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Matroids and Linking Systems

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With the help of the concept of a linking system, theorems relating matroids with bipartite graphs and directed graphs are deduced. In this way natural generalizations of theorems of Edmonds & Fulkerson, Perfect, Pym, Rado, Brualdi and Mason are obtained. Furthermore some other properties of these linking systems are investigated.

1. INTRODUCTION

In this article we give definitions of and theorems on "linking systems." The purpose of this is to give a more general form to some theorems relating matroids with bipartite graphs and directed graphs; a characteristic theorem of this kind is the following result of Perfect [20]:

Let (X, Y, E) be a bipartite graph and let $(X, \mathscr{I})$ be a matroid (where $\mathscr{I}$ is the collection of independent subsets of the matroid); define $\mathscr{J}$ as the collection of all subsets Y' of Y such that there is a matching in the bipartite graph between some independent subset of X and Y'; then $(Y, \mathscr{J})$ is again a matroid, with $\mathscr{J}$ as collection of independent subsets of Y.

This theorem is generalized by Brualdi [3] and Mason [14] to the case where the "medium" between X and Y is a directed graph instead of a bipartite graph. We generalize this to the case where the medium between X and Y is a so-called "linking system"; such a linking system links subsets of X with subsets of Y, or, more formally, a linking system is a triple (X, Y, Λ), with Λ a subset of $\mathscr{P}(X) \times \mathscr{P}(Y)$, satisfying certain axioms (see Definition 2.1). Theorems of Ore [18] and Pym [23] on bipartite graphs and directed graphs, respectively, imply that these graphical mediums satisfy the axioms of a linking system. Another class of linking systems arises from taking a matrix over some field, say with collection X of rows and collection Y of columns; then take for Λ the set of all pairs (X', Y') with the property that $X' \subset X$, $Y' \subset Y$ and the submatrix generated by the rows X' and columns Y' is nonsingular.

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As indicated, we prove that in the case where $(X, \mathscr{I})$ is a matroid, (X, Y, Λ) is a linking system and $\mathscr{J}$ is the collection of all subsets Y' of Y such that $(X', Y') \in \Lambda$ for some $X' \in \mathscr{I}$, then $(Y, \mathscr{J})$ is again a matroid (cf. Theorem 3.3).

Thus we have generalized theorems of Edmonds & Fulkerson [8], Perfect [19, 20] and Brualdi [3] on bipartite graphs and directed graphs to a theorem on linking systems. As a result, which seems to be new, we have that if a matroid is linked with a set via a matrix, the matroid and matrix together induce a matroid on that set. Other results of Mason [12] and Brualdi [2, 3, 4] find also their generalizations to linking systems (cf. Theorems 3.4 and 3.6).

Besides theorems on the linking of matroids by means of linking systems, we give some theorems on the structure of linking systems. In a natural way one can let correspond a matroid with a linking system (cf. Theorem 3.2) and one can define the product of two linking systems (X, Y, Λ_1) and (Y, Z, Λ_2) (cf. Theorem 3.5). In Section 4 we show some relations between a linking system (X, Y, Λ) and its "underlying" bipartite graph, that is the bipartite graph (X, Y, E_Λ), defined by

$$(x, y) \in E_\Lambda \qquad \text{if and only if} \quad (\{x\}, \{y\}) \in \Lambda.$$

We prove, among other things, that

(i) if there is exactly one matching in the underlying bipartite graph between sets $X' \subset X$ and $Y' \subset Y$, then $(X', Y') \in \Lambda$;

(ii) if $(X', Y') \in \Lambda$, then there exists at least one matching in the bipartite graph between X' and Y', (cf., Theorem 4.2 and 4.3).

In this article we suppose that the basic facts from matroid theory are known to the reader. For a survey on matroid theory we refer to Welsh [28] or Wilson [29]. Particularly in the proofs we shall frequently use well-known results on matroids, e.g., on the rank of the dual of a matroid, of the union of two matroids (Nash–Williams [17]) and of the restriction and contraction of a matroid. For a survey on matroids induced by digraphs we refer to Brualdi [5].

Finally we remark that in this paper we only consider *finite* structures (matroids, graphs, linking systems).

2. Definitions and Examples

In this section we give the definition of a linking system, show the equivalence of this definition with an alternative one and display some examples of these linking systems.

DEFINITION 2.1. A *linking system* is a triple (X, Y, Λ), where X and Y are finite sets and $\varnothing \neq \Lambda \subset \mathscr{P}(X) \times \mathscr{P}(Y)$, such that

(i) if $(X', Y') \in \Lambda$, then $|X'| = |Y'|$;

(ii) if $(X', Y') \in \Lambda$ and $X'' \subset X'$, then $(X'', Y'') \in \Lambda$ for some $Y'' \subset Y'$;

(iii) if $(X', Y') \in \Lambda$ and $Y'' \subset Y'$, then $(X'', Y'') \in \Lambda$ for some $X'' \subset X'$;

(iv) if $(X_1, Y_1) \in \Lambda$ and $(X_2, Y_2) \in \Lambda$, then there exists an $(X', Y') \in \Lambda$ such that $X_1 \subset X' \subset X_1 \cup X_2$ and $Y_2 \subset Y' \subset Y_1 \cup Y_2$.

From these axioms it follows that always $(\varnothing, \varnothing) \in \Lambda$.

Before we pass on to examples of linking systems, we give some further definitions. We call the elements of Λ *linked pairs*. For a linking system (X, Y, Λ) we define its *linking function* λ by

$$\lambda(X', Y') = \max\{|X''| \mid (X'', Y'') \in \Lambda \text{ for some } X'' \subset X' \text{ and } Y'' \subset Y'\},$$

for $X' \subset X$ and $Y' \subset Y$. A linking system is determined by its linking function, since, clearly:

$$(X', Y') \in \Lambda \qquad \text{if and only if} \quad \lambda(X', Y') = |X'| = |Y'|.$$

Just as we can define the notion of a matroid in terms of the rank function (instead of in terms of the collection of independent subsets, for instance), we can define the notion of a linking system in terms of the linking function (instead of in terms of the set of linked pairs).

(Alternative) DEFINITION 2.2. A *linking system* is a triple (X, Y, λ), where X and Y are finite sets and λ is an integer-valued function defined on $\mathscr{P}(X) \times \mathscr{P}(Y)$ such that

(i) $0 \leqslant \lambda(X', Y') \leqslant \min\{|X'|, |Y'|\}$ (for $X' \subset X$ and $Y' \subset Y$);

(ii) if $X'' \subset X'$ and $Y'' \subset Y'$, then $\lambda(X'', Y'') \leqslant \lambda(X', Y')$ (for $X' \subset X$ and $Y' \subset Y$);

(iii) $\lambda(X' \cap X'', Y' \cup Y'') + \lambda(X' \cup X'', Y' \cap Y'') \leqslant \lambda(X', Y') + \lambda(X'', Y'')$ (for $X', X'' \subset X$ and $Y', Y'' \subset Y$).

