 3.3. *$f: Y \to X$ is lattice-compact precisely when $f_!: \mathscr{C}Y \to \mathscr{C}X$ is $\mathscr{C}X$-linear.*

Proof. By definition $f: Y \to X$ is lattice-compact when $f_!$ preserves filtered infima and f is closed. The former says $f_!$ is a map of $\mathscr{C}$-lattices, since as a left adjoint $f_!$ preserves all joins; using Lemma 1.5 the latter says $f_!$ preserves the action of $\mathscr{C}X$. $\square$

The "extremal" instances of lattice-compact maps are closed embeddings $i: C \hookrightarrow X$ ($i_!$ preserves any infimum taken over an inhabited subset of $\mathscr{C}C$) and $p: Y \to 1$ with Y compact—each satisfies part of the definition trivially.

Proposition 3.4. *In a pullback square*

$$\begin{array}{ccc} P & \xrightarrow{k} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & X \end{array}$$

if f is lattice-compact, so is h, and $g^- f_! = h_! k^-$.

Proof. In the diagram

$$\begin{array}{ccc} \mathscr{C}Z & \xleftarrow{g^-} & \mathscr{C}X \\ {\scriptstyle h_!}\uparrow & & \uparrow{\scriptstyle f_!} \\ \mathscr{C}P & \xleftarrow{k^-} & \mathscr{C}Y \\ {\scriptstyle h^-}\uparrow & & \uparrow{\scriptstyle f^-} \\ \mathscr{C}Z & \xleftarrow{g^-} & \mathscr{C}X \end{array}$$

h^- is the base-extension of f^- along g^- (Lemma 2.2), with g^- and k^- unit maps for the adjunction. If $f_!$ is $\mathscr{C}X$-linear, it has itself an extension $h_!$ to a $\mathscr{C}Z$-linear map, the unique such making the top square commutative. But since base-extension preserves the order on maps, it preserves in particular the adjointness-relation $f_! \dashv f^-$, that is, $h_! \dashv h^-$. □

Proposition 3.4 implies that a lattice-compact map is stably closed. In particular, we have the following

Corollary 3.5. *Suppose Y is compact. Then for any Z the projection $\pi: Z \times Y \to Z$ is closed.* □

Stably closed maps are compact

We first obtain the converse of Corollary 3.5.

Lemma 3.6. *Let X be a locale, and $\mathscr{F}$ a filter of opens in X. Then there is an open inclusion of X into a locale $X +_{\mathscr{F}} \infty$ in which the closed complement of X is a point ∞ having $\{F \vee \{\infty\} \mid F \in \mathscr{F}\}$ as filter of neighbourhoods.*

Proof. The characteristic function $\chi_{\mathscr{F}}: \mathcal{O}X \to \mathcal{O}1$ of $\mathscr{F}$ preserves finite meets, hence defines a fringe map for glueing a closed point to X, to give a locale $X +_{\mathscr{F}} \infty$. By construction, $\infty \in W \in \mathcal{O}(X +_{\mathscr{F}} \infty)$ precisely when $X \wedge W \in \mathscr{F}$. □

Proposition 3.7 (After A. Tozzi and A. Pultr). *A locale Y for which the projection $\pi : Z \times Y \to Z$ is closed for arbitrary Z is compact.*

Proof. Let $s : S \twoheadrightarrow Y$ be the splitting cover of Y. Then, given an open directed cover $\mathscr{D}$ of Y, the set of open complements $\{S - s^{-}D \mid D \in \mathscr{D}\}$ is a base for a filter in $\mathcal{O}S$. Let Z be the locale obtained by glueing to S a closed point ∞ having the set $\{Z - s^{-}D \mid D \in \mathscr{D}\}$ as base of open neighbourhoods; explicitly, there is an open inclusion $i : S \hookrightarrow Z$, with $\{\infty\} = Z - S$, and such that $\infty \in U \in \mathcal{O}Z$ only if there exists some $D \in \mathscr{D}$ such that $Z - s^{-}D \subseteq U$. Now let $A = \bigvee\{(Z - s^{-}D) \times D \mid D \in \mathscr{D}\} \in \mathcal{O}(Z \times Y)$. Then

