PERGAMON

Applied Mathematics Letters 14 (2001) 285–290

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Strong Asymptotic Stability of a Compactly Coupled System of Wave Equations

M. AASSILA
Institut de Recherche Mathématique, Université de Strasbourg
7 rue René Descartes, 67084 Strasbourg, France
aassila@math.u-strasbg.fr

(Received March 1999; revised and accepted May 2000)

Abstract—We prove the well-posedness and study the strong asymptotic stability of a compactly coupled system of wave equations with a nonlinear feedback acting on one end only.

Keywords—Well-posedness, Strong asymptotic stability, Nonuniform stabilization.

1. INTRODUCTION

In this paper, we are concerned with the global existence and asymptotic stability of the evolutionary system

$$u_1'' - \Delta u_1 + \alpha(u_1 - u_2) = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$u_2'' - \Delta u_2 + \alpha(u_2 - u_1) = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.2}$$

$$u_1 = 0, \quad u_2 = 0, \qquad \text{on } \Gamma_0 \times \mathbb{R}_+, \tag{1.3}$$

$$u_1 = 0, \quad \frac{\partial u_2}{\partial \nu} + a u_2 + g(u_2') = 0, \qquad \text{on } \Gamma_1 \times \mathbb{R}_+, \tag{1.4}$$

$$u_1(0) = u_{10}, \quad u_1'(0) = u_{11}, \quad u_2(0) = u_{20}, \quad u_2'(0) = u_{21}, \qquad \text{in } \Omega, \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\{\Gamma_0, \Gamma_1\}$ is a partition of its boundary Γ, ν is the outward unit normal vector to Γ, $\mathbb{R}_+ = [0, +\infty)$, and $\alpha : \Omega \to \mathbb{R}_+$, $a : \Gamma_1 \to \mathbb{R}_+$, $g : \mathbb{R} \to \mathbb{R}$ are some given functions. Under suitable assumptions, we shall prove that system (1.1)–(1.5) is well posed, dissipative, and strongly stable.

This problem is motivated by an analogous problem in ordinary differential equations for coupled oscillators and has potential application in isolation of objects from outside disturbances. As an example in engineering, rubber and rubber-like materials are used to absorb vibration or shield structures from vibration. Modeling of structures such as beams, or plates sandwiched with rubber or similar materials, will lead to equations similar to those of (1.1) and (1.2).

When u_1 satisfies the same equation as u_2 on $\Gamma_1 \times \mathbb{R}_+$, this problem was studied by Komornik and Rao [1] by using two different approaches. The first one is based on the application of a

PII: S0893-9659(00)00150-6

compact perturbation theorem of Gibson [2], which has been applied successively in the study of the SCOLE model [3] and the Rayleigh beam equation [4]. This method allows them to establish the uniform decay rate of energy in the linear case for arbitrary nonnegative bounded measurable function α. The second one is a direct adaptation of the usual multiplier method [5]. This method leads to decay rate estimates both in the linear and nonlinear case, under the assumption that α is constant. The results of Komornik and Rao [1] were improved by the author in [6].

In this paper, as there is no damping acting on u_1 on Γ_1, we will see that there is a price paid for weakening this hypothesis compared with the one assumed by Komornik and Rao [1] and Aassila [6]. Only strong asymptotic stability will be proved, and in general, there is no exponential decay even if g is linear. System (1.1)–(1.5) is called a compactly coupled system since if we set $\mathcal{B}(u_1,u_2,v_1,v_2) := (0,0,\alpha(u_2-u_1),\alpha(u_1-u_2))$ with $D(\mathcal{B}) := \mathcal{H} = H_0^1(\Omega) \times H_{\Gamma_0}^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, then thanks to Rellich's Theorem, $\mathcal{B}$ is a compact linear operator in $\mathcal{H}$.

Throughout the paper, we shall make the following assumptions.

(H1) The domain Ω is of class C^2.