We give a proof of the equivalence of the two definitions, where the concepts are related, as said, by

$$\Lambda = \{(X', Y') \mid \lambda(X', Y') = |X'| = |Y'|\},$$

and

$$\lambda(X', Y') = \max\{|X''| \mid (X'', Y'') \in \Lambda \text{ for some } X'' \subset X' \text{ and } Y'' \subset Y'\},$$

for $X' \subset X$ and $Y' \subset Y$.

Proof of the equivalence of the Definitions 2.1 and 2.2.

(1) Necessity of the axioms of Definition 2.2.

Since axioms (i) and (ii) of Definition 2.2 follow easily from Definition 2.1 we prove only axiom (iii) of Definition 2.2.

Choose

(a) $(X_1, Y_1) \in \Lambda$ such that $X_1 \subset X' \cap X''$, $Y_1 \subset Y' \cup Y''$ and $|X_1| = |Y_1| = \lambda(X' \cap X'', Y' \cup Y'')$;

(b) $(X_2, Y_2) \in \Lambda$ such that $X_2 \subset X' \cup X''$, $Y_2 \subset Y' \cap Y''$ and $|X_2| = |Y_2| = \lambda(X' \cup X'', Y' \cap Y'')$.

By definition of λ this is always possible.

Now, by axiom (iv) of Definition 2.1, there exists an $(X_3, Y_3) \in \Lambda$ with

$$X_1 \subset X_3 \subset X_1 \cup X_2 \subset X' \cup X'',$$

and

$$Y_2 \subset Y_3 \subset Y_1 \cup Y_2 \subset Y' \cup Y''.$$

Using axiom (ii) of Definition 2.1, there is a $Y_4 \subset Y_3$ with the property

$$(X_3 \cap X', Y_4) \in \Lambda.$$

Axiom (iii) ensures the existence of an $X_4 \subset X_3 \cap X'$ satisfying

$$(X_4, Y_4 \cap Y') \in \Lambda.$$

Since $X_4 \subset X'$ and $Y_4 \cap Y' \subset Y'$ it is true that

$$|X_4| = |Y_4 \cap Y'| \leqslant \lambda(X', Y').$$

Now we have

$$\begin{aligned}\lambda(X', Y') &\geqslant |Y_4 \cap Y'| = |Y_4| - |Y_4 \backslash Y'| = |X_3 \cap X'| - |Y_4 \backslash Y'| \\ &\geqslant |X_3 \cap X'| - |Y_3 \backslash Y'|.\end{aligned}$$

(Note that $|Y_4| = |X_3 \cap X'|$ since $(X_3 \cap X', Y_4) \in \Lambda$, and that $Y_4 \subset Y_3$.) The same method, applied to X'' and Y'' instead of X' and Y', results in

$$\lambda(X'', Y'') \geqslant |X_3 \cap X''| - |Y_3 \backslash Y''|.$$

Hence

$$\begin{aligned}\lambda(X', Y') + \lambda(X'', Y'') &\geqslant |X_3 \cap X'| - |Y_3 \backslash Y'| + |X_3 \cap X''| - |Y_3 \backslash Y''| \\ &= |X_3| + |X_3 \cap X' \cap X''| - |Y_3| + |Y_3 \cap Y' \cap Y''| \\ &= |X_3 \cap X' \cap X''| + |Y_3 \cap Y' \cap Y''| \geqslant |X_1| + |Y_2| \\ &= \lambda(X' \cap X'', Y' \cup Y'') + \lambda(X' \cup X'', Y' \cap Y'').\end{aligned}$$

(Note that $X_3 \subset X' \cup X''$; $Y_3 \subset Y' \cup Y''$; $|X_3| = |Y_3|$; $X_1 \subset X_3 \cap X' \cap X''$; $Y_2 \subset Y_3 \cap Y' \cap Y''$.)

(2) Sufficiency of the axioms of Definition 2.2.

Let (X, Y, λ) be a system satisfying the axioms of Definition 2.2. We first prove that if $X' \subset X$ and $Y' \subset Y$, then there are $X'' \subset X'$ and $Y'' \subset Y'$ such that

$$\lambda(X', Y') = |X''| = |Y''| = \lambda(X'', Y'').$$

It is easy to see, by the symmetry of Definition 2.2 and by induction, that it is enough to prove that if $\lambda(X', Y') < |X'|$, then there is an $x \in X'$ such that

$$\lambda(X' \setminus \{x\}, Y') = \lambda(X', Y').$$

Suppose, to the contrary, that for all $x \in X'$ we have

$$\lambda(X' \setminus \{x\}, Y') \leqslant \lambda(X', Y') - 1.$$

Then, using axiom (iii) of Definition 2.2 and by induction on $|X''|$, one has

$$\lambda(X' \setminus X'', Y') \leqslant \lambda(X', Y') - |X''| \qquad \text{for} \quad X'' \subset X'.$$

(Use $\lambda(X' \setminus (X'' \cup \{x\}), Y') \leqslant \lambda(X' \setminus X'', Y') + \lambda(X' \setminus \{x\}, Y') - \lambda(X', Y')$, for $x \in X' \setminus X''$.) Hence also, putting $X'' = X'$,

$$0 \leqslant \lambda(\varnothing, Y') = \lambda(X' \setminus X', Y') \leqslant \lambda(X', Y') - |X'| < |X'| - |X'| = 0,$$

which is a contradiction.

We now prove the axioms of Definition 2.1, so define Λ by:

$$(X', Y') \in \Lambda \qquad \text{if and only if} \quad \lambda(X', Y') = |X'| = |Y'|.$$

Then axiom (i) of Definition 2.1 follows readily. The axioms (ii) and (iii) are symmetric; we prove only (ii).

Let $(X', Y') \in \Lambda$ and $X'' \subset X'$. Then

$$\lambda(X' \setminus X'', Y') + \lambda(X'', Y') \geqslant \lambda(X', Y') + \lambda(\varnothing, Y'),$$

by axiom (iii) of Definition 2.2, and therefore

$$\begin{aligned} |X''| &\geqslant \lambda(X'', Y') \geqslant \lambda(X', Y') + \lambda(\varnothing, Y') - \lambda(X' \setminus X'', Y') \\ &\geqslant |X'| + 0 - |X' \setminus X''| = |X''|, \end{aligned}$$

from which it follows that

$$\lambda(X'', Y') = |X''|.$$

Now, by the foregoing, there exists a $Y'' \subset Y'$ with the property

$$\lambda(X'', Y') = \lambda(X'', Y'') = |X''| = |Y''|,$$

or

$$(X'', Y'') \in \Lambda.$$

Finally we prove axiom (iv) of Definition 2.1. Let $(X_1, Y_1) \in \Lambda$ and $(X_2, Y_2) \in \Lambda$; without loss of generality we may suppose that (X_1, Y_1) is a maximal linked pair in $(X_1 \cup X_2, Y_1 \cup Y_2)$; that is, if $(X_1', Y_1') \in \Lambda$ such that $X_1 \subset X_1' \subset X_1 \cup X_2$ and $Y_1 \subset Y_1' \subset Y_1 \cup Y_2$, then $X_1 = X_1'$ and $Y_1 = Y_1'$. Similarly, we may suppose that (X_2, Y_2) is a maximal linked pair in $(X_1 \cup X_2, Y_1 \cup Y_2)$.

