Mackey functors and functorial extensions

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Abstract

Given a finite group $X$ and a normal subgroup $G$ of $X$, we show that any Mackey functor $M$ for $X$ induces another Mackey functor $\tilde{M}$ for $X$ associated to $G$. We then consider the question, whether there exists a map $\tilde{M} \to M$ extending elements from $M(G)$ to $M(X)$ and compatible with the restriction maps. In the case that the order of $G$ and the index of $G$ in $X$ are relatively prime, we give a sufficient condition for the existence of such a map, using canonical induction formulae.

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1. Introduction

Let $X$ be a finite group, and let $G$ be a normal subgroup of $X$. A classical result in the representation theory of finite groups says that, if the order of $G$ and the index of $G$ in $X$ are relatively prime, then any irreducible complex character of $G$ that is stable under conjugation with elements of $X$, can be extended to a character of $X$. Moreover, this extension is unique under the additional condition that the determinant of its restriction to a fixed complement of $G$ in $X$ is trivial. However, as Example 3.7 will show, this extension is not functorial, in the following sense.

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The rings of complex characters of $X$ and its subgroups form a Mackey functor $R$ for $X$. That is, every $R(Y)$, for $Y$ being a subgroup of $X$, is a $\mathbb{Z}$-module, and one has conjugation, restriction and induction maps between these modules, satisfying a number of axioms. In Section 2 we show, that the normal subgroup $G$ of $X$ induces another Mackey functor $\tilde{R}$ for $X$, where the $\mathbb{Z}$-module for a subgroup $Y$ of $X$ is given by $\tilde{R}(Y) = R(Y \cap G)^Y$, i.e., the virtual characters of $Y \cap G$ that are stable under $Y$. Now one can consider maps $e : \tilde{R} \to R$, i.e., families of maps $e_Y : \tilde{R}(Y) \to R(Y)$, one for each subgroup $Y$ of $X$. If we require additionally that $e_Y$ followed by $\text{res}_{Y \cap G}^Y$ is the identity on $\tilde{R}(Y)$ then such a map can be obtained by extending characters, as described above. But, in this functorial framework, the given answer is not satisfying, since these maps $e_Y : \tilde{R}(Y) \to R(Y)$ obtained by extending characters do not commute with the restriction maps of the Mackey functors $\tilde{R}$ and $R$.

In Section 3 we show how to construct a different family of maps, that do commute with conjugations and restrictions. We call such a family an $r$-extension morphism. The essential ingredients in its construction are the Adams operators on the ring of complex characters.

Now, if $M$ is any Mackey functor for $X$, we still can construct another Mackey functor $\tilde{M}$ for $X$ associated to $G$, where for a subgroup $Y$ of $X$ we set $\tilde{M}(Y) = M(Y \cap G)^Y$. Also, we still can hope to find an $r$-extension morphism. However, this question is much harder, and we only give a partial answer to it. First of all, we require the existence of a canonical induction formula for $M$. Such a formula for the ring of complex characters (and some other representation rings) has been given by Boltje in [1]. We briefly recall the most important facts about this theory in Section 4. Then we show that a canonical induction formula for $M$ induces also one for $\tilde{M}$. Finally, in Section 5 we give sufficient conditions for the existence of $r$-extension morphisms, using the canonical induction formula. As an application, we prove the existence of an $r$-extension morphism for the trivial source ring.

**Notation.** For a finite group $G$, we write $H \leq G$ to denote that $H$ is a subgroup of $G$. If $H \leq G$ and $x \in G$ we write $^xH$ for the conjugate subgroup $xHx^{-1}$. For subgroups $H, K \leq G$ we write $[G/H]$ respectively $[H\backslash G/K]$ for a set of representatives for the cosets respectively double cosets in $G$.

## 2. Mackey functors

Let $X$ be a finite group, and let $k$ be any commutative ring. We recall the following two equivalent definitions of Mackey functors. The first one is due to Green, our notations are from [1].

**2.1. Definition.** (a) A $k$-conjugation functor $M = (M, c)$ for $X$ is a family of $k$-modules $M(Y)$, for $Y \leq X$, together with $k$-module homomorphisms

$$c_{x, Y} : M(Y) \to M(^xY), \quad \text{for } x \in X, \ Y \leq X,$$

such that

$$c_{x, Y} = \text{id}_{M(Y)} \quad \text{for } y \in Y, \ Y \leq X,$$

(C1)
\(c_{x_1x_2,Y} = c_{x_1,Y} \circ c_{x_2,Y} \quad \text{for } x_1, x_2 \in X, \ Y \leq X. \quad \text{(C2)}\)

(b) A \(k\)-restriction functor \(M = (M, c, r)\) for \(X\) is a \(k\)-conjugation functor \((M, c)\) together with \(k\)-module homomorphisms

\[r_Y^Z : M(Y) \to M(Z) \quad \text{for } Z \leq Y \leq X,\]

such that

\[r_Y^Y = \text{id}_{M(Y)} \quad \text{for } Y \leq X, \quad \text{(R1)}\]

\[r_W^Z \circ r_Y^Z = r_W^Y \quad \text{for } W \leq Z \leq Y \leq X, \quad \text{(R2)}\]

\[c_x^Y \circ r_Y^Z = r_Y^Z \circ c_x^Y \quad \text{for } x \in X, \ Z \leq Y \leq X. \quad \text{(R3)}\]

(c) A \(k\)-Mackey functor \(M = (M, c, r, t)\) for \(X\) is a \(k\)-restriction functor \((M, c, r)\) together with \(k\)-module homomorphisms

\[t_Y^Z : M(Z) \to M(Y) \quad \text{for } Z \leq Y \leq X,\]

such that

\[t_Y^Y = \text{id}_{M(Y)} \quad \text{for } Y \leq X, \quad \text{(M1)}\]

\[t_Y^Y \circ t_W^Y = t_Y^W \quad \text{for } W \leq Z \leq Y \leq X, \quad \text{(M2)}\]

\[c_x^Y \circ t_Y^Z = t_Y^Z \circ c_x^Y \quad \text{for } x \in X, \ Z \leq Y \leq X, \quad \text{(M3)}\]

\[r_U^Y \circ t_Y^Z = \sum_{y \in \{U \cap Y \mid Y\}} t_U^y \circ t_U^{y\cap Z} \circ c_y^Z \quad \text{for } Z \leq Y \leq X, \ U \leq Y. \quad \text{(M4)}\]

We call the maps \(c_{x,Y}\) conjugations, the maps \(r_Y^Z\) restrictions and the maps \(t_Y^Z\) transfer or induction maps.

