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Sturm–Liouville problems with discontinuities at two points

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Abstract

In this paper we extend some spectral properties of regular Sturm–Liouville problems to those which consist of a Sturm–Liouville equation with piecewise continuous potentials together with eigenparameter-dependent boundary conditions and four supplementary transmission conditions. By modifying some techniques of [C.T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, Proc. Roy. Soc. Edinburgh Sect. A 77 (1977) 293–308; E. Tunç, O.Sh. Muhtarov, Fundamental solutions and eigenvalues of one boundary-value problem with transmission conditions, Appl. Math. Comput. 157 (2004) 347–355; O.Sh. Mukhtarov, E. Tunç, Eigenvalue problems for Sturm–Liouville equations with transmission conditions, Israel J. Math. 144 (2004) 367–380] and [O.Sh. Mukhtarov, M. Kadakal, F.Ş. Muhtarov, Eigenvalues and normalized eigenfunctions of discontinuous Sturm–Liouville problem with transmission conditions, Rep. Math. Phys. 54 (2004) 41–56], we give an operator-theoretic formulation for the considered problem and obtain asymptotic formulae for the eigenvalues and eigenfunctions.

Keywords: Sturm–Liouville problems; Transmission conditions; The asymptotic of eigenvalues and eigenfunctions; Discontinuous boundary-value problems; Eigenparameter-dependent boundary conditions

1. Introduction

Sturmian theory is one of the most extensively developing fields in theoretical and applied mathematics. Particularly, there has been increasing interest in the spectral analysis of boundary value problems with eigenvalue-dependent boundary conditions. There are quite substantial literatures on such problems. Here we mention the results of [1,5–12] and the corresponding references cited therein.

Basically, boundary-value problems with continuous coefficients at the highest derivative of the equation have been investigated. Note that discontinuous Sturm–Liouville problems with eigen-dependent boundary conditions and with two supplementary transmission conditions at the point(s) of discontinuity have been investigated in [2–4,13,14]. In this paper, we shall consider discontinuous eigenvalue problem which consist of the differential equation

$$\tau u := -u'' + q(x)u = \lambda u \tag{1.1}$$

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on $[a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$, with boundary condition at $x = a$

$$L_1 u := \alpha_1 u(a) + \alpha_2 u'(a) = 0, \tag{1.2}$$

with the four transmission conditions at the points of discontinuity $x = \xi_1$ and $x = \xi_2$,

$$L_2 u := \gamma_1 u(\xi_1 - 0) - \delta_1 u(\xi_1 + 0) = 0 \tag{1.3}$$

$$L_3 u := \gamma_1' u'(\xi_1 - 0) - \delta_1' u'(\xi_1 + 0) = 0 \tag{1.4}$$

$$L_4 u := \gamma_2 u(\xi_2 - 0) - \delta_2 u(\xi_2 + 0) = 0 \tag{1.5}$$

$$L_5 u := \gamma_2' u'(\xi_2 - 0) - \delta_2' u'(\xi_2 + 0) = 0 \tag{1.6}$$

and the eigen-dependent boundary condition at $x = b$

$$L_6(\lambda) u := \lambda \left[\beta_1' u(b) - \beta_2' u'(b)\right] + \left[\beta_1 u(b) - \beta_2 u'(b)\right] = 0 \tag{1.7}$$

where $q(x)$ is a given real-valued function continuous in $[a, \xi_1]$, $[\xi_1, \xi_2]$ and $[\xi_2, b]$ (that is, continuous in $[a, \xi_1)$, (ξ_1, ξ_2) and $(\xi_2, b]$ and has finite limits $q(\xi_1\pm) := \lim_{x\to\xi_1\pm} q(x)$, $q(\xi_2\pm) := \lim_{x\to\xi_2\pm} q(x)$); λ is a complex eigenvalue parameter; the coefficients of the boundary and transmission conditions are real numbers. We assume $|\alpha_1| + |\alpha_2| \neq 0$, $|\gamma_i| + |\delta_i| \neq 0$, $|\gamma_i'| + |\delta_i'| \neq 0\,(i = 1, 2)$ and $\rho = \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0$. In contrast to previous works, the eigenfunctions of this problem may have discontinuities.

Note that problems of such a type arise, as a rule, in the theory of heat and mass transfer problems, and in a varied assortment of physical transfer problems. (See [1,8] and [15] and corresponding references cited therein for various physical applications.)

2. Preliminaries

For convenience let us introduce the following notations:

$$\Omega_1 := [a, \xi_1], \quad \Omega_2 := [\xi_1, \xi_2], \quad \Omega_3 := [\xi_2, b], \quad u_{(1)}(x) := \begin{cases} u(x) & x \in [a, \xi_1) \\ \lim\limits_{x\to\xi_1-} u(x) & x = \xi_1, \end{cases}$$

$$u_{(2)}(x) := \begin{cases} u(x) & x \in (\xi_1, \xi_2) \\ \lim\limits_{x\to\xi_1+} u(x) & x = \xi_1, \end{cases} \qquad u_{(3)}(x) := \begin{cases} u(x) & x \in (\xi_1, \xi_2) \\ \lim\limits_{x\to\xi_2-} u(x) & x = \xi_2, \end{cases}$$

$$u_{(4)}(x) := \begin{cases} u(x) & x \in (\xi_2, b] \\ \lim\limits_{x\to\xi_2+} u(x) & x = \xi_2 \end{cases}$$

$$(u)_\beta := \lim_{x\to b} \left(\beta_1 u(x) - \beta_2 u'(x)\right), \quad (u)_\beta' := \lim_{x\to b} \left(\beta_1' u(x) - \beta_2' u'(x)\right), \quad \tilde{u}(x) = \begin{cases} u(x), & x \in [a, b) \\ (u)_\beta', & x = b. \end{cases}$$

Note that everywhere below, we shall assume that $\gamma_i \gamma_i' \delta_i \delta_i' > 0\,(i = 1, 2)$, and for the Lebesque measurable subsets $M \subset [a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$ with Lebesque measure $\mu_L(M)$, we shall define a new positive measure $\mu_\rho(M)$ by

$$\mu_\rho(M) := \frac{\gamma_1\gamma_1'}{\delta_1\delta_1'} \mu_L(M \cap [a, \xi_1)) + \mu_L(M \cap (\xi_1, \xi_2)) + \frac{\delta_2\delta_2'}{\gamma_2\gamma_2'} \mu_L(M \cap (\xi_2, b]) + \frac{\delta_2\delta_2'}{\gamma_2\gamma_2'} \frac{b(M)}{\rho}$$

where

$$b(M) := \begin{cases} 0 & \text{if } b \notin M \\ 1 & \text{if } b \in M. \end{cases}$$

Let $\langle \cdot, \cdot \rangle_{H_\rho}$ denote the scalar product in the Hilbert space $H_\rho := L^2\left([a, b]; \mu_\rho\right)$. In this space, we define a linear operator A by the domain of definition

$$D(A) := \Big\{u \in H_\rho | u_{(i)}, u'_{(i)} \text{ are absolutely continuous in } \Omega_i\ (i = 1, 2, 3),\ \tau\tilde{u} \in L^2[a, b]$$
$$\alpha_1\tilde{u}(a) + \alpha_2\tilde{u}'(a) = 0,\ \gamma_1\tilde{u}(\xi_1 - 0) = \delta_1\tilde{u}(\xi_1 + 0),\ \gamma_1'\tilde{u}'(\xi_1 - 0) = \delta_1'\tilde{u}'(\xi_1 + 0),$$
$$\gamma_2\tilde{u}(\xi_2 - 0) = \delta_2\tilde{u}(\xi_2 + 0),\ \gamma_2'\tilde{u}'(\xi_2 - 0) = \delta_2'\tilde{u}'(\xi_2 + 0)\Big\}$$

and

$$(A\tilde{u})(x) = \begin{cases} (\tau\tilde{u})(x) & \text{for } x \in [a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b) \\ -(\tilde{u})_\beta & \text{for } x = b. \end{cases}$$

Consequently, the considered problem (1.1)–(1.7) can be rewritten in operator form as

$$A\tilde{u} = \lambda\tilde{u}$$

i.e., the problem (1.1)–(1.7) can be considered as the eigenvalue problem for the operator A.