$$A \vee (S \times Y) = \bigvee\{(Z - s^{-}D) \times D \mid D \in \mathscr{D}\} \vee \bigvee\{S \times D \mid D \in \mathscr{D}\}$$

$$(\mathscr{D} \text{ covers } Y)$$

$$\geq \bigvee\{((Z - s^{-}D) \vee S) \times D \mid D \in \mathscr{D}\}$$

$$= \bigvee\{Z \times D \mid D \in \mathscr{D}\} = Z \times Y.$$

But π is closed, giving $(\pi_* A) \vee S = \pi_*(A \vee (S \times Y)) = \pi(Z \times Y) = Y$; it follows that $\infty \in \pi_* A$, so that there exists $D \in \mathscr{D}$ for which $S - s^{-}D \subseteq \pi_* A$, i.e. $(S - s^{-}D) \times Y \subseteq A$. Pulling back along $\langle i, s \rangle : S \hookrightarrow Z \times Y$ now gives

$$S - s^{-}D \leq \bigvee\{-s^{-}D \wedge D \mid D \in \mathscr{D}\} = 0,$$

i.e. $s^{-}D = S = s^{-}Y$; since s covers, this means $D = Y$, and so $Y \in \mathscr{D}$. □

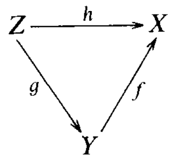
The proof of Proposition 3.7 is constructive. Since stable closedness is independent of a reference base, we conclude that any stably closed map is compact. This completes the proof of Theorem 3.1.

4. Stability properties

Composition and pullback

Closed localic inclusions, and thus in particular homeomorphisms are proper. Further, we have the following:

Proposition 4.1 (see also [8]). *In a commutative diagram*

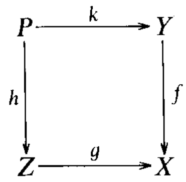


(i) *f and g proper $\Rightarrow$ h proper,*
(ii) *h proper and f a sublocale inclusion $\Rightarrow$ g proper,*
(iii) *h proper and g a surjection $\Rightarrow$ f proper.*

Proof. (i) and (ii) follow from the corresponding facts for closed maps. Property (iii) is just the interpretation in $\mathscr{E}X$ of the fact that the image of a compact locale is compact. $\square$

Concerning change of base, we record the following:

Proposition 4.2. *In a pullback square*



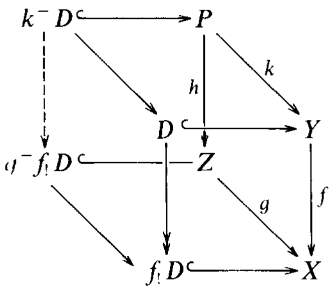
if f is proper (resp. a proper surjection), so is h.

Proof. If f is lattice-compact, so is h, and $g^- f_! = h_! k^-$ by Proposition 3.4. Thus, if f is also surjective, then

$$h[P] = h_! P = h_! k^- Y = g^- f_! Y = g^- f[Y] = g^- X = Z,$$

that is, h is surjective. $\square$

Using Proposition 4.1, the so-called *Beck–Chevalley condition* $g^- f_! = h_! k^-$ may conversely be deduced from the pullback-stability of proper surjections: consider the diagram

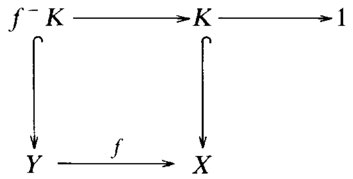


in which f is proper, D is closed in Y and the top-, bottom- and side squares are pullbacks.