(d) A morphism \(f : M \to N\) of \(k\)-conjugation (respectively restriction, respectively Mackey) functors \(M\) and \(N\) is a family of \(k\)-module homomorphisms

\[f_Y : M(Y) \to N(Y) \quad \text{for } Y \leq X,\]

that commute with conjugations (respectively conjugations and restrictions, respectively conjugations, restrictions and transfers).

Instead of \(c_{x,Y}(m)\), often we write just \(x^m\), and sometimes we write \(m|_Z\) instead of \(r_Y^Z(m)\).

\(A\) is called a subfunctor of \(M\) if \(A(Y) \subseteq M(Y)\) is a submodule for all \(Y \leq X\), and if the structure maps restricted to these submodules have images in the corresponding submodules. In this case we write \(A \subseteq M\).
For subgroups \( Y, Z \subseteq X \) with \( Z \) normal in \( Y \), denote by \( M(Z)^Y \) the set of \( Y \)-fixed points in \( M(Z) \), i.e.,

\[
M(Z)^Y = \{ m \in M(Z) \mid c_{y,Z}(m) = m \text{ for all } y \in Y \}.
\]

The second definition of Mackey functors in terms of \( X \)-sets is due to Dress, see for example [4].

2.2. Definition. A \( k \)-Mackey functor for \( X \) is a bivariant functor \( M \) from the category \( X \)-set of finite \( X \)-sets to the category \( k \)-Mod of \( k \)-modules, meaning that \( M \) consists of a covariant functor \( M^* \) and a contravariant functor \( M^* \) which coincide on objects, satisfying the following two axioms.

(A1) For any two \( X \)-sets \( S \) and \( T \), let \( i_S \) and \( i_T \) denote the respective injections from \( S \) and \( T \) into the disjoint union \( S \cup T \), then the maps \( M^*(i_S) \oplus M^*(i_T) \) and \( M^*(i_S) \oplus M^*(i_T) \) are mutually inverse isomorphisms between \( M(S \cup T) \) and \( M(S) \oplus M(T) \).

(A2) For any pullback square of \( X \)-sets

\[
\begin{array}{ccc}
S & \to & T \\
\downarrow{\delta} & & \downarrow{\alpha} \\
U & \to & V \\
\end{array}
\]

we have that \( M^*(\beta) \circ M_*(\alpha) = M_*(\delta) \circ M^*(\gamma) \).

2.3. Remark. If \( M \) is a Mackey functor in the sense of the second definition, then we obtain a Mackey functor \( (M, c, r, t) \) in the sense of the first definition by setting \( M(Y) = M(X/Y) \) for a subgroup \( Y \) of \( X \) and defining the structure maps as

\[
c_{x,Y} = M_*(\gamma_{x,Y}), \quad r^Y_Z = M^*(\pi^Y_Z) \quad \text{and} \quad t^Y_Z = M_*(\pi^Y_Z),
\]

where \( \pi^Y_Z \) is the natural surjection \( X/Z \to X/Y \), \( xZ \to xY \), and \( \gamma_{x,Y} \) is the map \( X/Y \to X/Y, \ gY \mapsto gx^{-1}(xY) \). Here and in the following we use the symbol \( M \) for both definitions; if we write \( M(Y) \) for a subgroup \( Y \) of \( X \) it is understood that we use the first one, while \( M(S) \) for an \( X \)-set \( S \) refers to the second one.

2.4. Theorem. Let \( (M, c, r, t) \) be a \( k \)-Mackey functor for \( X \), and let \( G \) be a normal subgroup of \( X \). Then

\[
\tilde{M}(Y) := M(Y \cap G)^Y, \quad \text{for } Y \subseteq X,
\]

\[
\tilde{c}_{x,Y} := c_{x,Y \cap G} \quad \text{for } x \in X, \ Y \subseteq X,
\]

\[
\tilde{r}^Y_Z := r^Y_Z \cap G \quad \text{for } Z \subseteq Y \subseteq X,
\]
\[ \tilde{t}_Z^Y := \sum_{y \in [Y/(Y \cap G)]} c_{y,Y \cap G} \circ t_{Y \cap G}^y \] for \( Z \leq Y \leq X, \)

defines a \( k \)-Mackey functor \((\tilde{M}, \tilde{c}, \tilde{r}, \tilde{t})\) for \( X \).

**Proof.** It is possible to prove the theorem by directly verifying the axioms of Definition 2.1 through elementary (though lengthy) computations. The following, more structural proof had been suggested by the referee.

We will first define a Mackey functor \( N \) in terms of \( X \)-sets, which will turn out to be just \( \tilde{M} \) as given in the statement. To define \( N \), let us first recall the Dress construction.

For any \( X \)-set \( S \) and any Mackey functor \( M \), there is a new Mackey functor \( MS \) defined by composing the bifunctor \( M \) with the endo-functor \( X \)-set \( \rightarrow \) \( X \)-set, \( U \mapsto U \times S \), so that \( MS(U) = M(U \times S) \). In our situation, applying the Dress construction with respect to the \( X \)-set \( S = X/G \) with \( G \) normal in \( X \), we obtain a Mackey functor \( MX/G \) such that \( MX/G(U) = M(U \times X/G) \) for an \( X \)-set \( U \).

Moreover, the \( k \)-module \( M(U \times X/G) \) is even a right \( kX \)-module, with the action of \( x \in X \) on \( m \in M(U \times X/G) \) given by

\[ m \cdot x = M(\pi_x)(m), \]

where \( \pi_x \) is the map of \( X \)-sets \( U \times X/G \rightarrow U \times X/G, (u, zG) \mapsto (u, zxG) \); in fact, if \( x_1, x_2 \in X \) then \( \pi_{x_2} \circ \pi_{x_1} = \pi_{x_1 x_2} \) and hence

\[ (m \cdot x_1) \cdot x_2 = M(\pi_{x_2})(M(\pi_{x_1})(m)) = M(\pi_{x_1 x_2})(m) = m \cdot (x_1 x_2). \]

Now, for any additive functor \( F : \text{Mod} \rightarrow k \text{-Mod} \), we claim that the composition \( F \circ MX/G : X \text{-set} \rightarrow k \text{-Mod} \) defines again a Mackey functor. In fact, we only need to verify the axioms (A1) and (A2) from Definition 2.2. But these follow immediately from the additivity of \( F \) and the corresponding properties of \( M \).