Theorem 2.1. *The operator A is symmetric.*

Proof. Let $f, g \in D(A)$. By two partial integrations, we get

$$\langle Af, g\rangle_{H_\rho} - \langle f, Ag\rangle_{H_\rho} = \frac{\gamma_1\gamma_1'}{\delta_1\delta_1'}(W(f, \bar{g}; \xi_1 - 0) - W(f, \bar{g}; a)) + (W(f, \bar{g}; \xi_2 - 0) - W(f, \bar{g}; \xi_1 + 0))$$
$$+ \frac{\delta_2\delta_2'}{\gamma_2\gamma_2'}(W(f, \bar{g}; b) - W(f, \bar{g}; \xi_2 + 0)) - \frac{\delta_2\delta_2'}{\gamma_2\gamma_2'}\frac{1}{\rho}((f)'_\beta(\bar{g})_\beta - (f)_\beta(\bar{g})'_\beta) \tag{2.1}$$

where, as usual,

$$W(f, g; x) = f(x)g'(x) - f'(x)g(x) \tag{2.2}$$

denotes the Wronskians of the functions f and g. Since f and $\bar{g}$ satisfy the boundary condition (1.2), it follows that

$$W(f, \bar{g}; a) = 0. \tag{2.3}$$

From the transmission conditions (1.3)–(1.6), we get

$$\gamma_i\gamma_i' W(f, g; \xi_i - 0) = \delta_i\delta_i' W(f, g; \xi_i + 0) \quad (i = 1, 2). \tag{2.4}$$

Further, it is easy to verify that

$$(f)_\beta(\bar{g})'_\beta - (f)'_\beta(\bar{g})_\beta = \rho W(f, \bar{g}; b). \tag{2.5}$$

Finally, substituting (2.2)–(2.5) in (2.1) yields the required equality

$$\langle Af, g\rangle_{H_\rho} = \langle f, Ag\rangle_{H_\rho} (f, g \in H_\rho). \quad \square \tag{2.6}$$

Corollary 2.1. *All eigenvalues of the problem (1.1)–(1.7) are real.*

We can now assume that all eigenfunctions are real-valued.

Corollary 2.2. *If λ_1 and λ_2 are two different eigenvalues of the problem (1.1)–(1.7), then corresponding eigenfunctions u_1 and u_2 of this problem satisfy the following equality:*

$$\frac{\gamma_1\gamma_1'}{\delta_1\delta_1'}\int_a^{\xi_1} u_1(x)u_2(x)\mathrm{d}x + \int_{\xi_1}^{\xi_2} u_1(x)u_2(x)\mathrm{d}x + \frac{\delta_2\delta_2'}{\gamma_2\gamma_2'}\int_{\xi_2}^{b} u_1(x)u_2(x)\,\mathrm{d}x + \frac{1}{\rho}\frac{\delta_2\delta_2'}{\gamma_2\gamma_2'}(u_1)'_\beta(u_2)'_\beta = 0. \tag{2.7}$$

In fact, this formula means the orthogonality of eigenfunctions u_1 and u_2 in the Hilbert space H_ρ.

We need the following lemma, which can be proved similarly to [2, Theorem 2].

Lemma 2.1. *Let the real-valued function $q(x)$ be continuous in $[a, b]$ where $f(\lambda)$, $g(\lambda)$ are given entire functions. Then for any $\lambda \in \mathbb{C}$ the equation*

$$-u'' + q(x)u = \lambda u, \quad x \in [a, b]$$

has a unique solution $u = u(x, \lambda)$ such that

$$u(a) = f(\lambda), \quad u'(a) = g(\lambda) \quad \big(\text{or } u(b) = f(\lambda), u'(b) = g(\lambda)\big),$$

and for each $x \in [a, b]$, $u(x, \lambda)$ is an entire function of λ.

We shall define two solutions

$$\phi_\lambda(x) = \begin{cases} \phi_{1\lambda}(x), & x \in [a, \xi_1) \\ \phi_{2\lambda}(x), & x \in (\xi_1, \xi_2) \\ \phi_{3\lambda}(x), & x \in (\xi_2, b]. \end{cases} \quad \text{and} \quad \chi_\lambda(x) = \begin{cases} \chi_{1\lambda}(x), & x \in [a, \xi_1) \\ \chi_{2\lambda}(x), & x \in (\xi_1, \xi_2) \\ \chi_{3\lambda}(x), & x \in (\xi_2, b] \end{cases}$$

of the Eq. (1.1) as follows: Let $\phi_{1\lambda}(x) = \phi_1(x, \lambda)$ be the solution of Eq. (1.1) on $[a, \xi_1]$, which satisfies the initial conditions

$$\begin{pmatrix} u(a) \\ u'(a) \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ -\alpha_1 \end{pmatrix}. \tag{2.8}$$

By virtue of Lemma 2.1, after defining this solution, we may define the solution $\phi_2(x, \lambda)$ of Eq. (1.1) on $[\xi_1, \xi_2]$ by means of the solution $\phi_1(x, \lambda)$ by the nonstandard initial conditions

$$\begin{pmatrix} u(\xi_1 + 0) \\ u'(\xi_1 + 0) \end{pmatrix} = \begin{pmatrix} \dfrac{\gamma_1}{\delta_1}\phi_{1\lambda}(\xi_1 - 0, \lambda) \\ \dfrac{\gamma_1'}{\delta_1'}\phi_{1\lambda}'(\xi_1 - 0, \lambda) \end{pmatrix}. \tag{2.9}$$

After defining this solution, we may define the solution $\phi_3(x, \lambda)$ of Eq. (1.1) on $[\xi_2, b]$ by means of the solution $\phi_2(x, \lambda)$ by the nonstandard initial conditions

$$\begin{pmatrix} u(\xi_2 + 0) \\ u'(\xi_2 + 0) \end{pmatrix} = \begin{pmatrix} \dfrac{\gamma_2}{\delta_2}\phi_{2\lambda}(\xi_2 - 0, \lambda) \\ \dfrac{\gamma_2'}{\delta_2'}\phi_{2\lambda}'(\xi_2 - 0, \lambda) \end{pmatrix}. \tag{2.10}$$

Hence, $\phi(x, \lambda)$ satisfies the Eq. (1.1) on $[a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$, the boundary condition (1.2), and the transmission conditions (1.3)–(1.6).