The basic stability properties above may be combined to produce some classically familiar facts about proper maps. For example, we have the following:

Corollary 4.3. *Proper maps preserve compactness of sublocales under pre-image.*

Proof. Consider the diagram



in which the horizontal maps are proper. □

A locale X is *Hausdorff* if its diagonal $\Delta : X \hookrightarrow X \times X$ is closed. The following fact produces a wealth of proper maps:

Corollary 4.4. *Any map $f: Y \to X$ from a compact to a Hausdorff locale is proper. In particular, a compact sublocale of a Hausdorff locale is closed.*

Proof. If X is Hausdorff, the graph $\langle \mathrm{id}, f \rangle : Y \hookrightarrow Y \times X$ of f is closed; if Y is compact, the projection $\pi : Y \times X \to X$ is proper. Since $f = \pi \circ \langle \mathrm{id}, f \rangle$, the result follows from Proposition 4.1(i). □

Note that Hausdorffness is again a stable, that is, "internal" property. It follows that compact Hausdorff maps, which by the results of [19] are exactly the compact regular maps (the proper maps of [8]) are stable.

Inverse limits

Proper maps of locales are stable under filtered (inverse) limits. To be precise, we have the following:

Theorem 4.5. *Let X be a locale, and $D: I^{\mathrm{op}} \to \mathscr{L}X$ a (small inversely) filtered diagram for which the objects are proper maps (resp. proper surjections). Then the limit of D is proper (resp. a proper surjection).*

Proof. A filtered limit over X corresponds to a filtered colimit of coframes under $\mathscr{C}X$, calculated in the category of $\mathscr{C}X$-modules under $\mathscr{C}X$. By Lemma 3.3, the result therefore follows through an application of Proposition 2.3. □

Theorem 4.5 has an obvious counterpart involving "parallel" limits (taken in the category $\mathbf{Loc}^2$), which is equivalent to Theorem 4.5 via Proposition 4.2. Although the transition maps d_α in Theorem 4.5 were not required to be proper (resp. proper surjective), we obtain the following:

Corollary 4.6. *Let $D: I^{\mathrm{op}} \to \mathbf{Loc}$ be a filtered diagram of locales and proper maps (resp. proper surjections). Then the canonical projections $l_i: \varprojlim D \to Di$, $i \in \mathrm{Ob}\, I$, are proper (resp. proper surjections).*

Proof. Fix $i \in I$. Since I is filtered, l_i is the limit of the induced filtered diagram $(I/i)^{\mathrm{op}} \to \mathbf{Loc}/Di$ sending $\gamma: i \to k$ to $D\gamma: Dk \to Di$. □

Now, let $f: Y \to X$ and $g: Z \to X$ be proper maps with common codomain, and form the pullback

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Z \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow{\;f\;} & X \end{array}$$

Then the composite $Y \times_X Z \to X$, that is, the product of f and g in $\mathscr{L}X$, is proper, using Proposition 4.2 and Proposition 4.1(i). Further, the projections are proper, and are surjective if both f and g are. By a standard argument, this extends to arbitrary products using Theorem 4.5:

Theorem 4.7. *Let X be a locale. Then the product (that is, common pullback) in $\mathscr{L}X$ of proper maps (resp. proper surjections) with codomain X is proper, with proper (resp. proper surjective) projections.*

Proof. The statement is true for finite products by the preceeding remarks, and the fact that homeomorphisms are proper. An arbitrary product may be constructed as a filtered limit of finite products; in particular, each projection is a filtered limit of projections from finite sub-products, and these are proper (resp. proper surjective). The result therefore follows from Theorem 4.5. □

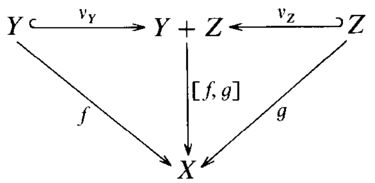
Taking $X \equiv 1$, we recover the localic Tychonoff Theorem [9, 12, 18]: a product of compact locales is compact.