Now, let \( N \) be the Mackey functor obtained from this construction for \( F \) being the fixed point functor \( \mathbf{-X} : \text{Mod} \rightarrow k \text{-Mod} \), so that for an \( X \)-set \( U \) we have

\[ N(U) = M(U \times X/G)^X, \]

or in the case that \( U = X/Y \) for some subgroup \( Y \) of \( X \),

\[ N(X/Y) = M(X/Y \times X/G)^X. \]

Since \( G \) is normal in \( X \), there is an isomorphism of \( X \)-sets

\[ X/Y \times X/G \cong \bigcup_{t \in X/Y G} X/(Y \cap G) \cong X/(Y \cap G) \times X/Y G. \]

This implies that

\[ M(X/Y \times X/G) \cong \bigoplus_{t \in X/Y G} M(Y \cap G) \]
as a $k$-module, and the right $X$-action on this module is such that

$$M(X/Y \times X/G) \cong \text{ind}_{YG}^X M(Y \cap G)$$

as $kX$-modules, where the $k(YG)$-module structure on $M(Y \cap G)$ is obtained from the conjugation action of $Y$ via the group isomorphism

$$\Psi : YG/G \cong Y/(Y \cap G).$$

Now an element

$$\sum_{z \in [X/YG]} z \otimes_{YG} m_z \in \text{ind}_{YG}^X M(Y \cap G)$$

is fixed under $X$ if and only if

$$m_z = m \in M(Y \cap G)^{YG} = M(Y \cap G)^Y$$

for all $z \in [X/YG]$. It follows that

$$N(X/Y) = M(X/Y \times X/G)^X \cong M(Y \cap G)^Y,$$

so that the evaluation of the Mackey functor $N$ is indeed the same as for $\tilde{M}$ in the statement of the theorem. It remains to show that the structure maps obtained for $N$ are the ones stated in the theorem.

The conjugation map $\tilde{c}_{x,Y}$ on $N(Y)$ is obtained by restricting the map

$$(MX/G)_+(Yx,Y) : M(X/Y \times X/G) \to M(X/(Yx) \times X/G)$$

to the $X$-fixed points. With the above isomorphisms,

$$M(Y \cap G)^Y \xrightarrow{\sim} (\text{ind}_{YG}^X M(Y \cap G))^X \xrightarrow{\sim} (\text{ind}_{YG}^X M(Y \cap G))^X \xrightarrow{\sim} M(Y \cap G)^X,$$

$$m \mapsto \sum_{z \in [X/YG]} z \otimes m \mapsto \sum_{z \in [X/(Yx)G]} z \otimes c_{x,Y,G}(m) \mapsto c_{x,Y,G}(m).$$

Hence the conjugation map on $N(Y)$ is indeed $\tilde{c}_{x,Y} = c_{YG}$.

A very similar argument shows that the restrictions maps are of the form $\tilde{r}_Z^Y = r_\cap Z\cap G$ for $Z \leq Y$. Finally, if again $Z \leq Y$ and $m \in M(Z \cap G)^Y$, then the element

$$\sum_{z \in [X/ZG]} z \otimes_{ZG} m \in (\text{ind}_{ZG}^X M(Z \cap G))^X$$

is mapped under the transfer map $\tilde{t}_Z^Y$ to

$$\sum_{z \in [X/ZG]} z \otimes_{ZG} m \in (\text{ind}_{ZG}^X M(Z \cap G))^X.$$
\[
\sum_{u \in [X/YG]} \sum_{z \in [X/ZG], z \in uYG} z \otimes_{kYG} i_{Y \cap G}^{Y \cap G}(m) = \sum_{x \in [X/YG]} \sum_{y \in [X/ZG], y \in u^{-1}ZG} u \otimes_{kYG} c_{y, Y \cap G} \circ i_{Y \cap G}^{Y \cap G}(m),
\]

(1)

where \(\psi(u^{-1}z)\) is any coset representative of \(\Psi(u^{-1}ZG)\). It is easily checked that for any \(u \in X\) we have a bijection of sets

\[
\{zZG \in [X/ZG] \mid u^{-1}Z \in YG\} \to Y/Z(Y \cap G),
\]

\[
zZG \mapsto \Psi(u^{-1}ZG) \cdot Z(Y \cap G)
\]

with inverse given by

\[
uZ(Y \cap G) \mapsto u\nuZG,
\]

and hence the element (1) equals

\[
\sum_{u \in [X/YG]} \sum_{y \in [Y/Z(Y \cap G)]} u \otimes_{kYG} c_{y, Y \cap G} \circ i_{Y \cap G}^{Y \cap G}(m).
\]

It follows that the transfer map is given by

\[
i_{Y}^{Y}(m) = \sum_{y \in [Y/Z(Y \cap G)]} c_{y, Y \cap G} \circ i_{Y \cap G}^{Y \cap G}(m)
\]

as required. □

3. Extension morphisms

Let \(X\) be a finite group, \(G\) a normal subgroup of \(X\), and let \((M, c, r, t)\) denote a \(k\)-Mackey functor for \(X\). Define \((\tilde{M}, \tilde{c}, \tilde{r}, \tilde{t})\) as in Theorem 2.4.

3.1. Lemma. The maps

\[
d : M \to \tilde{M}, \quad d_Y = r_Y^{Y \cap G} \quad \text{for } Y \subseteq X
\]

define a morphism of Mackey functors.

Proof. This can be verified by a few straightforward calculations, but we give the following alternative proof, suggested by the referee, which also gives the interpretation of the lemma in terms of \(X\)-sets.

As in Section 9 of [6], we define the morphism of Mackey functors

\[
D : M \to M_{X/G}
\]
at some $X$-set $S$ by

$$DS : M(S) \to M_{X/G}(S) = M(S \times X/G), \quad m \mapsto M^*(p_S)(m),$$

where $p_S : S \times X/G \to S$ is the projection map onto the first component. In view of the definition of the right $kX$-action on $M_{X/G}(S)$, it is clear that the image of $DS$ is fixed under $X$, so that $D$ is in fact a morphism of Mackey functors $M \to \tilde{M}$. We will see that $D = d$ as stated in the lemma. In fact, the map

$$DY : M(Y) \to M_{X/G}(Y) \cong \bigoplus_{x \in X/YG} M(Y \cap G)$$

is just the restriction map $r^{Y}_{Y \cap G}$ in each component, and by the isomorphism

$$\left( \bigoplus_{x \in X/YG} M(Y \cap G) \right)^X \to M(Y \cap G)^Y$$

given in the proof of Theorem 2.4, we see that $DY(m) = \text{res}^{Y}_{Y \cap G}(m) = d_Y(m)$ for $m \in M(Y)$.