(H2) The partition of Γ satisfies the condition $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

(H3) There exists a point $x_0 \in \mathbb{R}^n$ such that, putting $m(x) = x - x_0$, we have

$$m \cdot \nu \le 0, \quad \text{on } \Gamma_0, \qquad \text{and} \qquad \inf_{\Gamma_1} m \cdot \nu > 0, \quad \text{on } \Gamma_1.$$

(H4) The coefficient a is nonnegative and belongs to $C^1(\Gamma_1)$. Moreover, either $\Gamma_0 \neq \emptyset$ or $\inf_{\Gamma_1} a > 0$.

(H5) The function α is nonnegative and belongs to $L^\infty(\Omega)$.

(H6) The function g is continuous, nondecreasing $g(0) = 0$, and there exists a constant $c > 0$ such that

$$|g(x)| \le 1 + c|x|, \qquad \forall x \in \mathbb{R}. \tag{1.6}$$

Setting $H_{\Gamma_0}^1(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\}$, our main results are as follows.

THEOREM 1. *(Well-posedness and regularity.) Given $(u_{01},u_{11},u_{21},u_{22}) \in H_0^1(\Omega) \times L^2(\Omega) \times H_{\Gamma_0}^1(\Omega) \times L^2(\Omega)$ arbitrarily, problem (1.1)–(1.5) has a unique weak solution satisfying*

$$u_1 \in C\left(\mathbb{R}_+, H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}_+, L^2(\Omega)\right),$$
$$u_2 \in C\left(\mathbb{R}_+, H_{\Gamma_0}^1(\Omega)\right) \cap C^1\left(\mathbb{R}_+, L^2(\Omega)\right),$$

and its energy defined by

$$E(t) := \frac{1}{2}\int_\Omega u_1'^2 + |\nabla u_1|^2 + u_2'^2 + |\nabla u_2|^2 + \alpha(u_1-u_2)^2\,dx + \frac{1}{2}\int_{\Gamma_1} a u_2^2\,d\gamma$$

is nonincreasing.

Furthermore, if g is globally Lipschitz continuous and

$$(u_{01},u_{11},u_{21},u_{22}) \in \left(H^2 \cap H_0^1\right) \cap H_0^1 \times \left(H^2 \cap H_{\Gamma_0}^1\right) \times H_{\Gamma_0}^1$$

are such that

$$\frac{\partial u_{20}}{\partial \nu} + a u_{20} + g(u_{21}) = 0, \qquad \text{on } \Gamma_1,$$

then the solution is more regular and we have

$$u_1 \in L^\infty\left(\mathbb{R}_+, H^2(\Omega) \cap H_0^1(\Omega)\right) \cap W^{1,\infty}\left(\mathbb{R}_+, H_0^1(\Omega)\right) \cap W^{2,\infty}\left(\mathbb{R}_+, L^2(\Omega)\right),$$
$$u_2 \in L^\infty\left(\mathbb{R}_+, H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)\right) \cap W^{1,\infty}\left(\mathbb{R}_+, H_{\Gamma_0}^1(\Omega)\right) \cap W^{2,\infty}\left(\mathbb{R}_+, L^2(\Omega)\right).$$

THEOREM 2. *(Strong asymptotic stability.) Assume that $\alpha = \text{constant} = 1$ and $g(x) = x$, then we have*

$$E(t) \to 0, \qquad \text{as } t \to +\infty,$$

for all $(u_{10},u_{11},u_{21},u_{22}) \in H_0^1(\Omega) \times L^2(\Omega) \times H_{\Gamma_0}^1(\Omega) \times L^2(\Omega)$.

THEOREM 3. *(Nonuniform stability.) If Ω is an open bounded interval ($N = 1$), $g(x) = x$, and $\alpha = (1/2)(n^2 - m^2)\pi^2$, then (1.1)–(1.5) has an eigenvalue $\lambda = in\pi$, and hence, the system is not uniformly exponentially stable (n and m are positive integers).*

The paper is organized as follows. In Section 2, we will prove Theorem 1. In Section 3, Theorem 2, and in Section 4, Theorem 3.