Colimits

Proposition 4.8. *The coproduct in* **Loc**^{2} *of a family of proper maps is proper.*

Proof. Let $\{f_i : Y_i \to X_i \mid i \in I\}$ be a family of proper maps, with parallel coproduct $f: Y \to X$. Then $f_!$ is the product (as monotone mapping) of the family $\{f_{i!}: \mathscr{C}Y_i \to \mathscr{C}X_i \mid i \in I\}$, and in a canonical way inherits linearity over $\mathscr{C}X$ from linearity of each $f_{i!}$ over $\mathscr{C}X_i$. □

For coproducts of proper maps taken over a common codomain X we have to restrict to finite families. Formally, the empty coproduct $0 \hookrightarrow X$ is proper as a closed embedding; also, in the diagram



$[f, g]$ is proper if and only if both f and g are, since

$$\mathscr{C}Y \underset{\nu_{Y^-}}{\overset{\nu_{Y!}}{\rightleftarrows}} \mathscr{C}(Y+Z) \underset{\nu_{Z^-}}{\overset{\nu_{Z!}}{\leftrightarrows}} \mathscr{C}Z$$

is a sum in the category of $\mathscr{C}X$-modules. In particular, the coproduct-embeddings ν_Y, ν_Z are proper, i.e. closed. Using Proposition 4.1(iii), we obtain the following:

Proposition 4.9. *A map of locales is proper as soon as it is proper on (i.e. after pre-composition with each member of) a finite cover.* □

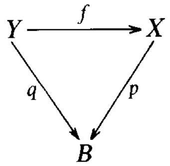
Thus, a finite colimit in $\mathscr{L}X$ of proper maps is proper (or a finite colimit of compact locales is compact). This of course does not mean that the coequalizer of a pair of proper maps is proper: for a rather trivial (discrete) counter-example, consider

$$\mathbb{Z} \underset{2.\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{Z} \overset{q}{\twoheadrightarrow} \mathbb{Z}_2.$$

Strong co-density and weakly fitted sublocales

We end this section by introducing an auxilliary concept with which reference to proper maps plays a rôle analogous to that of strong density [11] for open maps.

Let B be any (base) locale. A map



in $\mathscr{L}B$ is *strongly co-dense* (*over* B) if $q_! f^- = p_!$ (in general, $q_! f^- \le p_!$).

Lemma 4.10. *$f: \langle Y, q\rangle \to \langle X, p\rangle$ is strongly co-dense iff for all $E \in \mathscr{C}B$ and $C \in \mathscr{C}Y$,*

$$f^- C \le q^- E \quad \Rightarrow \quad C \le p^- E.$$

Proof. If $q_! f^- = p_!$, then $f^- C \le q^- E$ iff $p_! C = q_! f^- C \le C$ iff $C \le p^- E$. Conversely, if $f^- C \le q^- E \Rightarrow C \le p^- E$ for arbitrary C, E, then in particular since $f^- \le q^- q_! f^-$, id $\le p^- q_! f^-$, i.e. $p_! \le q_! f^-$. □

The following properties are straightforward to verify:

Proposition 4.11. *For $\langle Z, r\rangle \overset{g}{\to} \langle Y, q\rangle \overset{f}{\to} \langle X, p\rangle$ in $\mathscr{L}B$,*
(i) *f and g strongly co-dense $\Rightarrow$ fg strongly co-dense,*
(ii) *fg strongly co-dense $\Rightarrow$ f strongly co-dense,*
(iii) *f and g strongly co-dense, with f a sublocale inclusion $\Rightarrow$ g strongly co-dense,*
(iv) *f strongly co-dense $\Rightarrow$ p surjective iff q surjective.* □

We may call a sublocale $A \hookrightarrow X \overset{p}{\to} B$ over B *weakly fitted* if it is an intersection of sublocales of the form $U \vee p^- E$, $U \in \mathscr{O}X$ and $E \in \mathscr{C}B$, that is, if A is fixed by the closure operator

$$A \mapsto \bigwedge \{U \vee p^- E \mid A \le U \vee p^- E,\ U \in \mathscr{O}X,\ E \in \mathscr{C}B\}. \tag{9}$$

The weakly fitted sublocales of $\langle X, p\rangle$ are stable under pullback, and of course include the *fitted sublocales*, the name given in [6] to intersections of opens. By Lemma 4.10, the weakly fitted hull (9) of A is the least sublocale of X in which A is strongly co-dense over B. It is possible for all sublocales of $\langle X, p\rangle$ to be weakly fitted, as happens for example over any base when X is regular.