Analogically, first we define the solution $\chi_{3\lambda}(x) = \chi_3(x, \lambda)$ on $[\xi_2, b]$ by the initial conditions

$$\begin{pmatrix} u(b) \\ u'(b) \end{pmatrix} = \begin{pmatrix} \beta_2'\lambda + \beta_2 \\ \beta_1'\lambda + \beta_1 \end{pmatrix}. \tag{2.11}$$

Again, after defining this solution, we define the solution $\chi_{2\lambda}(x) := \chi_2(x, \lambda)$ of the Eq. (1.1) on $[\xi_1, \xi_2]$ by the initial conditions

$$\begin{pmatrix} u(\xi_2 - 0) \\ u'(\xi_2 - 0) \end{pmatrix} = \begin{pmatrix} \dfrac{\delta_1}{\gamma_1}\chi_{3\lambda}(\xi_2 + 0, \lambda) \\ \dfrac{\delta_1'}{\gamma_1'}\chi_{3\lambda}'(\xi_2 + 0, \lambda) \end{pmatrix}. \tag{2.12}$$

After defining this solution, we define the solution $\chi_{1\lambda}(x) := \chi_1(x, \lambda)$ of the Eq. (1.1) on $[\xi_2, b]$ by the initial conditions

$$\begin{pmatrix} u(\xi_1-0) \\ u'(\xi_1-0) \end{pmatrix} = \begin{pmatrix} \dfrac{\delta_2}{\gamma_2}\chi_{3\lambda}(\xi_1+0,\lambda) \\ \dfrac{\delta_2'}{\gamma_2'}\chi_{3\lambda}'(\xi_1+0,\lambda) \end{pmatrix}. \tag{2.13}$$

Hence, $\chi(x,\lambda)$ satisfies the equality (1.1) on $[a,\xi_1)\cup(\xi_1,\xi_2)\cup(\xi_2,b]$, the boundary condition (1.7) and the transmission conditions (1.3)–(1.6).

Further it follows from (1.1) that the Wronskians

$$\omega_i(\lambda) := W_\lambda(\phi_i,\chi_i;x) := \phi_i(x,\lambda)\chi_i'(x,\lambda) - \phi_i'(x,\lambda)\chi_i(x,\lambda), \quad x\in\Omega_i \ (i=1,2,3)$$

are independent of $x\in\Omega_i$. Moreover, these functions are entire functions of λ.

Lemma 2.2. *For each $\lambda\in\mathbb{C}$, $\gamma_1\gamma_1'\gamma_2\gamma_2'\omega_1(\lambda) = \delta_1\delta_1'\gamma_2\gamma_2'\omega_2(\lambda) = \delta_1\delta_2\delta_1'\delta_2'\omega_3(\lambda)$.*

Proof. In view of (2.9), (2.10), (2.12) and (2.13), a short calculation gives

$$\gamma_1\gamma_1'\gamma_2\gamma_2' W(\phi_1,\chi_1;\xi_1-0) = \delta_1\delta_1'\gamma_2\gamma_2' W(\phi_2,\chi_2;\xi_1+0) = \delta_1\delta_2\delta_1'\delta_2' W(\phi_3,\chi_3;\xi_2+0)$$

so $\gamma_1\gamma_1'\gamma_2\gamma_2'\omega_1(\lambda) = \delta_1\delta_1'\gamma_2\gamma_2'\omega_2(\lambda) = \delta_1\delta_2\delta_1'\delta_2'\omega_3(\lambda)$ for each $\lambda\in\mathbb{C}$. □

Now we may introduce the characteristic function

$$\omega(\lambda) := \gamma_1\gamma_1'\gamma_2\gamma_2'\omega_1(\lambda) = \delta_1\delta_1'\gamma_2\gamma_2'\omega_2(\lambda) = \delta_1\delta_2\delta_1'\delta_2'\omega_3(\lambda).$$

Theorem 2.2. *The eigenvalues of the problem (1.1)–(1.7) are the zeros of the function $\omega(\lambda)$.*

Proof. Let $\omega(\lambda_0)=0$. Then $W_{\lambda_0}(\phi_1,\chi_1;x)=0$ and therefore the functions $\phi_{1\lambda_0}(x)$ and $\chi_{1\lambda_0}(x)$ are linearly dependent, i.e.

$$\chi_{1\lambda_0}(x) = k_1\phi_{1\lambda_0}(x), \quad x\in[a,\xi_1]$$

for some $k_1\neq 0$. From this, it follows that $\chi(x,\lambda_0)$ satisfies also the first boundary condition (1.2), so $\chi(x,\lambda_0)$ is an eigenfunction for the eigenvalue λ_0.

Now let $u_0(x)$ be any eigenfunction corresponding to eigenvalue λ_0, but $\omega(\lambda_0)\neq 0$. Then the pair of the functions (ϕ_1,χ_1), (ϕ_2,χ_2) and (ϕ_3,χ_3) would be linearly independent on $[a,\xi_1]$, $[\xi_1,\xi_2]$ and $[\xi_2,b]$ respectively. Therefore $u_0(x)$ may be represented as

$$u_0(x) = \begin{cases} c_1\phi_1(x,\lambda_0)+c_2\chi_1(x,\lambda_0), & x\in[a,\xi_1) \\ c_3\phi_2(x,\lambda_0)+c_4\chi_2(x,\lambda_0), & x\in(\xi_1,\xi_2) \\ c_5\phi_3(x,\lambda_0)+c_6\chi_3(x,\lambda_0), & x\in(\xi_2,b]. \end{cases}$$

where at least one of the constants c_1,c_2,c_3,c_4,c_5,c_6 is not zero. Considering the equations

$$L_\nu(u_0(x)) = 0, \quad \nu=\overline{1,6} \tag{2.14}$$

as a system of linear equations of the variables c_1,c_2,c_3,c_4,c_5,c_6, and taking (2.9), (2.10), (2.12) and (2.13) into account, it follows that the determinant of this system is

$$\begin{vmatrix} 0 & \omega_1(\lambda_0) & 0 & 0 & 0 & 0 \\ \gamma_1\phi_{1\lambda_0}(\xi_1-0) & \gamma_1\chi_{1\lambda_0}(\xi_1-0) & -\delta_1\phi_{2\lambda_0}(\xi_1+0) & -\delta_1\chi_{2\lambda_0}(\xi_1+0) & 0 & 0 \\ \gamma_1'\phi_{1\lambda_0}'(\xi_1-0) & \gamma_1'\chi_{1\lambda_0}'(\xi_1-0) & -\delta_1'\phi_{2\lambda_0}'(\xi_1+0) & -\delta_1'\chi_{2\lambda_0}'(\xi_1+0) & 0 & 0 \\ 0 & 0 & \gamma_2\phi_{2\lambda_0}(\xi_2-0) & \gamma_2\chi_{2\lambda_0}(\xi_2-0) & -\delta_2\phi_{3\lambda_0}(\xi_2+0) & -\delta_2\chi_{3\lambda_0}(\xi_2+0) \\ 0 & 0 & \gamma_2'\phi_{2\lambda_0}'(\xi_2-0) & \gamma_2'\chi_{2\lambda_0}'(\xi_2-0) & -\delta_2'\phi_{3\lambda_0}'(\xi_2+0) & -\delta_2'\chi_{3\lambda_0}'(\xi_2+0) \\ 0 & 0 & 0 & 0 & \omega_3(\lambda_0) & 0 \end{vmatrix}$$

$$= -\delta_1\delta_2\delta_1'\delta_2'\omega_1(\lambda_0)\omega_2(\lambda_0)\omega^3(\lambda_0) \neq 0.$$

Therefore, the system (2.14) has only the trivial solution $c_1=c_2=c_3=c_4=c_5=c_6=0$. Thus we get a contradiction, which completes the proof. □

Lemma 2.3. *If $\lambda = \lambda_0$ is an eigenvalue, then $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are linearly dependent.*

Proof. Let $\lambda = \lambda_0$ be an eigenvalue. Then by virtue of Theorem 2.2

$$W\left(\phi_{i\lambda_0}, \chi_{i\lambda_0}; x\right) = \omega_i(\lambda_0) = 0$$

and therefore

$$\chi_{i\lambda_0}(x) = k_i \phi_{i\lambda_0}(x) \quad (i = 1, 2, 3) \tag{2.15}$$

for some $k_1 \neq 0$, $k_2 \neq 0$ and $k_3 \neq 0$. We must show that $k_1 = k_2 = k_3$. Suppose, if possible, that $k_1 \neq k_2$.