We first prove that $|X_1| = |X_2| = |Y_1| = |Y_2| = \lambda(X_1 \cup X_2, Y_1 \cup Y_2)$. By the maximality of (X_1, Y_1) we have for each $x \in X_2$ and $y \in Y_2$

$$\lambda(X_1 \cup \{x\}, Y_1 \cup \{y\}) = |X_1| = |Y_1|.$$

Then, by induction on $|X'|$ and using axiom (iii) of Definition 2.2, we have

$$\lambda(X_1 \cup X', Y_1 \cup \{y\}) = |X_1| = |Y_1|, \qquad \text{if} \quad X' \subset X_2 \text{ and } y \in Y_2,$$

and hence

$$\lambda(X_1 \cup X_2, Y_1 \cup \{y\}) = |X_1| = |Y_1|, \qquad \text{if} \quad y \in Y_2.$$

In the same manner one finds

$$\lambda(X_1 \cup X_2, Y_1 \cup Y_2) = |X_1| = |Y_1|.$$

Using similar arguments for X_2 and Y_2 instead of X_1 and Y_1, it follows that

$$\lambda(X_1 \cup X_2, Y_1 \cup Y_2) = |X_2| = |Y_2|.$$

Hence we know

$$|X_1| = |X_2| = |Y_1| = |Y_2| = \lambda(X_1 \cup X_2, Y_1 \cup Y_2).$$

Secondly we prove that $\lambda(X_1, Y_2) = |X_1| = |Y_2|$, from which it follows that

$$(X_1, Y_2) \in \Lambda.$$

We have for $y \in Y_2 \backslash Y_1$

$$\begin{aligned}\lambda(X_1, (Y_1 \cap Y_2) \cup \{y\}) &\geqslant \lambda(X_1, Y_1 \cup \{y\}) + \lambda(X_1 \cup X_2, (Y_1 \cap Y_2) \cup \{y\}) \\ &\quad - \lambda(X_1 \cup X_2, Y_1 \cup \{y\}) \\ &\geqslant |X_1| + |Y_1 \cap Y_2| + 1 - |Y_1| \\ &= |Y_1 \cap Y_2| + 1.\end{aligned}$$

Hence, by induction on $|Y'|$ and using axiom (iii) of Definition 2.2,

$$\lambda(X_1, (Y_1 \cap Y_2) \cup Y') \geqslant |Y_1 \cap Y_2| + |Y'| \qquad \text{for} \quad Y' \subset Y_2 \backslash Y_1;$$

therefore also, putting $Y' = Y_2 \backslash Y_1$,

$$\lambda(X_1, Y_2) \geqslant |Y_1 \cap Y_2| + |Y_2 \backslash Y_1| = |Y_2| = |X_1|. \quad \blacksquare$$

If a linking system is given as (X, Y, Λ), it is understood that Λ is its collection of linked pairs. Likewise the symbol λ will be reserved for the linking function.

Let (X, Y, Λ) be a linking system with linking function λ. Its *dual linking system* is the linking system (Y, X, Λ^*), with

$$\Lambda^* = \{(Y', X') \mid (X', Y') \in \Lambda\}.$$

The dual linking function is then λ^*, defined, for $X' \subset X$ and $Y' \subset Y$, by:

$$\lambda^*(Y', X') = \lambda(X', Y').$$

For $X_0 \subset X$ and $Y_0 \subset Y$ let $\Lambda_0 = \{(X', Y') \in \Lambda \mid X' \subset X_0 \text{ and } Y' \subset Y_0\}$. Then (X_0, Y_0, Λ_0) forms a *sub-linking system* of (X, Y, Λ) (of course, this is again a linking system); its linking function λ_0 is then, clearly, given by

$$\lambda_0(X', Y') = \lambda(X', Y'), \qquad \text{for} \quad X' \subset X_0 \text{ and } Y' \subset Y_0.$$

Let $(X_1, Y_1) \in \Lambda$ and $(X_2, Y_2) \in \Lambda$ be maximal linked pairs in (X, Y, Λ), i.e., if $(X_1', Y_1') \in \Lambda$ such that $X_1 \subset X_1'$ and $Y_1 \subset Y_1'$, then $X_1 = X_1'$ and $Y_1 = Y_1'$, and similarly for (X_2, Y_2). Then it is easy to prove that also $(X_1, Y_2) \in \Lambda$ and $(X_2, Y_1) \in \Lambda$; in particular, $|X_1| = |X_2| = |Y_1| = |Y_2|$. Of course, we have the same for all sub-linking systems of (X, Y, Λ).

Now we come to some examples of linking systems.

a. *Linking Systems, induced by Bipartite Graphs*

Let (X, Y, E) be a bipartite graph (i.e., X and Y are finite sets and $E \subset X \times Y$) and define Δ_E by:

> $(X', Y') \in \Delta_E$ if and only if there is a matching in E between $X' \subset X$ and $Y' \subset Y$ (or: a bijection $\sigma: X' \to Y'$ with $(x, \sigma(x)) \in E$ for all $x \in X'$).

Sometimes we shall say: X' is *matched with* Y' if there is a matching between X' and Y'.

Now, (X, Y, Δ_E) is a linking system, with Δ_E as set of linked pairs. Clearly, Δ_E satisfies the axioms (i), (ii) and (iii) of Definition 2.1; a theorem of Ore [18] (cf. Perfect & Pym [21]) implies axiom (iv).

We shall denote the linking function of (X, Y, Δ_E) by δ_E ; from König's theorem it follows that $\delta_E(X', Y')$ is the minimal cardinality of a subset of $X \cup Y$ meeting each edge between $X' \subset X$ and $Y' \subset Y$, or

$$\delta_E(X', Y') = \min_{X'' \subset X'} \{| X' \backslash X'' | + | E(X'') \cap Y' |\}$$
$$= \min_{Y'' \subset Y'} \{| Y' \backslash Y'' | + | E(Y'') \cap X' |\},$$

where, for $X'' \subset X$, $E(X'') = \{y \mid (x, y) \in E \text{ for some } x \in X''\}$, and for $Y'' \subset Y$, $E(Y'') = \{x \mid (x, y) \in E \text{ for some } y \in Y''\}$. Linking systems obtained in this way are called *deltoid linking systems.*

b. *Linking Systems, induced by Directed Graphs*

Let (Z, E) be a directed graph (i.e., Z is a finite set and $E \subset Z \times Z$) and let X and Y be subsets of Z. Define Γ_E by

> $(X', Y') \in \Gamma_E$ if and only if $X' \subset X$, $Y' \subset Y$, $| X' | = | Y' |$ and there are $| X' |$ pairwise vertex-disjoint paths starting in X' and ending in Y'.