Strong co-density behaves well under pullback along proper maps:

Proposition 4.12. *Let $f:\langle Y, q\rangle \to \langle X, p\rangle$ be a map in $\mathscr{L}B$. Then:*

(i) *If f is strongly co-dense, so is any pullback of f along a proper map.*

(ii) *If q is proper and f is strongly co-dense, then p is proper, and this situation is preserved under change of base.*

Proof. (i) In a pullback

$$\begin{array}{ccc} P & \xrightarrow{k} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & X \end{array}$$

with g proper, the condition $f^{-}g_! = k_!h^{-}$ gives $(qk)_!h^{-} = q_!(k_!h^{-}) = (q_!f^{-})g_!$, which is equal to $(pg)_! = p_!g_!$ when f is strongly co-dense.

(ii) Suppose q is proper, with f strongly co-dense. Then $p_! = q_!f^{-}$ holds as an equation of $\mathscr{C}B$-linear maps, preserved under extension of base. □

5. Proper surjections and equivalence relations

Identification under proper equivalence relations

Consider a diagram of locales

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X \overset{q}{\twoheadrightarrow} Q, \tag{10}$$

where $q\alpha = q\beta$. We say (10) is *left exact* if α, β are the kernel pair of q and *right exact* if q is the coequalizer of α and β.

Lemma 5.1. (10) *is right exact precisely when q is surjective, and (the closure operator) $q^{-}q_!$ the least among monotone operations $j:\mathscr{C}X \to \mathscr{C}X$ satisfying*

$$\mathrm{id} \le j, \qquad \alpha_!\beta^{-}j,\ \beta_!\alpha^{-}j \le j. \tag{11}$$

Proof. First note that for any map $h: X \to Z$, $h\alpha = h\beta$ iff the closure operator $h^{-}h_!$ satisfies the second part of (11), and, given that q is surjective, factors through q iff $q^{-}q_! \le h^{-}h_!$. It therefore only remains to show that if q is the coequalizer of α and β, $q^{-}q_! \le j$ for any order-preserving j satisfying (11). But for such j, given any $D \in \mathscr{C}X$, $D \le jD \in q^{-}[\mathscr{C}Q]$—it follows that $q^{-}q_!D = \bigwedge\{C \in q^{-}[\mathscr{C}Q] \mid D \le C\} \le jD$. □

It is not hard to see that when a relation

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X$$

is reflexive, id $\leq \alpha_!\beta^-$, $\beta_!\alpha^-$ follows, and that when it is symmetric, $\alpha_!\beta^- = \beta_!\alpha^-$. These are in particular the case for an equivalence relation, which we shall call *proper* if α (and then also β) is proper.

Lemma 5.2. *For a proper equivalence relation*

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X,$$

$\alpha_!\beta^- = \beta_!\alpha^-$ *is a closure operator.*

Proof. By the preceeding remarks, we only need to show that $\alpha_!\beta^-$ is idempotent. Form the pullback

$$\begin{array}{ccc} R \times_X R & \xrightarrow{\;a\;} & R \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle \beta} \\ R & \xrightarrow{\;\alpha\;} & X \end{array}$$

Then $\beta^-\alpha_! = a_!b^-$, since α is proper. Let $t: R \times_X R \to R$ be the transitivity of R, unique with the property that $\alpha t = \alpha a$ and $\beta t = \beta b$. Then

$$\begin{aligned} \alpha_!\beta^-\alpha_!\beta^- &= \alpha_! a_! b^- \beta^- = \alpha_! a_! t^- \beta^- \\ &\leq \alpha_! a_! t^- \alpha^- \alpha_! \beta^- = \alpha_! a_! a^- \alpha^- \alpha_! \beta \leq \alpha_! \beta^-. \qquad \square \end{aligned}$$

We have the following lemma:

Lemma 5.3. *Suppose*

$$R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X$$

X is a proper equivalence relation. Then (10) *is right exact precisely when q is surjective and satisfies* $q^-q_! = \alpha_!\beta^- = \beta_!\alpha^-$.