3.2. Definition. For $Y \subseteq X$ and $m \in \tilde{M}(Y) = M(Y \cap G)^Y$ we call $\hat{m} \in M(Y)$ an extension of $m$ to $M(Y)$ if $d_Y(\hat{m}) = m$. A morphism $e : \tilde{M} \to M$ of conjugation functors is called an extension morphism, if for all $Y \subseteq X$

$$d_Y \circ e_Y = \text{id}_{\tilde{M}(Y)}.$$  

If $e$ is even a morphism of restriction functors, we call $e$ an $r$-extension morphism.

3.3. Proposition. Assume that $[X : G] \in k^\times$. Then

$$e_Y : \tilde{M}(Y) \to M(Y), \quad m \mapsto \frac{1}{[Y : Y \cap G]} Y_{Y \cap G}(m)$$

defines an $r$-extension morphism. Even more, it defines a morphism of Mackey functors $e : \tilde{M} \to M$.

Proof. Again this can be shown through direct computations, but we give an alternative construction as in Theorem 2.4, suggested by the referee. Similarly to the construction of $\tilde{M}$, we consider $M_{X/G}$ as a right $kX$-module, and define a Mackey functor $M$ by composing now with the coinvariant functor $-X : \textbf{Mod}-kX \to k-\textbf{Mod}$. For any $kX$-module $V$, the module $V^X$ of $X$-coinvariants is defined to be the largest quotient of $V$ on which $X$ acts trivially. The composition $V^X \hookrightarrow V \to V^X$ of inclusion and natural epimorphism then
gives a natural map from invariants to coinvariants. Returning to our situation, we obtain a morphism of Mackey functors $g: \tilde{M} \to M$ from the natural map

$$\tilde{M}(S) = M(S \times X/G)^X \to M(S) = M(S \times X/G)_X$$

for any $X$-set $S$. Also, similarly to the construction of $D$ in the proof of Lemma 3.1, the projection $p_S: S \times X/G \to S$ induces a morphism of Mackey functors $f: M_{X/G} \to M$, defined for any $X$-set $S$ by

$$f_S: M_{X/G}(S) \to M(S), \quad m \mapsto M(p_S)(m).$$

Obviously $f_S$ factors through the coinvariants, and we get a morphism of Mackey functors $f: \tilde{M} \to M$. Composing the morphisms $g$ and $f$ we obtain $E: \tilde{M} \to M \to M$.

For a subgroup $Y$ of $X$, the map

$$f_Y: \bigoplus_{x \in X/YG} M(Y \cap G) \to M(Y)$$

is just the sum of the transfer maps $t^Y_{Y \cap G}$ from each component, so for $m \in \tilde{M}(Y)$ we have

$$E_Y(m) = \sum_{x \in X/YG} t^Y_{Y \cap G}(m) = [Y : Y \cap G] \cdot [X : YG] \cdot e_Y(m) = [X : G] \cdot e_Y(m),$$

hence $E = [X : G] e$. In particular, if $[X : G]$ is invertible in $k$ then $e$ is a morphism of Mackey functors. Moreover, applying the Mackey formula, we see that

$$d_Y(E_Y(m)) = \sum_{x \in X/YG} r^Y_{Y \cap G} \circ t^Y_{Y \cap G} = \sum_{x \in X/YG} \sum_{y \in Y \cap G} c_{x, Y \cap G}(m) = [X : G] \cdot m,$$

hence $d \circ e = d_{\tilde{M}(Y)}$. \hfill \Box

3.4. Remark. If $k = \mathbb{Z}$ then we can extend scalars to $\mathbb{Q}$, i.e., consider the $\mathbb{Q}$-Mackey functors $\mathbb{Q}M$ (by that we mean $\mathbb{Q} \otimes_{\mathbb{Z}} M$), and similarly $\mathbb{Q}\tilde{M}$. We can apply Proposition 3.3, but in general the image of $\tilde{M}$ under $e$ will not be in $M$ but only in $\mathbb{Q}M$.

Now we consider the Mackey functor $R = (R, c, \text{res}, \text{ind})$ that assigns to each $Y \subseteq X$ the ring $R(Y) = R_{\mathbb{C}}(Y)$ of complex characters of $Y$. Moreover, assume that $([G], [X : G]) = 1$, so that $G$ has a complement $S$ in $X$. In this situation, there is a well known result about extensions of characters:

3.5. Theorem [5, 13.3]. For any irreducible character $\chi$ of $G$ that is stable under $X$, there is a unique irreducible character $\hat{\chi}$ of $X$ such that $\hat{\chi}|_G = \chi$ and $\det(\hat{\chi}|_S) = 1$. Any other extension of $\chi$ is then of the form $\hat{\chi}\varphi$ for some $\varphi \in \text{Hom}(S, \mathbb{C}^\times)$. 
3.6. Corollary. If $\chi$ is an irreducible character of $G$ and $\text{Stab}_X(\chi) = Y < X$ then $\chi$ has an
extension $\hat{\chi}$ to an irreducible character of $Y$, and $\text{ind}_X^Y(\hat{\chi})$ is an extension of $\sum_{x \in [X/Y]} x \chi$
to a character of $X$. This induces an extension morphism

$$e_1 : \tilde{R} \to R.$$ 

However, $e_1$ does not commute with restrictions, as the following example shows:

3.7. Example. Let $G$ be the quaternion group of order 8, and let $X$ be a semidirect product
of $G$ and a cyclic group $S$ of order 3, such that $S$ acts non-trivially on $G$. Let $\chi$ be the irre-
ducible character of $G$ of degree 2, then $\chi \in \tilde{R}(X)$. Denote by $\phi$ and $\phi^2$ the irreducible
characters of $S$, then $\hat{\chi} = e_1(\chi)$, $\hat{\phi}$ and $\hat{\phi}^2$ are the different extensions of $\chi$ to $X$. One has

$$\text{res}_X^Y(\hat{\chi}_1) = 2 \cdot 1_S,$$

but

$$(e_1)_S(\text{res}_X^Y(\chi)) = \text{res}_G^Y(\chi) = \phi + \phi^2.$$ 

Hence in general $e_1 \circ \text{res} \neq \text{res} \circ e_1$.