(A path may consist of only one vertex.)

Now (X, Y, Γ_E) is a linking system. Again the axioms (i), (ii) and (iii) of Definition 2.1 are easily verified; axiom (iv) is in fact the finite "linkage theorem" of Pym [23] (cf. Brualdi & Pym [6], or McDiarmid [16]).

Let γ_E be the linking function of (X, Y, Γ_E); by Menger's theorem we have: $\gamma_E(X', Y')$ equals the minimal cardinality of a subset of Z meeting each path starting in $X' \subset X$ and ending in $Y' \subset Y$.

We call so-constructed linking systems *gammoid linking systems*; of course, each deltoid linking system is a gammoid linking system.

c. *Linking Systems, induced be Matrices*

Let (X, Y, ϕ) be a matrix over some field F (i.e., ϕ is an F-valued function defined on $X \times Y$; X and Y are the collections of rows and columns, respectively).

Let Λ_ϕ be as follows

> $(X', Y') \in \Lambda_\phi$ if and only if the submatrix generated by the rows X' and columns Y' is non-singular.

Then (X, Y, Λ_ϕ) is a linking system, with Λ_ϕ as set of linked pairs; using simple linear algebraic methods one proves the axioms of Definition 2.1. (Sketch of proof of axiom (iv): Let $M = (X, Y, \phi)$, $(X_1, Y_1) \in \Lambda_\phi$ and $(X_2, Y_2) \in \Lambda_\phi$. We may suppose that $M \mid X_1 \times Y_1$ and $M \mid X_2 \times Y_2$ are maximal nonsingular submatrices of $M \mid (X_1 \cup X_2) \times (Y_1 \cup Y_2)$. Hence each column of $M \mid (X_1 \cup X_2) \times (Y_1 \cup Y_2)$ is a linear combination of the columns of $M \mid (X_1 \cup X_2) \times Y_2$. But then also each column of $M \mid X_1 \times (Y_1 \cup Y_2)$ is a linear combination of the columns of $M \mid X_1 \times Y_2$. So the rank of $M \mid X_1 \times Y_2$ equals the rank of $M \mid X_1 \times (Y_1 \cup Y_2)$, and this equals $\mid X_1 \mid$, since $M \mid X_1 \times Y_1$ is nonsingular. In the same way one proves that the rank of $M \mid X_1 \times Y_2$ equals $\mid Y_2 \mid$. This means that $M \mid X_1 \times Y_2$ is nonsingular, i.e., $(X_1, Y_2) \in \Lambda_\phi$.) Writing λ_ϕ for the linking function of (X, Y, Λ_ϕ) we have that $\lambda_\phi(X', Y')$ equals the rank of the submatrix generated by the rows X' and columns Y'. We call linking systems obtained in this way *representable over F*.

All three examples are self-dual: the dual linking system of a deltoid linking system (or gammoid linking system, or linking system representable over a field F) is again a deltoid system (or gammoid linking system, or linking system representable over F, respectively).

3. Matroids and Linking Systems

There are close relations between the concepts of matroid and linking system and in this section we give some of these relations. First we notice:

Theorem 3.1. *Let (X, Y, Λ) be a linking system and $X' \subset X$. Let $\mathscr{I}$ be the collection of all $Y' \subset Y$ with $(X'', Y') \in \Lambda$ for some $X'' \subset X'$. Then $\mathscr{I}$ is the collection of all independent sets of a matroid $(Y, \mathscr{I})$; this matroid has as rank function the function σ given by*

$$\sigma(Y') = \lambda(X', Y') \qquad \textit{for} \quad Y' \subset Y.$$

Proof. The function σ defined in the theorem is indeed the rank function of a matroid; also one has

$$\sigma(Y') = \mid Y' \mid \qquad \text{if and only if} \quad (X'', Y') \in \Lambda \text{ for some } X'' \subset X',$$

whence $\mathscr{I}$ is the corresponding collection of independent sets of the matroid. ∎

A corollary of this is a theorem of Edmonds & Fulkerson [8]: if (X, Y, E) is a bipartite graph and

$$\mathscr{I} = \{Y' \subset Y \mid \text{there is a matching between some subset of } X \text{ and } Y'\},$$

then $(Y, \mathscr{I})$ is a matroid; these matroids are called *transversal matroids* and can be obtained from deltoid linking systems.

A second corollary is a theorem of Perfect [19] and Pym [24]: if (Z, E) is a directed graph, X and Y are subsets of Z and

$$\mathscr{J} = \{Y' \subset Y \mid \text{there are } |Y'| \text{ pairwise vertex-disjoint paths starting in } X \text{ and ending in } Y'\},$$

then $(Y, \mathscr{J})$ is a matroid; these matroids are called *gammoids* and can be obtained from gammoid linking systems (cf. Mason [14]).

Furthermore, matroids obtained as in Theorem 3.1 from linking systems representable over some field are also representable over that field (and conversely, each representable matroid can be obtained from a representable linking system).

Secondly we show that each linking system may be understood as a matroid with a fixed base, and conversely.

THEOREM 3.2. *Let X and Y be disjoint finite sets. Then there is a one-to-one relation between*

(1) *linking systems (X, Y, Λ),*

and

(2) *matroids $(X \cup Y, \mathscr{B})$ with the property: $X \in \mathscr{B}$ ($\mathscr{B}$ is the collection of bases of the matroid),*

given by

$$(X', Y') \in \Lambda \quad \text{if and only if} \quad (X \backslash X') \cup Y' \in \mathscr{B} \text{ for } X' \subset X \text{ and } Y' \subset Y.$$

The corresponding linking function λ and rank-function ρ are related by

$$\rho(X' \cup Y') = \lambda(X \backslash X', Y') + |X'| \quad \text{for} \quad X' \subset X \quad \text{and} \quad Y' \subset Y.$$

Proof. (1) Let (X, Y, Λ) be a linking system, with linking function λ, and define

$$\mathscr{B} = \{(X \backslash X') \cup Y' \mid (X', Y') \in \Lambda\},$$

and

$$\rho(X' \cup Y') = \lambda(X \backslash X', Y') + |X'| \quad \text{for} \quad X' \subset X \quad \text{and} \quad Y' \subset Y.$$

The fact that ρ is a rank-function of a matroid follows easily from the axioms of Definition 2.2. The rank of this matroid is

$$\rho(X \cup Y) = \lambda(\varnothing, Y) + |X| = |X|.$$

In order to prove that $\mathscr{B}$ is the collection of bases of this matroid it is sufficient to prove

$$X'' \cup Y' \in \mathscr{B} \quad \text{if and only if} \quad \rho(X'' \cup Y') = | X'' \cup Y' | = | X |,$$

for $X'' \cup X$ and $Y' \subset Y$, or, putting $X' = X \backslash X''$,

$$(X', Y') \in \Lambda \quad \text{if and only if } \lambda(X', Y') + | X'' | = | X'' | + | Y' | = | X |.$$

Now this last equality holds if and only if $\lambda(X', Y') = | Y' | = | X' |$, and this is true, by the definition of a linking function, if and only if $(X', Y') \in \Lambda$.