2. WELL-POSEDNESS

We will apply the standard theory of nonlinear semigroups [7]. Let us introduce the two following Hilbert spaces $H = L^2(\Omega) \times L^2(\Omega)$ and $V = H_0^1(\Omega) \times H_{\Gamma_0}^1(\Omega)$ endowed with the norms

$$\|(u_1, u_2)\|_H^2 = \int_\Omega u_1^2 + u_2^2 \, dx$$

and

$$\|(u_1, u_2)\|_V^2 = \int_\Omega |\nabla u_1|^2 + |\nabla u_2|^2 + \alpha(u_1 - u_2)^2 \, dx + \int_{\Gamma_1} a u_2^2 \, d\gamma.$$

One can easily verify that $\|\cdot\|_H$ and $\|\cdot\|_V$ are norms, and then we have $V \subset H \subset V'$ with dense and compact imbeddings.

Let $v = (v_1, v_2) \in V$ arbitrarily and assume for a moment that (1.1)–(1.5) has a smooth solution $u := (u_1, u_2)$. Multiplying equations (1.1),(1.2) with v_1, v_2, integrating their sum in Ω, and finally using boundary conditions (1.3),(1.4), we easily obtain that

$$\langle u'' + Au + Bu', v \rangle_{V',V} = 0, \qquad \forall v \in V,$$

where $A : V \to V'$ is the duality mapping and B is the nonlinear operator defined by

$$\langle Bv, z \rangle_{V',V} = \int_{\Gamma_1} g(v_2) z_2 \, d\gamma, \qquad v = (v_1, v_2), \quad z = (z_1, z_2) \in V,$$

which is well defined thanks to (H6).

Hence, (1.1)–(1.5) may be written abstractly as

$$U' + \mathcal{A}U = 0, \tag{1.7}$$
$$U(0) = U_0, \tag{1.8}$$

where

$$U = (u, v) := (u, u'), \qquad \mathcal{A}U = (-v, Au + Bv), \qquad U(0) = (u_{10}, u_{11}, u_{12}, u_{22}),$$

and

$$D(\mathcal{A}) := \{U = (u, v) \in V \times V,\ Au + Bv \in H\}.$$

LEMMA 2.1. *$\mathcal{A}$ is maximal monotone on $V \times H$.*

PROOF. Letting $U = (u, v)$ and $\tilde{U} = (\tilde{u}, \tilde{v}) \in D(\mathcal{A})$ arbitrarily, we have

$$\left(\mathcal{A}U - \mathcal{A}\tilde{U}\right)_{V \times H} = \int_{\Gamma_1} (g(v_2) - g(\tilde{v}_2))(v_2 - \tilde{v}_2) \, d\gamma \geq 0,$$

hence, $\mathcal{A}$ is monotone.

It remains to show that for any given $\tilde{U} = (\tilde{u}, \tilde{v}) \in V \times H$, there exists $U = (u, v) \in D(\mathcal{A})$ such that $(I + \mathcal{A})U = \tilde{U}$. It suffices to show that the map $I + A + B : V \to V'$ is onto. Indeed, then there exists $v \in V$ satisfying

$$(I + A + B)v = \tilde{v} - A\tilde{u}.$$

Setting $u = v + \tilde{u}$, we conclude easily that $\tilde{U} \in V \times V$, $Au + Bv = \tilde{v} - v \in H$ (hence, $U \in D(\mathcal{A})$), and $(I + \mathcal{A})U = \tilde{U}$.

Let us turn now to prove the surjectivity of $I + A + B : V \to V'$. Fix $f \in V'$ arbitrarily, set

$$G(t) = \int_0^t g(s)\,ds, \qquad t \in \mathbb{R},$$

and consider the map $F : V \to \mathbb{R}$ defined by the formula

$$F(u) = \frac{1}{2}\|u\|_H^2 + \frac{1}{2}\|u\|_V^2 + \int_{\Gamma_1} G(u_2)\,d\gamma - \langle f, u\rangle_{V',V}.$$

Thanks to (H6), F is well defined, continuously differentiable, and

$$\langle F'(u), v\rangle_{V',V} = \langle (I + A + B)u - f, v\rangle_{V',V},$$

for all $u, v \in V$. Furthermore, thanks to the nondecreasingness of g, F is convex, and hence, lower semicontinuous in V. Finally, we conclude from the inequality

$$F(v) \geq \left(\frac{1}{2}\|v\|_V - \|f\|_{V'}\right)\|v\|_V$$

that $F(v) \to +\infty$ if $\|v\|_V \to +\infty$, i.e., F is coercive. Hence, there is a point $u \in V$ minimizing F. It follows that $F'(u) = 0$, i.e., $(I + A + B)u = f$.