Let $\Psi^n_X : R(X) \to R(X)$ denote the $n$th Adams operator, for any integer $n$, i.e.,

$$\Psi^n_X(\chi)(x) = x^n$$

for any $\chi \in R(X)$ and $x \in X$.

3.8. Proposition. There is a unique $r$-extension morphism

$$\text{ext} : \tilde{R} \to R$$

with the property that for all cyclic subgroups $C$ of $X$ and all $\phi \in \tilde{R}(C)$ one has $\text{ext}(\phi) = \phi \cdot 1$. Moreover, for any $Y \leq X$, the map $\text{ext}_Y : \tilde{R}(Y) \to R(Y)$ is given by

$$\text{ext}_Y(\chi) = \Psi^n_X(\chi_1),$$

where $\chi_1 \in R(Y)$ is any extension of $\chi \in \tilde{R}(Y)$, and $n$ is an integer such that

$$n \equiv 1 \mod |G| \quad \text{and} \quad n \equiv 0 \mod [X : G].$$

Proof. Let $\pi$ denote the set of prime divisors of $|G|$, then every $x \in X$ can be written as

$$x = x_{\pi} \cdot x_{\pi'},$$

with $x_{\pi} \in G$ and $x_{\pi'}$ being an element of $\pi'$-order.

Now let $Y \leq X$. If $\chi_1$ and $\chi_2$ are two extensions of $\chi \in \tilde{R}(Y)$, and if $y \in Y$ then

$$\Psi^n_Y(\chi_1)(y) = \chi_1(y^n) = \chi_1(y \pi) = \chi_2(y \pi) = \Psi^n_Y(\chi_2)(y).$$
Hence, ext is well defined by Eq. (2), and clearly ext commutes with conjugations. Moreover, for $Z \leq Y \leq X$, let $\chi_1 \in R(Y)$ be any extension of $\chi \in \tilde{R}(Y)$, and let $\psi_1 \in R(Z)$ be any extension of $\psi = \chi|_{Z \cap G}$, then one has for any $z \in Z$ that
\[
\text{res}_Z^Y (\text{ext}^Y (\chi))(z) = \text{ext}^Y (\chi)(z) = \Psi^n(\chi_1)(z) = \chi_1(z^n) = \chi(z \pi) = \chi|_{Z \cap G}(z \pi) \\
= \psi(z \pi) = \psi_1(z^n) = \Psi^n(\psi_1)(z) = \text{ext}_Z^Z (\text{res}_Z^Y (\chi))(z),
\]
hence ext is indeed an $r$-extension morphism. If $C$ is a cyclic subgroup of $X$, $\varphi \in \tilde{R}(C)$ and $y \in C$ then
\[
\text{ext}^C(\varphi)(y) = \varphi(y \pi) = (\varphi \cdot 1)(y \pi \cdot y \pi') = (\varphi \cdot 1)(y).
\]
Finally, to prove uniqueness, assume that $e_0$ is another morphism satisfying the stated conditions. Then, for any $Y \leq X$, any $\chi \in \tilde{R}(Y)$ and any $y \in Y$ one has
\[
e_0^Y(\chi)(y) = r_0^Y(e_0^Y(\chi))(y) = e_0^Y(\text{res}_Y^Y(\chi))(y) \\
= (\text{res}_{[Y]}^Y(\chi) \cdot 1)(y) = \text{ext}_Y^Y(\text{res}_Y^Y(\chi))(y) = \text{ext}_Y^Y(\chi)(y),
\]
hence $e_0 = \text{ext}$.

4. Canonical induction formulae

We will use the theory of canonical induction formulae, as described in [1]. In the following we will briefly recall the construction, but refer the reader to [1] for details.

Throughout, we will fix a finite group $X$, and $G$ will always denote a normal subgroup of $X$, as in the previous section.

4.1. Let $(M, c, r)$ be a $k$-restriction functor for the finite group $X$. For every $Y \leq X$ we define
\[
S_M(Y) := \bigoplus_{Z \leq Y} M(Z),
\]
and
\[
M_+(Y) := S_M(Y)/I(kY) \cdot S_M(Y),
\]
where $I(kY)$ denotes the augmentation ideal of $kY$, i.e., the kernel of the algebra homomorphism $\epsilon : kY \to k$ mapping $\sum \alpha_y y$ to $\sum \alpha_y$, where $y \in Y$ and $\alpha_y \in k$. We write
\[
[Z, m]_Y := m + I(kY) \cdot S_M(Y)
\]
for the elements in $M_+(Y)$, where $Z \leq Y$ and $m \in M(Z)$. Then $M_+$ is a Mackey functor with the following structure maps:
\[ c_{+, Y}: M_+ (Y) \to M_+ (Y), \quad [U, m]_Y \mapsto \left[ U^+ m \right]_Y, \]
\[ r_{+, Z}: M_+ (Y) \to M_+ (Z), \quad [U, m]_Y \mapsto \sum_{y \in [Z, Y/U]} \left[ Z \cap \mathbb{Y}U, r_{Z \cap U}(\mathbb{Y}) m \right]_Z, \]
\[ t_{+, Z}: M_+ (Z) \to M_+ (Y), \quad [V, m]_Z \mapsto [V, m]_Y. \]

In fact, the assignment \( M \mapsto M_+ \) is a functor from the category of restriction functors to the category of Mackey functors. It is left adjoint to the obvious forgetful functor from the category of Mackey functors to the category of restriction functors, see Proposition 5.2 in [1]. We can also describe \( M_+ \) in terms of \( X \)-sets by
\[
M_+ (S) = \left( \bigoplus_{Y \subseteq X} M(Y) \otimes kS^Y \right)_X
\]
for any \( X \)-set \( S \).