(2) Let $(X \cup Y, \mathscr{B})$ be a matroid with collection of bases $\mathscr{B}$, rank-function ρ and $X \in \mathscr{B}$. Define

$$\Lambda = \{(X', Y') \mid (X \backslash X') \cup Y' \in \mathscr{B}, X' \subset X \text{ and } Y' \subset Y\},$$

and

$$\lambda(X', Y') = \rho(X \backslash X') \cup Y') - | X \backslash X' | \qquad \text{for} \quad X' \subset X \quad \text{and} \quad Y' \subset Y.$$

Now the fact that λ is the linking function of a linking system follows from the axioms for the rank-function of a matroid. Again, it is easy to prove that

$$(X', Y') \in \Lambda \qquad \text{if and only if} \quad \lambda(X', Y') = | X' | = | Y' |. \quad \blacksquare$$

This relation between linking systems and "based" matroids is such that gammoid linking systems are related with gammoids (cf. Mason [14]) and linking systems representable over some field are related with matroids representable over that field. The deltoid linking systems (X, Y, Λ) are related with "fundamental transversal matroids" (or "principal matroids," or "strict deltoids") with principal basis X (cf. Bondy & Welsh [1]).

A consequence of Theorem 3.2 is

COROLLARY 3.2a. *Let (X, Y, Λ) be a gammoid linking system. Then there exists a natural number N such that (X, Y, Λ) is representable over each field F with $| F | \geqslant N$.*

Proof. Ingleton & Piff [10] and Mason [14] proved that each gammoid is representable over each sufficiently large field. The corollary follows from the remark following the proof of Theorem 3.2. $\blacksquare$

We shall use Theorem 3.2 to show how a matroid can be linked with a linking system, forming a new matroid, and how two linking systems can be linked, generating a new linking system.

THEOREM 3.3. *Let $(X, \mathscr{I})$ be a matroid (with $\mathscr{I}$ the collection of inde-*

pendent sets and ρ *the rank-function) and let* (X, Y, Λ) *be a linking system (with* λ *as linking function). Put*

$$\mathscr{I} * \Lambda = \{Y' \subset Y \mid (X', Y') \in \Lambda \text{ for some } X' \in \mathscr{I}\}.$$

Then $(Y, \mathscr{I} * \Lambda)$ *is again a matroid with* $\mathscr{I} * \Lambda$ *the collection of independent sets and with rank-function* $\rho * \lambda$ *given by*

$$(\rho * \lambda)(Y') = \min_{X' \subset X} (\rho(X \backslash X') + \lambda(X', Y')) \qquad \text{for} \quad Y' \subset Y.$$

Proof. We may suppose that X and Y are disjoint sets (otherwise take disjoint copies of X and Y). Let $(X \cup Y, \mathscr{B})$ be the matroid related with (X, Y, Λ) as in Theorem 3.2, and let ρ' be its rank-function. Let M be the matroid union of the matroids $(X, \mathscr{I})$ and $(X \cup Y, \mathscr{B})$; this matroid is defined on $X \cup Y$ by taking as independent sets all unions of an independent set of $(X, \mathscr{I})$ and an independent set of $(X \cup Y, \mathscr{B})$. We now prove that the contraction $M \cdot Y$ of M to Y (i.e., we contract X) has as collection of independent sets the collection $\mathscr{I} * \Lambda$, as defined above.

Let Y' be an independent set of $M \cdot Y$. Then, since X is independent in M, we have that $X \cup Y'$ is independent in M. Then there exists an $X' \subset X$ such that $X' \in \mathscr{I}$ and $(X \backslash X') \cup Y' \in \mathscr{B}$, or $(X', Y') \in \Lambda$; hence $Y' \in \mathscr{I} * \Lambda$. Following the same steps in the reverse order one proves: if $Y' \in \mathscr{I} * \Lambda$, then Y' is independent in $M \cdot Y$.

Let $\rho * \lambda$ be the rank-function of the matroid $(Y, \mathscr{I} * \Lambda)$, i.e., of the matroid $M \cdot Y$, and let ρ'' be the rank-function of the matroid M. Then

$$\begin{aligned}
(\rho * \lambda)(Y') &= \rho''(X \cup Y') - \rho''(X) \\
&= \min\{\rho(X') + \rho'(X' \cup Y'') + |X \backslash X'| \\
&\qquad + |Y' \backslash Y''| - |X| \mid X' \subset X \text{ and } Y'' \subset Y'\} \\
&= \min\{\rho(X') + \lambda(X \backslash X', Y'') + |X'| + |X \backslash X'| \\
&\qquad + |Y' \backslash Y''| - |X| \mid X' \subset X \text{ and } Y'' \subset Y'\} \\
&= \min\{\rho(X') + \lambda(X \backslash X', Y'') + |Y' \backslash Y''| \mid X' \subset X \text{ and } Y'' \subset Y'\} \\
&= \min_{X' \subset X} (\rho(X') + \lambda(X \backslash X', Y')) \\
&= \min_{X' \subset X} (\rho(X \backslash X') + \lambda(X', Y')).
\end{aligned}$$

In this we have used well-known theorems on the rank of the contraction of a matroid and on the rank of the union of two matroids. ∎

As straightforward corollaries we have theorems of Rado [25] and Perfect [20] (where the linking system is obtained from a bipartite graph), theorems of Brualdi [3] and Mason [14] (in case the linking system is obtained from a directed graph) and a theorem on the linking of matroids by matrices.

Now we generalize theorems of Mason [12] and Brualdi [3] on the product of a matroid and a bipartite graph or a directed graph.

THEOREM 3.4. *Let (X, Y, Λ) be a linking system and let $(X, \mathscr{I})$ be a matroid. Let B be a base of $(Y, \mathscr{I} * \Lambda)$ and $A \in \mathscr{I}$ such that $(A, B) \in \Lambda$. Let $[A]$ be the closure (span) of A in the matroid $(X, \mathscr{I})$. Set furthermore:*

$$\mathscr{I}' = \{X' \in \mathscr{I} \mid X' \subset [A]\},$$

and

$$\Lambda' = \{(X', Y') \in \Lambda \mid X' \subset [A], Y' \subset Y\}.$$

Then $\mathscr{I} * \Lambda = \mathscr{I}' * \Lambda'$, i.e., $Y' \in \mathscr{I} * \Lambda$ if and only if $(X', Y') \in \Lambda$ for some $X' \in \mathscr{I}$ with $X' \subset [A]$.

Proof. Let D be a base of $(Y, \mathscr{I} * \Lambda)$. We have to prove the existence of an independent subset C of $[A]$ such that $(C, D) \in \Lambda$.