Taking into account the definitions of the solutions $\phi_i(x, \lambda)$ and $\chi_i(x, \lambda)$ and the equalities (2.15), we have

$$\begin{aligned} \delta_1(k_1 - k_2)\phi_{2\lambda}(\xi_1 + 0) &= \delta_1 k_1 \phi_{2\lambda}(\xi_1 + 0) - \delta_1 k_2 \phi_{2\lambda}(\xi_1 + 0) \\ &= k_1 \gamma_1 \phi_{1\lambda}(\xi_1 - 0) - k_2 \delta_1 \phi_{2\lambda}(\xi_1 + 0) \\ &= \gamma_1 \chi_{1\lambda}(\xi_1 - 0) - \delta_1 \chi_{2\lambda}(\xi_1 + 0) = 0. \end{aligned}$$

Hence

$$\phi_{2\lambda_0}(\xi_1 + 0) = 0. \tag{2.16}$$

Analogically, starting from $\delta_1'(k_1 - k_2)\phi_{2\lambda}'(\xi_1 + 0)$ and following the same procedure, we can derive that

$$\phi_{2\lambda_0}'(\xi_1 + 0) = 0. \tag{2.17}$$

From the fact that $\phi_{2\lambda_0}(x)$ is a solution of the differential equation (1.1) on $[\xi_1, \xi_2]$ and satisfies the initial conditions (2.16) and (2.17), it follows that $\phi_{2\lambda}(x) = 0$ identically on $[\xi_1, \xi_2]$. Making use of (2.9), (2.10), (2.16) and (2.17), we may also derive that

$$\phi_{1\lambda_0}(\xi_1 - 0) = \phi_{1\lambda_0}'(\xi_1 - 0) = 0$$

and

$$\phi_{3\lambda_0}(\xi_2 + 0) = \phi_{3\lambda_0}'(\xi_2 + 0) = 0$$

respectively. From this, by the same argument as for $\phi_{2\lambda_0}(x)$, it follows that $\phi_{1\lambda_0}(x) = 0$ identically on $[a, \xi_1]$ and $\phi_{3\lambda_0}(x) = 0$ identically on $[\xi_2, b]$. Hence $\phi(x, \lambda_0) = 0$ identically on $[a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$. But this contradicts (2.8), since $|\alpha_1| + |\alpha_2| \neq 0$. □

Corollary 2.3. *If $\lambda = \lambda_0$ is an eigenvalue, then both $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are eigenfunctions corresponding to this eigenvalue.*

Lemma 2.4. *All eigenvalues λ_n are simple zeros of $\omega(\lambda)$.*

Proof. Using the well known Lagrange's formula (cf. [16], p. 6–7), it can be shown that

$$(\lambda - \lambda_n)\left(\int_a^{\xi_1} \phi_\lambda(x)\phi_{\lambda_n}(x)\mathrm{d}x + \int_{\xi_1}^{\xi_2} \phi_\lambda(x)\phi_{\lambda_n}(x)\mathrm{d}x + \int_{\xi_2}^{b} \phi_\lambda(x)\phi_{\lambda_n}(x)\mathrm{d}x\right) = W(\phi_\lambda, \phi_{\lambda_n}; b) \tag{2.18}$$

for any λ. Recall that

$$\chi_{\lambda_n}(x) = k_n \phi_{\lambda_n}(x), \quad x \in [a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]$$

for some $k_n \neq 0$, $n = 1, 2, \ldots$. Using this equality for the right side of (2.18), we have

$$\begin{aligned} W\left(\phi_\lambda, \phi_{\lambda_n}; b\right) &= \frac{1}{k_n} W\left(\phi_\lambda, \chi_{\lambda_n}; b\right) = \frac{1}{k_n}\left(\lambda_n (\phi_\lambda)_\beta' + (\phi_\lambda)_\beta\right) \\ &= \frac{1}{k_n}\left[\omega(\lambda) + (\lambda - \lambda_n)(\phi_\lambda)_\beta'\right] \\ &= (\lambda - \lambda_n)\frac{1}{k_n}\left[\frac{\omega(\lambda)}{\lambda - \lambda_n} - (\phi_\lambda)_\beta'\right]. \end{aligned}$$

Substituting this formula in (2.18) and letting $\lambda \to \lambda_n$, we get

$$\int_a^{\xi_1} \left(\phi_{\lambda_n}(x)\right)^2 \mathrm{d}x + \int_{\xi_1}^{\xi_2} \left(\phi_{\lambda_n}(x)\right)^2 \mathrm{d}x + \int_{\xi_2}^{b} \left(\phi_{\lambda_n}(x)\right)^2 \mathrm{d}x = \frac{1}{k_n}\left(\omega'(\lambda_n) - (\phi_{\lambda_n})'_\beta\right). \tag{2.19}$$

Now putting

$$(\phi_{\lambda_n})'_\beta = \frac{1}{k_n}(\chi_{\lambda_n})'_\beta = \frac{\rho}{k_n}$$

in (2.19) seems that $\omega'(\lambda_n) \neq 0$. □

3. Asymptotic approximate formulas of $\omega(\lambda)$ for four distinct cases

We begin by proving some lemmas.

Lemma 3.1. *Let $\phi(x,\lambda)$ be the solutions of Eq. (1.1) defined in Section 2, and let $\lambda = s^2$. Then the following integral equations hold:*

$$\begin{aligned}\phi_{1\lambda}^{(k)}(x) &= \alpha_2\left(\cos s(x-a)\right)^{(k)} - \alpha_1\frac{1}{s}\left(\sin s(x-a)\right)^{(k)} \\ &\quad + \frac{1}{s}\int_a^x \left(\sin s(x-y)\right)^{(k)} q(y)\phi_{1\lambda}(y)\mathrm{d}y, \quad k=0,1,\end{aligned} \tag{3.1$_k$}$$

$$\begin{aligned}\phi_{2\lambda}^{(k)}(x) &= \frac{\gamma_1}{\delta_1}\phi_{1\lambda}(\xi_1-0)\left(\cos s(x-\xi_1)\right)^{(k)} + \frac{1}{s}\frac{\gamma_1'}{\delta_1'}\phi_{1\lambda}'(\xi_1-0)\left(\sin s(x-\xi_1)\right)^{(k)} \\ &\quad + \frac{1}{s}\int_{\xi_1}^x \left(\sin s(x-y)\right)^{(k)} q(y)\phi_{2\lambda}(y)\mathrm{d}y, \quad k=0,1,\end{aligned} \tag{3.2$_k$}$$

$$\begin{aligned}\phi_{3\lambda}^{(k)}(x) &= \frac{\gamma_2}{\delta_2}\phi_{2\lambda}(\xi_2-0)\left(\cos s(x-\xi_2)\right)^{(k)} + \frac{1}{s}\frac{\gamma_2'}{\delta_2'}\phi_{2\lambda}'(\xi_2-0)\left(\sin s(x-\xi_2)\right)^{(k)} \\ &\quad + \frac{1}{s}\int_{\xi_2}^x \left(\sin s(x-y)\right)^{(k)} q(y)\phi_{3\lambda}(y)\mathrm{d}y, \quad k=0,1,\end{aligned} \tag{3.3$_k$}$$

where $(\bullet)^{(k)} = \frac{\mathrm{d}^k}{\mathrm{d}x^k}(\bullet)$.