From now on, we will always consider the case \( k = \mathbb{Z} \), unless otherwise mentioned. Let \( (M, c, r, t) \) denote a \( \mathbb{Z} \)-Mackey functor for \( X \), and \( A \subseteq M \) a restriction subfunctor. We will make the following

**4.2. Assumptions.** (i) \( A(Y) \) is a free abelian group, for every \( Y \subseteq X \).

(ii) \( A \) has a stable basis \( (B(Y))_{Y \subseteq X} \), such that for all \( Z \subseteq Y \subseteq X \) and \( b \in B(Y) \) the element \( r_Y^Z (b) \) is a linear combination of the basis elements in \( B(Z) \) with non-negative coefficients.

**4.3.** Define \( \tilde{M} \) as in Theorem 2.4 with respect to the normal subgroup \( G \). It is easy to see that setting
\[
\tilde{A}(Y) := A(Y \cap G)^Y
\]
for every subgroup \( Y \) of \( X \) defines a restriction subfunctor \( \tilde{A} \) of \( \tilde{M} \). If \( M \) and \( A \) satisfy 4.2, then so do \( \tilde{M} \) and \( \tilde{A} \), since \( \tilde{A}(Y) \), \( Y \subseteq X \), then has the stable basis
\[
\tilde{B}(Y) = \left\{ \sum_{y \in [Y, Stab_Y(b)]}^{\gamma} b \mid b \in B(Y \cap G) \right\}.
\]

**4.4.** For \( Y \subseteq X \) define
\[
\mathcal{M}_Y := \{(Z, b) \mid Z \subseteq Y, \ b \in B(Z)\},
\]
then \( Y \) acts on \( \mathcal{M}_Y \) by \( \gamma(Z, b) = (\gamma(Z), \gamma(b)) \). Moreover,
\[
(Z_1, b_1) = y (Z_2, b_2) :\iff (Z_1, b_1) = (Z_2, b_2) \quad \text{for some } y \in Y.
\]
defines an equivalence relation on $\mathcal{M}_Y$. By $[Z, b]_Y$ we denote the equivalence class of $(Z, b)$, and by $\mathcal{M}_Y / Y$ the set of equivalence classes. Note that $A_+(Y)$ is then the free abelian group with basis $\mathcal{M}_Y / Y$. Similarly, $\tilde{A}_+(Y)$ is the free abelian group with basis $\mathcal{M}_Y / Y$, which is the set of $Y$-conjugacy classes of pairs $(Z, b')$ where $Z \leq Y$ and $b' \in \tilde{B}(Z)$. We denote the basis elements in $\tilde{A}_+$ again by $[Z, b']_Y$.

For $b \in \mathcal{B}(Z)$ and $W \leq Z \leq Y$, write $r_{W}^{Z}(b)$ as a linear combination of the basis elements in $\mathcal{B}(W)$, and call the (non-negative) coefficient $m_{(W, c)}^{(Z, b)}$ at $c \in \mathcal{B}(W)$ the multiplicity of $(W, c)$ in $(Z, b)$. Then

$$(W, c) \leq (Z, b) : \iff W \leq Z \text{ and } m_{(W, c)}^{(Z, b)} > 0$$

defines a partial ordering on $\mathcal{M}_Y$.

4.5. We define the induction morphism

$$b = b^M : M_+ \to M$$

by

$$b_Y : M_+(Y) \to M(Y), \quad [U, m]_Y \mapsto t_Y(U)(m).$$

Then $b$ is a morphism of Mackey functors (see [1, 3.1]). In fact, this is just the counit of the adjunction mentioned at the end of 4.1.

A canonical induction formula $a : M \to A_+$ is a morphism of restriction functors such that $b \circ a = \text{id}_M$.

4.6. Let $p : M \to A$ be a morphism of $\mathbb{Z}$-conjugation functors. Define

$$a_Y = a_Y^{(M, A, p)} : QM(Y) \to QA_+(Y),$$

$$a_Y(m) = \frac{1}{|Y|} \sum_{V < U < Y} |V| \mu(V, U) [L, r_Y(U)(p_U(r_Y(U)(m)))]_Y,$$

where $\mu$ denotes the Möbius function on the poset of subgroups of $X$. Then $a$ is a morphism of $Q$-restriction functors.

Conversely, given a morphism $a : M \to A_+$ of $\mathbb{Z}$-restriction functors, composition with the morphism of conjugation functors

$$\pi : A_+ \to A,$$

$$\pi_Y([U, m]_Y) := \begin{cases} m, & \text{if } Y = U, \\ 0, & \text{otherwise}, \end{cases}$$

gives a morphism $p = \pi \circ a : M \to A$ of $\mathbb{Z}$-conjugation functors, called the residue of $a$. In fact, this defines mutually inverse isomorphisms between the set of morphisms $a : M \to A_+$ of $Q$-restriction functors and the set of morphisms $p : M \to A$ of $Q$-conjugation functors (see [1, 6.1]).
4.7. Proposition (Boltje [1, 6.4]). The morphism $a$ defined in 4.6 is a canonical induction formula if and only if for all $Y \subseteq X$ and $m \in M(Y)$ one has $p_Y(m) - m \in \sum_{Z \leq Y} \mathfrak{i}_Z^Y(M(Z))$.

However, even if this condition is satisfied, the definition of the map $a$ contains denominators, so the image of $M(Y)$ under $a_Y$ will in general not be contained in $A_+(Y)$. We need an extra integrality condition:

4.8. Theorem (Boltje [1, 9.3]). Let $Y \subseteq X$, and assume that for a set $\pi$ of primes the following condition holds:

\[(\ast)_\pi \quad \text{For all } m \in M(Y) \text{ and all } T \triangleleft U \leq Y \text{ such that } U/T \text{ is a cyclic } \pi \text{-group, and all } b \in B(T) \text{ which are fixed under } U, \text{ the coefficients of the two elements } p_T(r_T^U(m)) \text{ and } r_T^U(p_U(r_Y^U(m))) \text{ in } A(T) \text{ with respect to the basis } B(T) \text{ coincide.}\]

Then

\[\text{im}(|Y|_{\pi'}, a_Y^{(M, A, p)}) \subseteq A_+(Y).\]

In particular, if condition $(\ast)_\pi$ is satisfied for all $Y \subseteq X$ and the set $\pi$ of all primes, then $a$ defines an integral canonical induction formula.