The regularity of solutions can be proved in a standard way, so we omit the details here.

3. STRONG ASYMPTOTIC STABILITY

For the proof of Theorem 2, we need the following useful theorem.

THEOREM. *(See [8].) Let $\mathcal{A}$ be a maximal monotone linear operator in a complex Hilbert space $V \times H$ and assume that*

(a) *$\mathcal{A}$ has a compact resolvent;*
(b) *$\mathcal{A}$ does not have purely imaginary eigenvalues.*

Then problem (1.7),(1.8) is strongly stable.

The proof of (a) follows from the compactness of the imbedding $D(\mathcal{A}) \subset V \times H$, a consequence of Rellich's Theorem. For the proof of (b), let $U_0 = (u_1, u_2, v_1, v_2)$ be an eigenfunction of $\mathcal{A}$ having a purely imaginary eigenvalue $i\omega$:

$$\mathcal{A}U_0 = i\omega U_0, \qquad \omega \in \mathbb{R}, \quad U_0 \in D(\mathcal{A}).$$

Using the definition of $\mathcal{A}$, it follows that

$$-v_1 = i\omega u_1, \qquad \text{in } \Omega, \tag{3.1}$$
$$-v_2 = i\omega u_2, \qquad \text{in } \Omega, \tag{3.2}$$
$$-\Delta u_1 + (u_1 - u_2) = i\omega v_1, \qquad \text{in } \Omega, \tag{3.3}$$
$$-\Delta u_2 + (u_2 - u_1) = i\omega v_2, \qquad \text{in } \Omega, \tag{3.4}$$
$$u_1 = 0, \qquad \text{on } \Gamma_0, \tag{3.5}$$
$$u_2 = 0, \qquad \text{on } \Gamma_0, \tag{3.6}$$
$$u_1 = 0, \qquad \text{on } \Gamma_1, \tag{3.7}$$
$$\frac{\partial u_2}{\partial \nu} + au_2 + v_2 = 0, \qquad \text{on } \Gamma_1. \tag{3.8}$$

Furthermore, $U(t) := e^{i\omega t}U_0$ has obviously constant energy

$$E(t) = \frac{1}{2}\|U(t)\|^2_{V\times H},$$

and therefore, $E' = 0$, that is, $v_2 = 0$ on Γ_1.

If $\omega \neq 0$, we conclude by Carleman's unique continuation theorem (cf., for example, [9]) that $u_1 = u_2$ in Ω. Taking into account (3.1),(3.2), we obtain that $v_1 = v_2$ in Ω, and therefore, $U_0 = 0$.

If $\omega = 0$, then multiplying (3.3) with u_1, integrating by parts in Ω, we obtain that

$$\int_\Omega u_1(u_1 - u_2)\,dx + \int_\Omega |\nabla u_1|^2\,dx = 0.$$

Similarly, we also have

$$\int_\Omega u_2(u_1 - u_2)\,dx + \int_\Omega |\nabla u_2|^2 + \int_{\Gamma_1} a|u_2|^2\,d\gamma = 0.$$

Hence, $\|(u_1, u_2)\|^2_V = 0$, and therefore, $u_1 = u_2 = 0$, and we conclude that $v_1 = v_2$ in Ω. Thus, we have $U_0 = 0$.

It follows that $\mathcal{A}$ cannot have purely imaginary eigenvalues.