Proof. The fact (Lemma 5.2) that $\alpha_!\beta^- = \beta_!\alpha^-$ is a closure operator is easily seen to imply that it is the least monotone operation on $\mathscr{C}X$ satisfying (11); the statement thus follows from Lemma 5.1. $\square$

We are now able to state the following:

Proposition 5.4. *Proper surjections are coequalizers.*

Proof. Suppose (10) is left exact and q a proper surjection. Then $\langle \alpha, \beta \rangle$ is an equivalence relation, which by Proposition 3.4 is proper, while $q^- q_! = \alpha_! \beta^-$. Now use Lemma 5.3. □

Proposition 5.5. *The coequalizer of any proper equivalence relation is proper, and stable under change of base.*

Proof. Lemma 5.3 again applies. Since q^-, $\alpha_!$ and β^- all preserve filtered infima and commute with the action of $\mathscr{C}Q$, and since q^- is an embedding, the fact that $q^- q_! = \alpha_! \beta^-$ implies that $q_!$ is a morphism of $\mathscr{C}Q$-modules; all this data is preserved under base-extension. □

Combining the last two results with Proposition 4.1, we obtain, using Proposition 1.8, the following theorem:

Theorem 5.6. *Proper surjections are effective descent morphisms in the category of locales.* □

Descent of properties down proper surjections

The fact that proper surjections are effective descent morphisms leads one to consider *properties* of locales (as maps) which descend. That is, if

$$\begin{array}{ccc} E & \xrightarrow{h} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{s} & Q \end{array} \tag{12}$$

is a pullback with s proper surjective, we consider properties of p transferred to q. For example, surjectivity trivially descends down any surjective map. But in this case we also have the following:

Proposition 5.7. *Inclusions, hence homeomorphisms descend down proper surjections.*

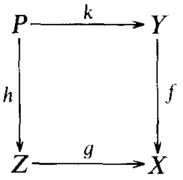
Proof. If p in (12) is an inclusion, $h_! p^- = q^- s_!$ is surjective, which means q^- is surjective, that is, q is an inclusion. □

It follows immediately from the stability of proper surjections that empty inclusions descend. Therefore, we have the following proposition:

Proposition 5.8. *Complemented inclusions descend down proper surjections.* □

We turn to open and proper maps.

Lemma 5.9. *Suppose in a pullback*

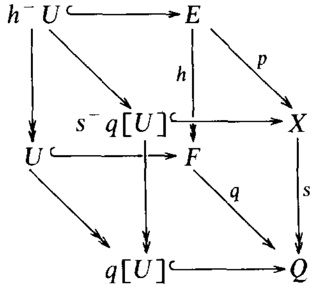


that f is proper, g surjective, and the image of k open. Then k is surjective.

Proof. Let C be the closed complement of the image of k. Then $k^- C = 0$, and so $g^- f_! C = h_! k^- C = 0$. Since g is surjective, $f_! C = 0$, i.e. $C \leq f^- 0 = 0$. □

Proposition 5.10. *Openness and compactness descend down proper surjections.* (*Entirely similarly, compactness and openness descend down open surjections.*)

Proof. We refer to (12). If p is proper, $qh = sp$ is proper; cancelling q leaves h proper Proposition 4.1. So compactness descends. In particular closed, hence by Proposition 5.8, open inclusions descend. Now, if p is an arbitrary open map, then given any open $U \hookrightarrow F$ we consider the diagram

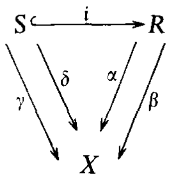


with front-, back- and side squares pullbacks and the vertical arrows proper surjections. Applying Lemma 5.9 to the left-hand pullback gives $s^- q[U] = p[h^- U]$, which is open. But then, since open inclusions descend, $q[U]$ is open. □

Corollary 5.11. *Discreteness and compact Hausdorffness descend down proper surjections.* □

Effectivity of proper equivalence relations

Let X be a locale, and consider an inclusion of proper equivalence relations,



(13)

We say $i:S \hookrightarrow R$ is strongly co-dense over X if $\gamma_! i^- = \alpha^-$ (or equivalently $\delta_! i^- = \beta^-$). Recall that an equivalence relation is said to be *effective* if it is a kernel pair. The following is essentially a reformulation of Lemma 5.3.