Now let us assume we are given a canonical induction formula $a : M \to A_+$ with residue $p : M \to A$ satisfying the integrality condition $(\ast)_\pi$ of Theorem 4.8 for the set $\pi$ of all primes. Under suitable conditions, we will construct an integral canonical induction formula $\tilde{a} : \tilde{M} \to \tilde{A}_+$ with residue $\tilde{p}$ given by

$$\tilde{p} : \tilde{M} \to \tilde{A}, \quad \tilde{p}_Y = p_{Y \cap G} \quad \text{for } Y \leq X.$$

As in 4.6, we define $\tilde{a} : \tilde{M} \to \tilde{A}_+$ by

$$\tilde{a}_Y(m) = \frac{1}{|Y|} \sum_{V \prec U \preceq Y} |V| \mu(Y, U) \left[ L, \mathfrak{i}_V^U \left( \tilde{p}_U (r_Y^U(m)) \right) \right]_Y, \quad (3)$$

for $Y \leq X$ and $m \in \tilde{M}(Y)$.

4.9. Theorem. Let $(M, c, r, t)$ be a $\mathbb{Z}$-Mackey functor for $X$, $A \subseteq M$ a subfunctor satisfying the assumptions in 4.2, and let $p : M \to A$ be a morphism of conjugation functors, such that condition $(\ast)_\pi$ from Theorem 4.8 is satisfied for the set $\pi$ of all primes. Moreover, let $G$ be a normal subgroup of $X$ such that the order of $G$ and the index in $X$ are relatively prime. Define $\tilde{M}$ as in Theorem 2.4. Then $(\ast)_\pi$ holds also for the Mackey functor $\tilde{M}$, its subfunctor $\tilde{A}$ and the morphism $\tilde{p}$, and the morphism $\tilde{a}$, defined by Eq. (3), is an integral canonical induction formula.
Proof. Clearly $\tilde{p}$ is a morphism of conjugation functors, hence defining $\tilde{a}$ by 4.6 certainly yields a morphism of restriction functors $\mathbb{Q}M \to \mathbb{Q}A$. Now, according to Proposition 4.7, for $\tilde{a}$ to be a canonical induction formula, we need to show that for any $Y \leq X$ and $m \in \mathbb{Q}M(Y)$, we have

$$m - \tilde{p}_Y(m) \in \sum_{Z \leq Y} \tilde{r}_Z^Y(\mathbb{Q}M(Z)).$$

Since $a$ is a canonical induction formula, again by 4.7, we know that

$$m - \tilde{p}_Y(m) = m - p_{Y \cap G}(m) = \sum_{H < Y \cap G} \tilde{r}_H^{Y \cap G}(m_H)$$

for some elements $m_H \in \mathbb{Q}M(H)$, thus

$$\sum_{y \in [Y : Y \cap G]} \chi(m - \tilde{p}_Y(m)) = \sum_{y \in [Y : Y \cap G]} \sum_{H < Y \cap G} \tilde{r}_H^{Y \cap G}(m_H) = \sum_{H < Y \cap G} \sum_{y \in [Y : (Y \cap G)H]} \tilde{r}_H^Y(m_H).$$

Since $m$ is stable under $Y$, and since $\tilde{p}$ is a morphism of conjugation functors, also $m - \tilde{p}_Y(m)$ is stable under $Y$, so we obtain

$$[Y : Y \cap G](m - \tilde{p}_Y(m)) \in \sum_{H < Y \cap G} \tilde{r}_H^Y(\mathbb{Q}M(H))$$

and dividing by $[Y : Y \cap G]$ shows that $\tilde{a}$ is indeed a canonical induction formula.

To show integrality, we have to verify condition $(*)_π$ from Theorem 4.8 for the set $π$ of all primes. So let $T \triangleleft U \leq Y$ such that $U / T$ is cyclic, let $b \in \mathcal{B}(T)^U$, and let $m \in \mathbb{Q}M(Y)$ be arbitrary. We need to show that the coefficients of the two elements $\tilde{p}_T(\tilde{r}_T^U(m))$ and $\tilde{r}_T^U(\tilde{p}_U(\tilde{r}_U^T(m)))$ in $\mathbb{Q}A(T)$ at the basis element $b$ coincide. We can write

$$b = \sum_{t \in T/S} t b_1$$

for some $b_1 \in \mathcal{B}(T \cap G)$, where $S = \text{Stab}_T(b_1)$.

First we claim that $b_1$ is stable under $U \cap G$. Set $V = \text{Stab}_{U \cap G}(b_1)$, then clearly the element

$$b' = \sum_{u \in [U \cap G : V]} u b_1$$

is stable under $U \cap G$, and so is $b$ by assumption. Comparing the definition of $b'$ and the description of $b$ in (4), both are linear combinations of basis elements in $\mathcal{B}(T \cap G)$, so the summands in $b'$ must all appear in $b$, and $b$ must be a sum of conjugates of $b'$. However, the
sum for \(b\) has \([T : S]\) terms, and the sum for \(b'\) has \([U \cap G : V]\) terms, and these numbers are relatively prime, by hypothesis. Thus \(V = U \cap G\).

Now,

\[
\tilde{p}_T(r^T_Y(m)) = p_{T \cap G}(r^Y_{T \cap G}(m))
\]

and

\[
r^Y_U(\tilde{p}_U(r^T_Y(m))) = r^U_{T \cap G}(p_{U \cap G}(r^Y_{U \cap G}(m))).
\]

Moreover, \(T \cap G \triangleleft U \cap G\) and \(U \cap G / T \cap G\) is cyclic, hence the coefficients of the two elements at \(b_1 \in B(T \cap G)U \cap G\) coincide, since condition \((*)_\pi\) holds for \(p\). However, these are the same coefficients as at \(b\), so \((*)_\pi\) holds also for \(\tilde{p}\). \(\square\)

4.10. Remark. We just mention that there is an explicit formula for \(a\) without denominators, but refer the interested reader to [1, formula (9.5a)]. Then it is easy to write down a similar formula for \(\tilde{a}\).

5. Extensions for Mackey functors

Throughout this section, we assume that \(X\) is a finite group and \(G\) a normal subgroup of \(X\) such that \((|G|, [X : G]) = 1\). Let \((M, c, r, t)\) be a \(\mathbb{Z}\)-Mackey functor for \(X\), let \(A \subseteq M\) be a subfunctor, and let \(p : M \to A\) be a morphism of conjugation functors satisfying the hypothesis of Proposition 4.7, and also satisfying condition \((*)_\pi\) from Theorem 4.8 for the set \(\pi\) of all primes, so that \(p\) corresponds to an integral canonical induction formula \(a : M \to A_+\). Let \(\tilde{M}\) and its subfunctor \(\tilde{A}\) be defined as in Theorem 2.4.