Proof. It is enough to substitute $s^2\phi_{1\lambda}(y)+\phi_{1\lambda}''(y)$, $s^2\phi_{2\lambda}(y)+\phi_{2\lambda}''(y)$ and $s^2\phi_{3\lambda}(y)+\phi_{3\lambda}''(y)$ instead of $q(y)\phi_{1\lambda}(y)$, $q(y)\phi_{2\lambda}(y)$ and $q(y)\phi_{3\lambda}(y)$ in the integral terms of the (3.1$_k$), (3.2$_k$) and (3.3$_k$), respectively, and integrate by parts twice. □

Lemma 3.2. *Let $\lambda = s^2$, $\operatorname{Im} s = t$. Then the functions $\phi_{i\lambda}(x)$ have the following asymptotic representations for $|\lambda| \to \infty$, which hold uniformly for $x \in \Omega_i$ (for $i = 1,2,3$):*

$$\phi_{1\lambda}^{(k)}(x) = \alpha_2\left(\cos s(x-a)\right)^{(k)} + O\left(|s|^{k-1}\mathrm{e}^{|t|(x-a)}\right) \tag{3.4$_k$}$$

$$\begin{aligned}\phi_{2\lambda}^{(k)}(x) &= \alpha_2\left[\frac{\gamma_1}{\delta_1}\left(\cos s(x-\xi_1)\right)^{(k)}\cos s(\xi_1-a)\right. \\ &\quad \left. - \frac{\gamma_1'}{\delta_1'}\left(\sin s(x-\xi_1)\right)^{(k)}\sin s(\xi_1-a)\right] + O\left(|s|^{k-1}\mathrm{e}^{|t|[(x-\xi_1)+(\xi_1-a)]}\right)\end{aligned} \tag{3.5$_k$}$$

$$\begin{aligned}\phi_{3\lambda}^{(k)}(x) &= \alpha_2\left\{\frac{\gamma_2}{\delta_2}\left(\cos s(x-\xi_2)\right)^{(k)}\left[\frac{\gamma_1}{\delta_1}\cos s(\xi_2-\xi_1)\cos s(\xi_1-a)\right.\right. \\ &\quad \left. - \frac{\gamma_1'}{\delta_1'}\sin s(\xi_2-\xi_1)\sin s(\xi_1-a)\right] - \frac{\gamma_2'}{\delta_2'}\left(\sin s(x-\xi_2)\right)^{(k)}\left[\frac{\gamma_1}{\delta_1}\sin s(\xi_2-\xi_1)\cos s(\xi_1-a)\right. \\ &\quad \left.\left. + \frac{\gamma_1'}{\delta_1'}\cos s(\xi_2-\xi_1)\sin s(\xi_1-a)\right]\right\} + O\left(|s|^{k-1}\mathrm{e}^{|t|[(x-\xi_2)+(\xi_2-\xi_1)+(\xi_1-a)]}\right)\end{aligned} \tag{3.6$_k$}$$

if $\alpha_2 \neq 0$,

$$\phi_{1\lambda}^{(k)}(x) = -\frac{1}{s}\alpha_1 \left[\sin s(x-a)\right]^{(k)} + O\left(|s|^{k-2} \, \mathrm{e}^{|t|(x-a)}\right) \tag{3.7$_k$}$$

$$\begin{aligned}\phi_{2\lambda}^{(k)}(x) \;=\; & -\frac{\alpha_1}{s}\left[\frac{\gamma_1}{\delta_1}\left(\cos s(x-\xi_1)\right)^{(k)} \sin s(\xi_1-a)\right.\\ & \left. + \; \frac{\gamma_1'}{\delta_1'}\left(\sin s(x-\xi_1)\right)^{(k)} \cos s(\xi_1-a)\right] + O\left(|s|^{k-2}\, \mathrm{e}^{|t|[(x-\xi_1)+(\xi_1-a)]}\right)\end{aligned} \tag{3.8$_k$}$$

$$\begin{aligned}\phi_{3\lambda}^{(k)}(x) \;=\; & -\frac{\alpha_1}{s}\left\{\frac{\gamma_2}{\delta_2}\left(\cos s(x-\xi_2)\right)^{(k)}\left[\frac{\gamma_1}{\delta_1}\cos s(\xi_2-\xi_1)\sin s(\xi_1-a)\right.\right.\\ & \left. + \; \frac{\gamma_1'}{\delta_1'}\sin s(\xi_2-\xi_1)\cos s(\xi_1-a)\right] - \frac{\gamma_2'}{\delta_2'}\left(\sin s(x-\xi_2)\right)^{(k)}\left[\frac{\gamma_1}{\delta_1}\sin s(\xi_2-\xi_1)\sin s(\xi_1-a)\right.\\ & \left.\left. - \; \frac{\gamma_1'}{\delta_1'}\cos s(\xi_2-\xi_1)\cos s(\xi_1-a)\right]\right\} + O\left(|s|^{k-2}\, \mathrm{e}^{|t|[(x-\xi_2)+(\xi_2-\xi_1)+(\xi_1-a)]}\right)\end{aligned} \tag{3.9$_k$}$$

if $\alpha_2 = 0$.

Proof. Since the proof of the formulae for $\phi_{1\lambda}(x)$ are identical to Titchmarsh's proof of similar results for $\phi_\lambda(x)$ (see [17], Lemma 1.7 p. 9–10), we may formulate them without proving them here. But the similar formulae for $\phi_{2\lambda}(x)$ and $\phi_{3\lambda}(x)$ need individual consideration, since the last solutions are defined by the initial conditions of these special nonstandard forms. We shall only prove the formula (3.5_0) for $k = 0$.

Let $\alpha_2 \neq 0$. Then according to (3.4_k)

$$\phi_{1\lambda}(\xi_1 - 0) = \alpha_2 \cos s(\xi_1 - a) + O\left(|s|^{-1}\, \mathrm{e}^{|t|(\xi_1-a)}\right)$$

and

$$\phi_{1\lambda}'(\xi_1 - 0) = -s\alpha_2 \sin s(x-a) + O\left(\mathrm{e}^{|t|(\xi_1-a)}\right).$$

Substituting these asymptotic expressions into (3.2_0), we get

$$\begin{aligned}\phi_{2\lambda}(x) \;=\; & \alpha_2\left[\frac{\gamma_1}{\delta_1}\cos s(x-\xi_1)\cos s(\xi_1-a) - \frac{\gamma_1'}{\delta_1'}\sin s(x-\xi_1)\sin s(\xi_1-a)\right]\\ & + \frac{1}{s}\int_{\xi_1}^{x}\sin s(x-y)q(y)\phi_{2\lambda}(y)\mathrm{d}y + O\left(|s|^{-1}\, \mathrm{e}^{|t|[(x-\xi_1)+(\xi_1-a)]}\right).\end{aligned} \tag{3.10}$$