Proposition 5.12. *Suppose that in* (13), $\langle \alpha, \beta \rangle$ *is the kernel pair of* $q: X \to Q$. *Then* q *is the coequalizer of* γ *and* δ *precisely when it is a proper surjection, and* $i:S \hookrightarrow R$ *is strongly co-dense over* X.

Proof. If q is the coequalizer of γ and δ, q is proper by Proposition 5.5, and $\gamma_! i^- \beta^- = \gamma_! \delta^- = q^- q_!$ by Lemma 5.3. But then, since $\gamma_! i^-$ is $\mathscr{C}X$-linear, it is the module extension of $q_!$, which is $\alpha_!$:

$$\begin{array}{ccc} C(X \times_Q X) & \xleftarrow{\ \beta^-\ } & \mathscr{C}X \\ \Big\downarrow{\scriptstyle \alpha_!} & & \Big\downarrow{\scriptstyle q_!} \\ \mathscr{C}X & \xleftarrow{\ q^-\ } & \mathscr{C}Q \end{array}$$

Conversely, if $\gamma_! i^- = \alpha_!$ and q is a proper surjection, then $\gamma_! \delta^- = \gamma_! i^- \beta^- = \gamma_! \delta^- = q^- q_!$, and right-exactness follows from Lemma 5.3. □

Corollary 5.13. *The equivalence relations* (13) *have the same quotient iff* $i:S \hookrightarrow R$ *is strongly co-dense over* X. □

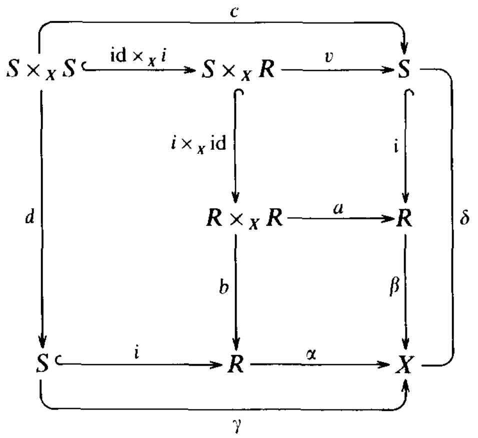
Corollary 5.14. *A proper equivalence relation on* X *which is weakly fitted in* $X \times X$ *over* X (*by either projection*) *is effective. In particular, open equivalence relations with proper projections are effective.* □

The pullback-stability of coequalizers of proper equivalence relations (Proposition 5.5) may also be explained in terms of Proposition 5.12, by observing that kernel pairs, proper surjections (Proposition 3.4) and strongly co-dense inclusions in compact objects Proposition 4.12(ii) are all preserved under change of base.

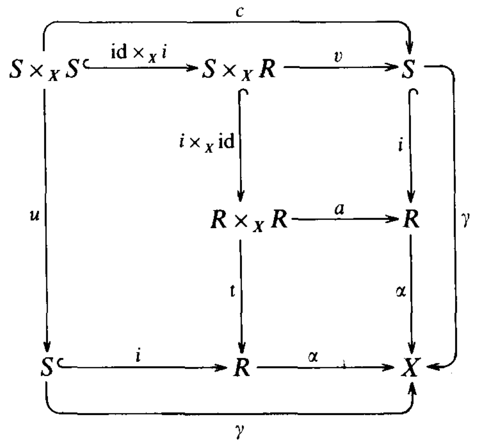
Lemma 5.15. *In* (13), *put* $s = \gamma_! \delta^- = \delta_! \gamma^-$, *and let* $\mathscr{F} \subseteq \mathscr{C}S$ *be a (down-)filtered set of which the members have the form* $\alpha^- C \vee \beta^- D$, *with* $C, D \in \mathscr{C}X$. *Then*

$$\bigwedge \mathscr{F} \leq S \quad \Rightarrow \quad \bigwedge \{\alpha^- sC \vee \beta^- D \mid \alpha^- C \vee \beta^- D \in \mathscr{F}\} \leq S.$$