Suppose that $C \subset X$ is such that:

(1) $C \in \mathscr{I}$ and $(C, D) \in \Lambda$;

(2) $\rho(C \cup A)$ is minimal with property (1) (ρ is the rank-function of $(X, \mathscr{I})$);

(3) $|C \cap A|$ is maximal with the properties (1) and (2).

If $\rho(C \cup A) = |A|$ then $C \subset [A]$, which was to be proved. Therefore suppose

$$\rho(C \cup A) > |A| = |B| = |C| = |D|.$$

Hence $C \cup \{x\} \in \mathscr{I}$ for some $x \in A \backslash C$. Then

$$\begin{aligned} &\lambda((A \cap C) \cup \{x\}, D) \\ &\quad \geqslant \lambda(C \cup \{x\}, D) + \lambda((A \cap C) \cup \{x\}, B \cup D) - \lambda(C \cup \{x\}, B \cup D) \\ &\quad = |D| + |(A \cap C) \cup \{x\}| - |C| = |(A \cap C) \cup \{x\}|. \end{aligned}$$

Therefore, $((A \cap C) \cup \{x\}, D') \in \Lambda$ for some $D' \subset D$. Since $(C, D) \in \Lambda$ there exists (by axiom (iv) of Definition 2.1) a $C' \subset C \cup \{x\}$, such that $(A \cap C) \cup \{x\} \subset C'$ and $(C', D) \in \Lambda$. But then

$$C' \in \mathscr{I}, \qquad \rho(C' \cup A) \leqslant \rho(C \cup A) \qquad \text{and} \qquad |C' \cap A| > |C \cap A|,$$

contradicting condition (1), (2) and (3) above. ∎

This theorem generalizes theorems of Mason [12, 13] (on the linking of matroids by bipartite graphs) and Brualdi [3] (on the linking by directed graphs).

Theorem 3.3 gave a kind of product of a matroid and a linking system; in the same way the following theorem gives a product of two linking systems.

THEOREM 3.5. *Let (X, Y, Λ_1) and (Y, Z, Λ_2) be two linking systems, with linking functions λ_1 and λ_2, respectively. Define*

$$\Lambda_1 * \Lambda_2 = \{(X', Z') \mid (X', Y') \in \Lambda_1 \text{ and } (Y', Z') \in \Lambda_2 \text{ for some } Y' \subset Y\}.$$

Then $(X, Z, \Lambda_1 * \Lambda_2)$ is again a linking system, with linking function given by

$$(\lambda_1 * \lambda_2)(X', Z') = \min_{Y' \subset Y} (\lambda_1(X', Y') + \lambda_2(Y \backslash Y', Z')).$$

Proof. Without restrictions on the generality we may suppose that X, Y and Z are pairwise disjoint sets.

Let $(X \cup Y, \mathscr{B}_1)$ and $(Y \cup Z, \mathscr{B}_2)$ be the matroids related to the linking systems (X, Y, Λ_1) and (Y, Z, Λ_2), respectively (as in Theorem 3.2).

Let $(X \cup Y \cup Z, \mathscr{B})$ be the union of these two matroids. It is easy to verify that

$$(X', Z') \in \Lambda_1 * \Lambda_2 \quad \text{if and only if} \quad (X \backslash X') \cup Y \cup Z' \in \mathscr{B}.$$

Since Y is independent in the matroid $(X \cup Y \cup Z, \mathscr{B})$, this last holds if and only if $(X \backslash X') \cup Z'$ is a base of the contraction of $(X \cup Y \cup Z, \mathscr{B})$ to $X \cup Z$ (i.e., contracting Y). Since X is also a base of this contraction, we have, using Theorem 3.2, that $(X, Z, \Lambda_1 * \Lambda_2)$ is a linking system.

Now let ρ_1, ρ_2 and ρ be the rank functions of $(X \cup Y, \mathscr{B}_1)$, $(Y \cup Z, \mathscr{B}_2)$ and $(X \cup Y \cup Z, \mathscr{B})$, respectively, and let ρ' be the rank function of the contraction of $(X \cup Y \cup Z, \mathscr{B})$ to $X \cup Z$. Then by the foregoing and Theorem 3.2

$$(\lambda_1 * \lambda_2)(X', Z') = \rho'((X \backslash X') \cup Z') - | X \backslash X' |.$$

Using well-known formulas for the rank of a contraction of a matroid and for the rank of the union of two matroids we know

$$\begin{aligned} \rho'((X \backslash X') \cup Z') - | X \backslash X' | &= \rho((X \backslash X') \cup Y \cup Z') - \rho(Y) - | X \backslash X' | \\ &= \min\{\rho_1(X'' \cup Y'') + \rho_2(Y'' \cup Z'') + | X \backslash (X' \cup X'')| \\ &\qquad + | Y \backslash Y'' | + | Z' \backslash Z'' | - | Y | - | X \backslash X' | \mid \\ &\qquad X'' \subset X \backslash X', Y'' \subset Y, Z'' \subset Z'\}. \end{aligned}$$

But this last equals, again by Theorem 3.2:

$$\begin{aligned} &\min\{\lambda_1(X \backslash X'', Y'') + \lambda_2(Y \backslash Y'', Z'') + | Z' \backslash Z'' | \mid X'' \subset X \backslash X', Y'' \subset Y, Z'' \subset Z'\} \\ &= \min_{\substack{X'' \subset X \backslash X' \\ Y'' \subset Y}} (\lambda_1(X \backslash X'', Y'') + \min_{Z'' \subset Z'} (\lambda_2(Y \backslash Y'', Z'') + | Z' \backslash Z'' |)) \\ &= \min_{\substack{X'' \subset X \backslash X' \\ Y'' \subset Y}} (\lambda_1(X \backslash X'', Y'') + \lambda_2(Y \backslash Y'', Z')) \\ &= \min_{Y'' \subset Y} (\lambda_1(X', Y'') + \lambda_2(Y \backslash Y'', Z')). \quad \blacksquare \end{aligned}$$

It is evident that if $(X, \mathscr{I})$ is a matroid and (X, Y, Λ_1), (Y, Z, Λ_2) and (Z, U, Λ_3) are linking systems, then:

$$(\mathscr{I} * \Lambda_1) * \Lambda_2 = \mathscr{I} * (\Lambda_1 * \Lambda_2)$$

and

$$(\Lambda_1 * \Lambda_2) * \Lambda_3 = \Lambda_1 * (\Lambda_2 * \Lambda_3).$$

Piff & Welsh [22] have proved that for each pair of matroids there exists a natural number N, such that if F is a field with more than N elements and both matroids are representable over F, then the union of the two matroids again is representable over F. In the light of the proofs of the Theorems 3.3 and 3.5 this result implies:

(i) if $(X, \mathscr{I})$ is a matroid and (X, Y, Λ) is a linking system, then there is a natural number N such that: if F is a field with more than N elements and both $(X, \mathscr{I})$ and (X, Y, Λ) are representable over F, then also the matroid $(Y, \mathscr{I} * \Lambda)$ is representable over F;

(ii) if (X, Y, Λ_1) and (Y, Z, Λ_2) are linking systems, then there is a natural number N such that: if F is a field with more than N elements and both (X, Y, Λ_1) and (Y, Z, Λ_2) are representable over F, then also the linking system $(X, Z, \Lambda_1 * \Lambda_2)$ is representable over F.