Multiplying through by $\mathrm{e}^{-|t|[(x-\xi_1)+(\xi_1-a)]}$, and denoting

$$F_{2\lambda}(x) := \mathrm{e}^{-|t|[(x-\xi_1)+(\xi_1-a)]}\phi_{2\lambda}(x)$$

we have

$$\begin{aligned}F_{2\lambda}(x) \;:=\; & \alpha_2 \mathrm{e}^{-|t|[(x-\xi_1)+(\xi_1-a)]}\left[\frac{\gamma_1}{\delta_1}\cos s(x-\xi_1)\cos s(\xi_1-a)\right.\\ & \left. - \; \frac{\gamma_1'}{\delta_1'}\sin s(x-\xi_1)\sin s(\xi_1-a)\right] + \frac{1}{s}\int_{\xi_1}^{x}\sin s(x-y)q(y)\mathrm{e}^{-|t|[(x-\xi_1)+(\xi_1-a)]}F_{2\lambda}(y)\mathrm{d}y\\ & + O\left(|s|^{-1}\right).\end{aligned}$$

Denoting $M(\lambda) := \max_{x\in[\xi_1,\xi_2]} |F_{2\lambda}(x)|$ from the last formula, it follows that

$$M(\lambda) \leqslant \left|\frac{\alpha_2\gamma_1}{\delta_1}\right| + \left|\frac{\alpha_2\gamma_1'}{\delta_1'}\right| + \frac{M(\lambda)}{|s|}\int_{\xi_1}^{\xi_2}|q(y)|\,\mathrm{d}y + \frac{M_0}{|s|}$$

for some $M_0 > 0$. From this, it follows that $M(\lambda) = O(1)$ as $\lambda \to \infty$, so

$$\phi_{2\lambda}(x) = O\left(\mathrm{e}^{|t|[(x-\xi_1)+(\xi_1-a)]}\right).$$

Substituting this back into the integral on the right of (3.10) yields (3.5_0). The other assertions can be proved similarly. □

Theorem 3.1. *Let* $\lambda = s^2, t = \operatorname{Im} s$. *Then the characteristic function* $\omega(\lambda)$ *has the following asymptotic representations:*

Case 1: If $\beta_2' \neq 0, \alpha_2 \neq 0$, *then*

$$\omega_3(\lambda) = \alpha_2\beta_2' s^3 \left[\frac{\gamma_2}{\delta_2}\sin s(b-\xi_2)\left(\frac{\gamma_1}{\delta_1}\cos s(\xi_2-\xi_1)\cos s(\xi_1-a) - \frac{\gamma_1'}{\delta_1'}\sin s(\xi_2-\xi_1)\sin s(\xi_1-a)\right)\right.$$
$$\left. + \frac{\gamma_2'}{\delta_2'}\cos s(b-\xi_2)\left(\frac{\gamma_1}{\delta_1}\sin s(\xi_2-\xi_1)\cos s(\xi_1-a) + \frac{\gamma_1'}{\delta_1'}\cos s(\xi_2-\xi_1)\sin s(\xi_1-a)\right)\right]$$
$$+ O\left(|s|^2 \exp|t|\,[(b-\xi_2)+(\xi_2-\xi_1)+(\xi_1-a)]\right). \tag{3.11}$$

Case 2: If $\beta_2' \neq 0, \alpha_2 = 0$, *then*

$$\omega_3(\lambda) = -\alpha_1\beta_2' s^2 \left[\frac{\gamma_2}{\delta_2}\sin s(b-\xi_2)\left(\frac{\gamma_1}{\delta_1}\cos s(\xi_2-\xi_1)\sin s(\xi_1-a) + \frac{\gamma_1'}{\delta_1'}\sin s(\xi_2-\xi_1)\cos s(\xi_1-a)\right)\right.$$
$$\left. + \frac{\gamma_2'}{\delta_2'}\cos s(b-\xi_2)\left(\frac{\gamma_1}{\delta_1}\sin s(\xi_2-\xi_1)\sin s(\xi_1-a) - \frac{\gamma_1'}{\delta_1'}\cos s(\xi_2-\xi_1)\cos s(\xi_1-a)\right)\right]$$
$$+ O\left(|s| \exp|t|\,[(b-\xi_2)+(\xi_2-\xi_1)+(\xi_1-a)]\right). \tag{3.12}$$

Case 3: If $\beta_2' = 0, \alpha_2 \neq 0$, *then*

$$\omega_3(\lambda) = \alpha_2\beta_1' s^2 \left[\frac{\gamma_2}{\delta_2}\cos s(b-\xi_2)\left(\frac{\gamma_1}{\delta_1}\cos s(\xi_2-\xi_1)\cos s(\xi_1-a) - \frac{\gamma_1'}{\delta_1'}\sin s(\xi_2-\xi_1)\sin s(\xi_1-a)\right)\right.$$
$$\left. - \frac{\gamma_2'}{\delta_2'}\sin s(b-\xi_2)\left(\frac{\gamma_1}{\delta_1}\sin s(\xi_2-\xi_1)\cos s(\xi_1-a) + \frac{\gamma_1'}{\delta_1'}\cos s(\xi_2-\xi_1)\sin s(\xi_1-a)\right)\right]$$
$$+ O\left(|s| \exp|t|\,[(b-\xi_2)+(\xi_2-\xi_1)+(\xi_1-a)]\right). \tag{3.13}$$

Case 4: If $\beta_2' = 0, \alpha_2 = 0$, *then*

$$\omega_3(\lambda) = -\alpha_1\beta_1' s \left[\frac{\gamma_2}{\delta_2}\cos s(b-\xi_2)\left(\frac{\gamma_1}{\delta_1}\cos s(\xi_2-\xi_1)\sin s(\xi_1-a) + \frac{\gamma_1'}{\delta_1'}\sin s(\xi_2-\xi_1)\cos s(\xi_1-a)\right)\right.$$
$$\left. + \frac{\gamma_2'}{\delta_2'}\sin s(b-\xi_2)\left(\frac{\gamma_1}{\delta_1}\sin s(\xi_2-\xi_1)\sin s(\xi_1-a) - \frac{\gamma_1'}{\delta_1'}\cos s(\xi_2-\xi_1)\cos s(\xi_1-a)\right)\right]$$
$$+ O\left(\exp|t|\,[(b-\xi_2)+(\xi_2-\xi_1)+(\xi_1-a)]\right). \tag{3.14}$$

Proof. The proof is immediate by substituting (3.6_k) and (3.9_k) into the representation

$$\omega_3(\lambda) = \lambda\left(\beta_1'\phi_{3\lambda}(b) - \beta_2'\phi_{3\lambda}'(b)\right) + \left(\beta_1\phi_{3\lambda}(b) - \beta_2\phi_{3\lambda}'(b)\right)$$
$$= -\lambda\beta_2'\phi_{3\lambda}'(b) + \lambda\beta_1'\phi_{3\lambda}(b) + \beta_1\phi_{3\lambda}(b) - \beta_2'\phi_{3\lambda}'(b). \quad \square \tag{3.15}$$

Corollary 3.1. *The eigenvalues of the problem* (1.1)–(1.7) *are bounded below.*

Proof. Putting $s = it$ $(t > 0)$ in the above formulae, it follows that $\omega_3(-t^2) \to \infty$ as $t \to \infty$. Hence, $\omega_3(\lambda) \neq 0$ for λ negative and sufficiently large. □

4. Asymptotic formulae for eigenvalues and eigenfunctions

Now we can obtain the asymptotic approximation formula for the eigenvalues of the considered problem (1.1)–(1.7).