Proof. Complete the diagram



where all rectangles are pullbacks. Let $g = b \circ i \times_X \mathrm{id}$ be the proper, surjective pullback of δ along α. By the transitivity of R and S there are maps $t: R \times_X R \to R$ and $u: S \times_X S \to S$ (unique) with the property that $\alpha t = \alpha a$, $\beta t = \beta b$ and $\gamma u = \gamma c$, $\delta u = \delta d$. Moreover, by the symmetry of R and S, each corresponding commutative square is a pullback. Thus, since $t \circ i \times_X i = i \circ u$, all rectangles in the diagram



are pullbacks. Let $h = t \circ i \times_X \mathrm{id}$. Then

$$h^{-}S \simeq g^{-}S \simeq S \times_X S \hookrightarrow S \times_X R. \tag{14}$$

Also, given any $C, D \in \mathscr{C}X$,

$$\alpha^{-}sC \vee \beta^{-}D = \alpha^{-}\delta_{!}\gamma^{-}C \vee \beta^{-}D$$

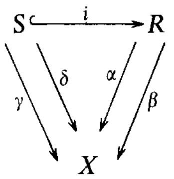
$$= g_{!}v^{-}\gamma^{-}C \vee \beta^{-}D \quad \text{(since } \delta \text{ is proper)}$$

$$= g_!(v^-\gamma^-C \vee g^-\beta^-D) \quad \text{(since } g \text{ is surjective)}$$
$$= g_!h^-(\alpha^-C \vee \beta^-D) \tag{15}$$

Suppose now $\bigwedge \mathscr{F} \leq S$. Then

$$\begin{aligned}
&\bigwedge\{\alpha^-sC \vee \beta^-D \mid \alpha^-C \vee \beta^-D \in \mathscr{F}\}\\
&\quad = \bigwedge\{g_!h^-(\alpha^-C \vee \beta^-D) \mid \alpha^-C \vee \beta^-D \in \mathscr{F}\} \quad \text{(by (15))}\\
&\quad = g_!h^-\bigwedge\mathscr{F} \qquad \text{(since } g \text{ is proper)}\\
&\quad \leq g[h^-S] \qquad \text{(by assumption)}\\
&\quad = g[g^-S] = S. \quad \text{(by (14)).} \qquad \square
\end{aligned}$$

Theorem 5.16. *Let*



be an inclusion of proper equivalence relations, with i closed. Then, if R and S have the same quotient, i is an isomorphism.

Proof. Let $q: X \to Q$ be the common quotient of R and S. By Lemma 5.3, $s \equiv \gamma_!\delta^- = q^-q_!$. Now, suppose $S \leq \alpha^-C \vee \beta^-D$ with $C, D \in \mathscr{C}X$. Then, if $e: X \to R$ is the common splitting of $\langle \alpha, \beta \rangle$,

$$X = e^-S \leq e^-\alpha^-C \vee e^-\beta^-D = C \vee D.$$

It follows that

$$\begin{aligned}
\alpha^-sC \vee \beta^-sD &= \alpha^-q^-q_!C \vee \beta^-q^-q_!D = \alpha^-q^-q_!C \vee \alpha^-q^-q_!D\\
&= \alpha^-q^-q_!(C \vee D) = \alpha^-q^-q_!X \geq \alpha^-X = R.
\end{aligned}$$

But then, since S is closed,

$$\begin{aligned}
S &= \bigwedge\{\alpha^-C \vee \beta^-D \mid S \leq \alpha^-C \vee \beta^-D,\ C, D \in \mathscr{C}X\}\\
&\geq \bigwedge\{\alpha^-sC \vee \beta^-sD \mid S \leq \alpha^-C \vee \beta^-D,\ C, D \in \mathscr{C}X\}\\
&\qquad \text{(by applying Lemma 5.15 twice)}\\
&\geq R. \qquad \square
\end{aligned}$$

Corollary 5.17. *Closed proper equivalence relations (in particular closed equivalence relations on a compact locale) are effective.* □

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