In general it is not true that the product of two linking systems generated by two matrices (as in example c) equals the linking system generated by the product of the two matrices. Also, one can not state in general that the product of two linking systems generated by two bipartite graphs (X, Y, E_1) and (Y, Z, E_2) equals the linking system generated by the product (X, Z, E_1E_2) of the two bipartite graphs (here, $(x, z) \in E_1E_2$ if and only if $(x, y) \in E_1$ and $(y, z) \in E_2$ for some $y \in Y$). It is not even true that the product of two linking systems representable over some field F is again representable over F, nor that the product of two deltoid linking systems is again a deltoid linking system. It is easy to prove that the product of two gammoid linking systems is again a gammoid linking system.

Finally, we generalize theorems of Brualdi [2, 4] on matroids and graphs to a theorem on matroids and linking systems.

THEOREM 3.6. *Let (X, Y, Λ) be a linking system and let $(X, \mathscr{I})$ and $(Y, \mathscr{J})$ be matroids. Then*

$$\max\{|Y'| \mid X' \in \mathscr{I}, Y' \in \mathscr{J} \text{ and } (X', Y') \in \Lambda\} = \min_{\substack{X' \subset X \\ Y' \subset Y}} (\rho(X') + \sigma(Y') + \lambda(X \backslash X', Y \backslash Y')),$$

where ρ and σ are the rank-functions of $(X, \mathscr{I})$ and $(Y, \mathscr{J})$, respectively.

Proof. As is done by Welsh [27], proving Brualdi's theorem, we use Edmonds' intersection theorem [7]:

If $(X, \mathscr{I}_1)$ and $(X, \mathscr{I}_2)$ are matroids, with rank-functions ρ_1 and ρ_2, respectively, then

$$\max\{|X'| \mid X' \in \mathscr{I}_1 \text{ and } X' \in \mathscr{I}_2\} = \min_{X' \subset X} (\rho_1(X') + \rho_2(X \setminus X')).$$

We have

$$\max\{|Y'| \mid X' \in \mathscr{I},\ Y' \in \mathscr{J} \text{ and } (X', Y') \in \Lambda\}$$
$$= \max\{|Y'| \mid Y' \in \mathscr{J} \text{ and } Y' \in \mathscr{I} * \Lambda\}.$$

This last equals, by Edmonds' intersection theorem,

$$\min_{Y' \subset Y} (\sigma(Y') + (\rho * \lambda)(Y \setminus Y')).$$

Following Theorem 3.3 this is the same as

$$\min_{Y' \subset Y} (\sigma(Y') + \min_{X' \subset X} (\rho(X') + \lambda(X \setminus X', Y \setminus Y')))$$
$$= \min_{\substack{X' \subset X \\ Y' \subset Y}} (\rho(X') + \sigma(Y') + \lambda(X \setminus X', Y \setminus Y')). \quad \blacksquare$$

From this theorem results of Brualdi on matroids, bipartite graphs ([2], cf. Welsh [27]) directed graphs ([4], cf. McDiarmid [16]) follow easily.

4. Bipartite Graphs and Linking Systems

In this last chapter we give some relations between linking systems and bipartite graphs. Since each linking system can be understood as a matroid with a fixed base (Theorem 3.2) this gives also results for matroids. Related work has been done by Krogdahl [11].

In Section 1 we have already defined for each bipartite graph (X, Y, E), a deltoid linking system (X, Y, Δ_E). Now we define for each linking system (X, Y, Λ) the "underlying" bipartite graph (X, Y, E_Λ), where E_Λ is the subset of $X \times Y$ with the property

$$(x, y) \in E_\Lambda \qquad \text{if and only if} \quad (\{x\}, \{y\}) \in \Lambda.$$

(Without loss of generality we may suppose that X and Y are disjoint sets.)

The two basic results of this section are:

(i) if $X' \subset X$ and $Y' \subset Y$ are such that there is exactly one matching in (X, Y, E_Λ) between X' and Y', then $(X', Y') \in \Lambda$;

(ii) if $(X', Y') \in \Lambda$, then there is at least one matching in (X, Y, E_Λ) between X' and Y'.

Clearly, a linking system is a deltoid linking system if and only if $\Lambda = \Delta_{E_\Lambda}$.

Let $(X \cup Y, \mathscr{B})$ be the matroid, related to the linking system (X, Y, Λ) (cf. Theorem 3.2), with fixed base X. Thne for each $y \in Y$ the set

$$\{y\} \cup \{x \mid (x, y) \in E_\Lambda\}$$

is the unique circuit of the matroid $(X \cup Y, \mathscr{B})$ contained in $X \cup \{y\}$. Similarly, for each $x \in X$ the set

$$\{x\} \cup \{y \mid (x, y) \in E_\Lambda\}$$

is the unique cocircuit contained in $Y \cup \{x\}$.

First we prove a theorem, which was inspired by a result of Greene (cf. Greene [9], Woodall [30] or McDiarmid [15]) and which we need for Theorem 4.2.

THEOREM 4.1. *Let (X, Y, Λ) be a linking system and $(X', Y') \in \Lambda$. Furthermore let $X'' \subset X'$. Then $(X'', Y'') \in \Lambda$ and $(X' \backslash X'', Y' \backslash Y'') \in \Lambda$ for some $Y'' \subset Y'$.*

Proof. Let $M_1 = (Y', \mathscr{I}_1)$ be the matroid on Y' with

$$\mathscr{I}_1 = \{Y_0' \subset Y' \mid (X_0', Y_0') \in \Lambda \text{ for some } X_0' \subset X''\}.$$

Similarly, let $M_2 = (Y', \mathscr{I}_2)$ be the matroid on Y' with

$$\mathscr{I}_2 = \{Y_0' \subset Y' \mid (X_0', Y_0') \in \Lambda \text{ for some } X_0' \subset X' \backslash X''\}.$$

If we have that Y' is a base of the union $M_1 \vee M_2$ of M_1 and M_2, then there exists a $Y'' \subset Y'$ such that

$$(X'', Y'') \in \Lambda \qquad \text{and} \qquad (X' \backslash X'', Y' \backslash Y'') \in \Lambda.$$

Edmonds' theorem implies that Y' is a base of $M_1 \vee M_2$ if and only if

$$\rho_1(Y_0') + \rho_2(Y_0') \geqslant |Y_0'|, \qquad \text{for each} \quad Y_0' \subset Y',$$

where ρ_1 and ρ_2 are the rank-functions of M_1 and M_2, respectively.