4. NONUNIFORM STABILITY

For simplicity, we assume that $\Omega = (0,1)$ and $a = 0$. The problem

$$(\mathcal{A} - \lambda I)\,(u_1, \tilde{u}_1, v_1, \tilde{v}_2)^\top = 0$$

is equivalent to

$$\tilde{u}_1 - \lambda u_1 = 0, \tag{4.1}$$

$$u_1'' - \alpha u_1 - \lambda \tilde{u}_1 + \alpha v_1 = 0, \tag{4.2}$$

$$\tilde{v}_2 - \lambda v_1 = 0, \tag{4.3}$$

$$\alpha u_1 + v_1'' - \alpha v_1 - \lambda \tilde{v}_2 = 0, \tag{4.4}$$

$$u_1(0) = u_1(1) = 0, \tag{4.5}$$

$$v_1(0) = 0, \qquad v_1'(1) = -\tilde{v}_2(1). \tag{4.6}$$

If we set

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{\lambda t}\begin{pmatrix} u_1(x) \\ v_1(x) \end{pmatrix}, \tag{4.7}$$

and owing to the proof of Theorem 2, we can easily deduce that $E'(t) = 0$, and hence, $\binom{u_1}{v_1}$ is a solution to

$$u_1'' - \lambda^2 u_1 = \alpha(u_1 - v_1), \tag{4.8}$$

$$v_1'' - \lambda^2 v_1 = \alpha(v_1 - u_1), \tag{4.9}$$

$$u_1(0) = u_1(1) = 0, \tag{4.10}$$

$$v_1(0) = v_1(1) = v_1'(1) = 0. \tag{4.11}$$

Setting $\phi = u + v$, $\psi = u - v$, (4.8)–(4.11) can be rewritten as

$$\phi'' - \lambda^2 \phi = 0, \tag{4.12}$$

$$\psi'' - (\lambda^2 + 2\alpha)\psi = 0, \tag{4.13}$$

$$\phi(0) = \phi(1) = \psi(0) = \psi(1) = 0. \tag{4.14}$$

The solutions to the above equations satisfying the boundary conditions at $x = 0$ are of the form

$$\phi(x) = c_1 \sinh \lambda x, \qquad \psi(x) = c_2 \sinh \sqrt{\lambda^2 + 2\alpha}\, x.$$

In order to satisfy the boundary conditions at $x = 1$, with $c_1 \neq 0$ and $c_2 \neq 0$, we should have

$$\lambda = in\pi, \quad \sqrt{\lambda^2 + 2\alpha} = im\pi, \qquad n, m \in \mathbb{N}.$$

Hence, we deduce that if $\alpha = (1/2)(n^2 - m^2)\pi^2$, then the system has an eigenvalue $\lambda = in\pi$, and consequently, it is not uniformly exponentially stable.

REMARKS.

(1) The results of Theorems 1–3 hold true if we assume that u_1 satisfies the Neumann boundary condition on Γ_1.

(2) Theorem 2 remains valid (with a simple modification of the proof) for the nonlinear case if the function g is locally Lipschitz continuous and if it satisfies the following two conditions:

$$\begin{aligned} g(x) &\neq 0, && \text{if } x \neq 0, \\ \exists\, c > 0 : |g(x)| &\leq 1 + c|x|^{N/(N-2)}, && \forall\, x \in \mathbb{R}. \end{aligned}$$

REFERENCES

1. V. Komornik and B. Rao, Boundary stabilization of compactly coupled wave equations, *Asymptotic Anal.* **14**, 339–359 (1997).
2. J.S. Gibson, A note on stabilization of infinite dimensional linear oscillators by compact linear feedback, *SIAM J. Control Optim.* **18**, 311–316 (1980).
3. B. Rao, Stabilisation uniforme d'un système hybride en élasticité, *C. R. Acad. Sci. Paris* **316**, 261–266 (1993).
4. B. Rao, A compact perturbation method for the boundary stabilization of the Rayleigh beam equation, *Appl. Math. Optim.* **33**, 253–264 (1996).
5. V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, Masson, Paris, (1994).
6. M. Aassila, A note on the boundary stabilization of a compactly coupled system of wave equations, *Appl. Math. Lett.* **12** (3), 19–24 (1999).
7. H. Brézis, *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, (1973).
8. C.D. Benchimol, A note on weak stabilizability of contraction semigroups, *SIAM J. Control Optim.* **16**, 373–379 (1978).
9. J.-L. Lions, *Contrôlabilité Exacte et Stabilisation de Systèmes Distribués, Vol. 1/2*, Masson, Paris, (1988).