Since the eigenvalues coincide with the zeros of the entire function $\omega_3(\lambda)$, it follows that they have no finite limit. Moreover, we know from Corollaries 2.1 and 2.2 that all eigenvalues are real and bounded below. Therefore, we may renumber them as $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, listed according to their multiplicity.

In this section, for the sake of simplicity we shall assume that $\gamma_i \delta_i' = \gamma_i' \delta_i$ $(i = 1, 2)$.

Theorem 4.1. *The eigenvalues $\lambda_n = s_n^2, n = 0, 1, 2, \ldots$ of the problem (1.1)–(1.7) have the following asymptotic representation for $n \to \infty$:*

Case 1: If $\beta_2' \neq 0, \alpha_2 \neq 0$, then

$$s_n = \frac{1}{(b-a)} \pi (n-1) + O\left(\frac{1}{n}\right). \tag{4.1}$$

Case 2: If $\beta_2' \neq 0, \alpha_2 = 0$, then

$$s_n = \frac{1}{(b-a)} \pi \left(n - \frac{1}{2}\right) + O\left(\frac{1}{n}\right). \tag{4.2}$$

Case 3: If $\beta_2' = 0, \alpha_2 \neq 0$, then

$$s_n = \frac{1}{(b-a)} \pi \left(n - \frac{1}{2}\right) + O\left(\frac{1}{n}\right). \tag{4.3}$$

Case 4: If $\beta_2' = 0, \alpha_2 = 0$, then

$$s_n = \frac{1}{(b-a)} \pi n + O\left(\frac{1}{n}\right). \tag{4.4}$$

Proof. We shall only consider the first case (the other cases may be considered analogically).

Denoting $\tilde{\omega}(s) := \omega_3(s^2) = \omega_3(\lambda)$,

$$\tilde{\omega}_1(s) = \frac{1}{2} \frac{\alpha_2 \beta_2' \gamma_1 \gamma_2}{\delta_1 \delta_2} s^3 \sin((b-a)s)$$

and $\tilde{\omega}_2(s) := \tilde{\omega}(s) - \tilde{\omega}_1(s)$, we write $\tilde{\omega}(s)$ as $\tilde{\omega}(s) = \tilde{\omega}_1(s) + \tilde{\omega}_2(s)$. In view of (3.11), from elementary considerations we have

$$\tilde{\omega}_2(s) = O\left(|s|^2 \exp |t| \left[(b-a)\right]\right).$$

We shall apply the well-known Rouche theorem, which asserts that if $f(s)$ and $g(s)$ are analytic inside and on a closed contour C, and $|g(s)| < |f(s)|$ on C, then $f(s)$ and $f(s) + g(s)$ have the same number zeros inside C, provided that each zero is counted according to their multiplicity.

It is readily shown that $|\tilde{\omega}_1(s)| > |\tilde{\omega}_2(s)|$ on the contours

$$C_n := \left\{ s \in \mathbb{C} \mid |s| = \frac{1}{(b-a)} \left(n + \frac{1}{2}\right) \pi \right\}$$

for sufficiently large n.

Let $\lambda_0 \leqslant \lambda_1 \leqslant \ldots$ be zeros of $\omega(\lambda)$ and $\lambda_n = s_n^2$. Since inside the contour C_n, $\tilde{\omega}_1(s)$ has zeros at points $s = 0$ (with multiplicity 4) and $s = \frac{1}{(b-a)} \frac{\pi k}{2}, k = \pm 1, \pm 2, \ldots, \pm n$ (with multiplicity 1), the number of zeros is $2n + 4$, and it follows that

$$s_n = \frac{1}{(b-a)} (n-1)\pi + \delta_n \tag{4.5}$$

where $\delta_n = O(1)$, more precisely $|\delta_n| < \frac{1}{(b-a)} \frac{\pi}{4}$ for sufficiently large n. By substituting this in (3.11), we derive that $\delta_n = O\left(\frac{1}{n}\right)$, which completes the proof. □

The next approximation for the eigenvalues may be obtained by the following procedure. For this, we shall suppose that $q(y)$ is of bounded variation in $[a, b]$.

We only consider the case $\beta_2' \neq 0$ and $\alpha_2 \neq 0$ (since the other cases may be considered similarly). Putting $x = \xi_1$ in (3.1$_k$), $x = \xi_2$ in (3.2$_k$) and then substituting in (3.3$_k$), we derive that

$$\begin{aligned}\phi_{3\lambda}'(b) &= -s\alpha_2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\sin((b-a)s) - \alpha_1\frac{\gamma_1'\gamma_2'}{\delta_1'\delta_2'}\cos((b-a)s)\\ &\quad + \frac{\gamma_1\gamma_2'}{\delta_1\delta_2'}\int_a^{\xi_1}\cos((b-y)s)\,q(y)\phi_{1\lambda}(y)\mathrm{d}y + \frac{\gamma_2}{\delta_2}\int_{\xi_1}^{\xi_2}\cos((b-y)s)\,q(y)\phi_{2\lambda}(y)\mathrm{d}y\\ &\quad + \int_{\xi_2}^{b}\cos((b-y)s)\,q(y)\phi_{3\lambda}(y)\mathrm{d}y.\end{aligned}$$

Substituting (3.4$_k$), (3.5$_k$) and (3.6$_k$) into the right side of the last integral equality then gives

$$\begin{aligned}\phi_{3\lambda}'(b) &= -s\alpha_2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\sin((b-a)s) - \alpha_1\frac{\gamma_1'\gamma_2'}{\delta_1'\delta_2'}\cos((b-a)s)\\ &\quad + \alpha_2\frac{\gamma_1\gamma_2'}{\delta_1\delta_2'}\int_a^{\xi_1}\cos((b-y)s)\,q(y)\cos(s(y-a))\,\mathrm{d}y + \alpha_2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\\ &\quad \times\int_{\xi_1}^{\xi_2}\cos((b-y)s)\,q(y)\cos((y-a)s)\,\mathrm{d}y\\ &\quad + \alpha_2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\int_{\xi_2}^{b}\cos((b-y)s)\,q(y)\cos((y-a)s)\,\mathrm{d}y + O\left(|s|^{-1}\exp|t|\,[(b-a)]\right).\end{aligned}$$

On the other hand, from (3.6$_k$), it follows that

$$\phi_{3\lambda}(b) = \alpha_2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\cos((b-a)s) + O\left(|s|^{-1}\exp|t|\,[(b-a)]\right).$$

Putting these formulae into (3.15), we have

$$\begin{aligned}\omega_3(\lambda) &= s^3\beta_2'\alpha_2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\sin((b-a)s) + s^2\left[\left(\alpha_2\beta_1'\frac{\gamma_1\gamma_2}{\delta_1\delta_2} + \beta_2'\alpha_1\frac{\gamma_1'\gamma_2'}{\delta_1'\delta_2'}\right)\cos((b-a)s)\right.\\ &\quad - \beta_2'\frac{\gamma_1\gamma_2'}{\delta_1\delta_2'}\int_a^{\xi_1}\cos((b-y)s)\,q(y)\mathrm{d}y - \beta_2'\frac{\gamma_2}{\delta_2}\int_{\xi_1}^{\xi_2}\cos((b-y)s)\,q(y)\phi_{2\lambda}(y)\mathrm{d}y\\ &\quad \left. - \beta_2'\int_{\xi_2}^{b}\cos((b-y)s)\,q(y)\phi_{3\lambda}(y)\mathrm{d}y\right] + O(|s|\exp|t|\,[(b-a)]).\end{aligned}$$