We shall prove that this last inequality holds. Let $Y_0' \subset Y'$. Then, by axiom (iii) of Definition 2.1, $(X_0', Y_0') \in \Lambda$ for some $X_0' \subset X'$. Now it is easy to see that

$$\rho_1(Y_0') \geqslant | X_0' \cap X'' | \qquad \text{and} \qquad \rho_2(Y_0') \geqslant | X_0' \backslash X'' |.$$

Hence

$$\rho_1(Y_0') + \rho_2(Y_0') \geqslant | X_0' \cap X'' | + | X_0' \backslash X'' | = | X_0' | = | Y_0' |.$$

As this is true for each $Y_0' \subset Y'$, we have shown that Y' is a base of $M_1 \vee M_2$. ∎

Theorem 4.1 is helpful to prove

THEOREM 4.2. *Let (X, Y, Λ) be a linking system and (X, Y, E_Λ) its underlying bipartite graph. Then for each pair $(X', Y') \in \Lambda$ there exists a matching in (X, Y, E_Λ) between X' and Y'.*

Proof. We proceed by induction on $| X' |$. If $X' = \varnothing$ the result is trivial. Suppose $X' \neq \varnothing$, and the theorem holds for all pairs $(X'', Y'') \in \Lambda$ with $| X'' | < | X' |$. Take $x \in X'$. Then, by Theorem 4.1, we can find a $y \in Y'$ such that $(\{x\}, \{y\}) \in \Lambda$ and $(X' \backslash \{x\}, Y' \backslash \{y\}) \in \Lambda$. Now, by induction, there is a matching in (X, Y, E_Λ) between $X' \backslash \{x\}$ and $Y' \backslash \{y\}$; since $(x, y) \in E_\Lambda$, also a matching exists between X' and Y'. ∎

Here we proved that for a linking system (X, Y, Λ) we always have $\Lambda \subset \Delta_{E_\Lambda}$ (or $\lambda \leqslant \delta_{E_\Lambda}$, where λ is the linking function of the linking system). It means that Δ_E is the maximum (under inclusion) of all linking systems with underlying bipartite graph (X, Y, E).

Next we prove that if there is exactly one matching in the underlying bipartite graph between two sets X' and Y' then (X', Y') is a linked pair. For this we need a lemma.

LEMMA. *Let (X, Y, Λ) be a linking system with underlying bipartite graph (X, Y, E_Λ). Suppose X' and X'' are disjoint subsets of X, and Y' and Y'' are disjoint subsets of Y. Suppose furthermore: $| X' | = | Y' |$ and $| X'' | = | Y'' |$, and there is no edge between X' and Y'' (i.e., $E_\Lambda \cap (X' \times Y'') = \varnothing$). Then:*

$$((X' \cup X''), (Y' \cup Y'')) \in \Lambda \quad \textit{if and only if } (X', Y') \in \Lambda \textit{ and } (X'', Y'') \in \Lambda.$$

Proof. (1) Suppose $(X' \cup X'', Y' \cup Y'') \in \Lambda$. By Theorem 4.1 there exists a subset Y_0 of $Y' \cup Y''$ with the properties

$$(X', Y_0) \in \Lambda \qquad \text{and} \qquad (X'', (Y' \cup Y'') \backslash Y_0) \in \Lambda.$$

By Theorem 4.2 there is a matching in E_Λ between X' and Y_0; since there is no edge between X' and Y'' it follows that $Y_0 \subset Y'$. But $|Y'| = |X'| = |Y_0|$; hence $Y_0 = Y'$ and

$$(X', Y') \in \Lambda \quad \text{and} \quad (X'', Y'') \in \Lambda.$$

(2) Suppose $(X', Y') \in \Lambda$ and $(X'', Y'') \in \Lambda$. By axiom (iv) of Definition 2.1 there is a $(X_0, Y_0) \in \Lambda$ such that

$$X' \subset X_0 \subset X' \cup X'' \quad \text{and} \quad Y'' \subset Y_0 \subset Y' \cup Y''.$$

According to Theorem 4.2 there is a matching in E_Λ between X_0 and Y_0. Since there is no edge between X' and Y'', one has $|X_0 \backslash X'| \geqslant |Y''|$, or $X_0 = X' \cup X''$ and $Y_0 = Y' \cup Y''$. Therefore $(X' \cup X'', Y' \cup Y'') \in \Lambda$. ∎

As a consequence we have

THEOREM 4.3. *Let (X, Y, Λ) be a linking system with underlying bipartite graph (X, Y, E_Λ). Let $X' \subset X$ and $Y' \subset Y$ be such that there is exactly one matching in (X, Y, E_Λ) between X' and Y'. Then $(X', Y') \in \Lambda$.*

Proof. Again, we prove the theorem by induction on $|X'|$. If $X' = \varnothing$ the theorem is trivial. Suppose $X' \neq \varnothing$, and the theorem holds for all pairs (X'', Y'') with $|X''| < |X'|$. Since there is exactly one matching between X' and Y', there exists, by a theorem of Ryser [26] on the number of matchings in a bipartite graph, an $x \in X'$ such that there is only one $y \in Y'$ with $(x, y) \in E_\Lambda$. Consequently, there is exactly one matching between $X' \backslash \{x\}$ and $Y' \backslash \{y\}$. By induction we know

$$(X' \backslash \{x\}, Y' \backslash \{y\}) \in \Lambda.$$

Also $(\{x\}, \{y\}) \in \Lambda$ and there is no edge in E_Λ between $\{x\}$ and $Y' \backslash \{y\}$. Hence, by the foregoing lemma, $(X', Y') \in \Lambda$. ∎

In general it is not true that there is a minimum of all linking systems with underlying bipartite graph (X, Y, E); in particular the set of all pairs (X', Y') with the properties

> $X' \subset X$, $Y' \subset Y$ and there is exactly one matching in (X, Y, E) between X' and Y',

in general does not form the set of linked pairs of a linking system.

To conclude this section we mention another consequence of the lemma, which says that a linking system is completely determined by the sub-linking systems on the connected components of the underlying bipartite graph. This notion of component coincides with that of component usual in the matroid related to the linking system (in the sense of Theorem 3.2).

THEOREM 4.4. *Let (X, Y, Λ) be a linking system with underlying bipartite graph (X, Y, E_Λ). Let $X_1 \cup Y_1, \ldots, X_n \cup Y_n$ be the connected components of this bipartite graph (where $X_1, \ldots, X_n \subset X$ and $Y_1, \ldots, Y_n \subset Y$). For each $i = 1, \ldots, n$, let $X_i' \subset X_i$ and $Y_i' \subset Y_i$. Then*

$$\left(\bigcup_{i=1}^{n} X_i', \bigcup_{i=1}^{n} Y_i'\right) \in \Lambda \text{ if and only if } (X_i', Y_i') \in \Lambda \text{ for each } i = 1, \ldots, n.$$

Proof. Left to the reader. ∎

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