Putting (4.1) in the last equality we find that

$$\sin\delta_n = -\frac{\cos\delta_n}{s_n}\left[\frac{\beta_1'}{\beta_2'} + \frac{\alpha_1}{\alpha_2} - \frac{Q}{2} + O\left(\frac{1}{n}\right)\right] + O\left(|s_n|^{-2}\right) \tag{4.6}$$

where $Q := \int_a^{\xi_1} q(y)\mathrm{d}y + \int_{\xi_1}^{\xi_2} q(y)\mathrm{d}y + \int_{\xi_2}^{b} q(y)\mathrm{d}y$. Recalling that $q(y)$ is of bounded variation in $[a, b]$, and applying the well-known Riemann–Lebesque Lemma (see [18], p. 48, Theorem 4.12) to the third integral on the right in (4.6), this term is $O\left(\frac{1}{n}\right)$. Consequently, from (4.6) it follows that

$$\delta_n = -(b-a)\frac{1}{\pi(n-1)}\left[\frac{\beta_1'}{\beta_2'} + \frac{\alpha_1}{\alpha_2} - \frac{Q}{2}\right] + O\left(\frac{1}{n^2}\right).$$

Substituting in (4.5), we have

$$s_n = \frac{1}{(b-a)}\pi(n-1) - \frac{1}{\pi(n-1)}\left[\frac{\beta_1'}{\beta_2'} + \frac{\alpha_1}{\alpha_2} - \frac{Q}{2}\right] + O\left(\frac{1}{n^2}\right).$$

Similar formulae in the other cases are as follows:

In case 2

$$s_n = \frac{1}{(b-a)}\pi\left(n-\frac{1}{2}\right) + \frac{1}{\pi\,(n-1/2)}\left[\frac{\beta_1'}{\beta_2'} - \frac{Q}{2}\right] + O\left(\frac{1}{n^2}\right).$$

In case 3

$$s_n = \frac{1}{(b-a)}\pi\left(n-\frac{1}{2}\right) + \frac{1}{\pi\,(n-1/2)}\left[\frac{\alpha_1}{\alpha_2} - \frac{\beta_2}{\beta_1'} - \frac{Q}{2}\right] + O\left(\frac{1}{n^2}\right).$$

In case 4

$$s_n = \frac{1}{(b-a)}\pi n + \frac{1}{\pi n}\left[\frac{\beta_2}{\beta_1'} + \frac{Q}{2}\right] + O\left(\frac{1}{n^2}\right).$$

Recalling that $\phi(x,\lambda_n)$ is an eigenfunction according to eigenvalue λ_n, by putting (4.1) into the (3.4_k), (3.5_k) and (3.6_k) we derive that

$$\phi_{1\lambda_n}(x) = \alpha_2\cos\left(\frac{(x-a)}{(b-a)}\pi(n-1)\right) + O\left(\frac{1}{n}\right),$$

$$\phi_{2\lambda_n}(x) = \alpha_2\frac{\gamma_1}{\delta_1}\cos\left(\frac{(x-a)}{(b-a)}\pi(n-1)\right) + O\left(\frac{1}{n}\right)$$

and

$$\phi_{3\lambda_n}(x) = \alpha_2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\cos\left(\frac{(x-a)}{(b-a)}\pi(n-1)\right) + O\left(\frac{1}{n}\right)$$

in the first case. Hence, if $\beta_2' \neq 0$ and $\alpha_2 \neq 0$, then the eigenfunction $\phi(x,\lambda_n)$ has the asymptotic representation

$$\phi(x,\lambda_n) = \begin{cases} \alpha_2\cos\left(\frac{(x-a)}{(b-a)}\pi(n-1)\right) + O\left(\frac{1}{n}\right), & x\in[a,\xi_1) \\ \alpha_2\frac{\gamma_1}{\delta_1}\cos\left(\frac{(x-a)}{(b-a)}\pi(n-1)\right) + O\left(\frac{1}{n}\right), & x\in(\xi_1,\xi_2) \\ \alpha_2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\cos\left(\frac{(x-a)}{(b-a)}\pi(n-1)\right) + O\left(\frac{1}{n}\right), & x\in(\xi_2,b] \end{cases}$$

which hold uniformly for $x \in [a,\xi_1)\cup(\xi_1,\xi_2)\cup(\xi_2,b]$.

Similar formulae in the other cases are as follows:

In case 2

$$\phi(x,\lambda_n) = \begin{cases} -\alpha_1(b-a)\frac{1}{\pi(n-1/2)}\sin\left(\frac{(x-a)}{(b-a)}\pi\left(n-\frac{1}{2}\right)\right) + O\left(\frac{1}{n^2}\right), & x\in[a,\xi_1) \\ -\alpha_1\frac{\gamma_1}{\delta_1}(b-a)\frac{1}{\pi(n-1/2)}\sin\left(\frac{(x-a)}{(b-a)}\pi\left(n-\frac{1}{2}\right)\right) + O\left(\frac{1}{n^2}\right), & x\in(\xi_1,\xi_2) \\ -\alpha_1\frac{\gamma_1\gamma_2}{\delta_1\delta_2}(b-a)\frac{1}{\pi(n-1/2)}\sin\left(\frac{(x-a)}{(b-a)}\pi\left(n-\frac{1}{2}\right)\right) + O\left(\frac{1}{n^2}\right), & x\in(\xi_2,b]. \end{cases}$$

In case 3

$$\phi(x,\lambda_n) = \begin{cases} \alpha_2\cos\left(\frac{(x-a)}{(b-a)}\pi\left(n-\frac{1}{2}\right)\right) + O\left(\frac{1}{n}\right), & x\in[a,\xi_1) \\ \alpha_2\frac{\gamma_1}{\delta_1}\cos\left(\frac{(x-a)}{(b-a)}\pi\left(n-\frac{1}{2}\right)\right) + O\left(\frac{1}{n}\right), & x\in(\xi_1,\xi_2) \\ \alpha_2\frac{\gamma_1\gamma_2}{\delta_1\delta_2}\cos\left(\frac{(x-a)}{(b-a)}\pi\left(n-\frac{1}{2}\right)\right) + O\left(\frac{1}{n}\right), & x\in(\xi_2,b] \end{cases}$$

In case 4

$$\phi(x,\lambda_n)=\begin{cases}-\alpha_1(b-a)\dfrac{1}{\pi n}\sin\left(\dfrac{(x-a)}{(b-a)}\pi n\right)+O\left(\dfrac{1}{n^2}\right), & x\in[a,\xi_1)\\ -\alpha_1\dfrac{\gamma_1}{\delta_1}(b-a)\dfrac{1}{\pi n}\sin\left(\dfrac{(x-a)}{(b-a)}\pi n\right)+O\left(\dfrac{1}{n^2}\right), & x\in(\xi_1,\xi_2)\\ -\alpha_1\dfrac{\gamma_1\gamma_2}{\delta_1\delta_2}(b-a)\dfrac{1}{\pi n}\sin\left(\dfrac{(x-a)}{(b-a)}\pi n\right)+O\left(\dfrac{1}{n^2}\right), & x\in(\xi_2,b].\end{cases}$$

All these asymptotic approximations hold uniformly for $x\in[a,\xi_1)\cup(\xi_1,\xi_2)\cup(\xi_2,b]$.

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