Our goal is to generalize the result of Proposition 3.8, so we want to give sufficient conditions for the existence of an \(r\)-extension morphism

\[
\text{ext} : \tilde{M} \to M.
\]

Our first result allows us to reduce the question to the subfunctor \(A\):

5.1. Proposition. If there exists a morphism \(\text{ext}_+ : \tilde{A} \to M\) of restriction functors such that

\[
d_Y(\text{ext}_+^Y(m)) = m
\]

for all \(m \in \tilde{A}(Y)\) and all \(Y \leq X\), then

\[
\text{ext} = b \circ \text{ext}_+ \circ \tilde{a}
\]

defines an \(r\)-extension morphism \(\tilde{M} \to M\), where

\[
\text{ext}_+^Y([Z, m])_Y := [Z, \text{ext}_+^Y(m)]_Y
\]

for \([Z, m])_Y \in \tilde{A}_+(Y)\) and \(Y \leq X\).
Proof. Clearly everything commutes with conjugations, and for $Z, U \subseteq Y \subseteq X$, and $m \in \tilde{A}(Z)$ one has

$$
\text{ext}^Y_U (r_Y^U ([Z, m]_Y)) = \text{ext}^Y_U \left( \sum_{x \in [U \setminus Y]_Z} [U \cap x \cdot Z, (r^Z_{U \cap x \cdot Z} (m))]_U \right)
$$

$$
= \sum_{x \in [U \setminus Y]_Z} [U \cap x \cdot Z, \text{ext}^U_{U \cap x \cdot Z} (r^Z_{U \cap x \cdot Z} (m))]_U
$$

$$
= \sum_{x \in [U \setminus Y]_Z} [U \cap x \cdot Z, \text{ext}^Z_{U \cap x \cdot Z} (\text{ext}^U_Y (m))]_U
$$

$$
= r_Y^U ([Z, \text{ext}^U_Y (m)]_Y) = r_Y^U (\text{ext}^Y_U ([Z, m]_Y)).
$$

Thus ext is a morphism of restriction functors, and

$$
d_Y \circ \text{ext}^Y_U = r_Y^{Y \cap G} \circ b_Y \circ \text{ext}^Y_U \circ \tilde{a}_Y = b_Y^{Y \cap G} \circ \text{ext}^Y_{U \cap Y \cap G} \circ \tilde{a}_Y^{Y \cap G} \circ r_Y^{Y \cap G}
$$

$$
= b_Y^{Y \cap G} \circ \tilde{a}_Y^{Y \cap G} = b_Y^{Y \cap G} \circ a_Y^{Y \cap G} = \text{id}_{M(Y \cap G)}. \quad \Box
$$

We will have to assume that elements of the subfunctor $\tilde{A}$ of $\tilde{M}$ have extensions, more precisely, the need the following condition

5.2. For every $Y \subseteq X$ and every basis element $b \in B(Y \cap G)$ there exists a (distinguished) extension $b_1 \in A(\text{Stab}_Y (b))$, such that for any $x \in X$, the element $^x b_1$ coincides with the distinguished extension of the basis element $^x b$ in $B(^x Y \cap G)$.

Additionally, we need an analog to the Adams operators, that is:

5.3. There is a morphism $\Psi : M \to M$ of restriction functors, such that $d_Y (\Psi (m)) = m$ for all $Y \subseteq X$ and $m \in \tilde{A}(Y)$, and such that we have $\Psi (m_1) = \Psi (m_2)$ for any two extensions $m_1, m_2 \in M(Y)$ of $m$.

5.4. Theorem. Suppose that conditions 5.2 and 5.3 are satisfied. Then there exists an $r$-extension morphism $\text{ext} = \text{ext}^{M, \tilde{A}} : \tilde{M} \to M$.

Proof. Let $Z \subseteq Y \subseteq X$. Every basis element in $\tilde{B}(Z)$ is of the form $b' = \sum_{z \in Z} z \cdot b$, where $b \in B(Z \cap G)$ and $S = \text{Stab}_Z (b)$. Let $b_1 \in A(S)$ be the distinguished extension of $b$ to $S$, then $\hat{b} = \iota^Z_S (b_1)$ is an extension of $b'$ to $Y$. Now define

$$
\text{ext}^Y_U : \tilde{A}(Y) \to M_* (Y),
$$

$$
\text{ext}^Y_U ([Z, b']_Y) = [Z, \Psi (\hat{b})]_Y.
$$

Then clearly $\text{ext}^Y_U$ commutes with conjugations, and
\[ r_+^Y_U(\text{ext}^X_U([Z, b'], Y)) = r_+^Y_U([Z, \Psi(\hat{b})]_Y) \]

\[ = \sum_{x \in [U \setminus Y/Z]} [x \cap U, r_{xU}^Z \cdot \Psi(\hat{b})]_U \]

\[ = \sum_{x \in [U \setminus Y/Z]} [x \cap U, \Psi(r_{xU}^Z(\hat{b}))]_U. \]

But \( r_{xU}^Z(\hat{b}) \) is an extension of \( r_{xU}^Z \cdot \hat{b} \), hence the above equals

\[ \sum_{x \in [U \setminus Y/Z]} \text{ext}^U_U([x \cap U, r_{xU}^Z \cdot \hat{b}]_U) \]

\[ = \text{ext}^U_U\left( \sum_{x \in [U \setminus Y/Z]} [x \cap U, r_{xU}^Z \cdot \hat{b}]_U \right) \]

\[ = \text{ext}^U_U\left( \sum_{x \in [U \setminus Y/Z]} [x \cap U, r_{xU}^Z(\hat{b})]_U \right) \]

\[ = \text{ext}^U_U\left( r_+^Y_U([Z, b']_Y) \right). \]

Therefore \( \text{ext} := b \circ \text{ext}_+ \circ a \) is an \( r \)-extension morphism by Proposition 5.1.

5.5. Remark. If \( M = R \) is the character ring over the complex numbers with the usual conjugation, restriction and induction maps, one can easily give another proof of Proposition 3.8, using the canonical induction formula from [1]. Here the subfunctor \( A = R_{ab} \), where for \( Y \leq X \), we denote by \( R_{ab}(Y) \) the free abelian group with basis \( \hat{Y} = \text{Hom}(\hat{Y}, \mathbb{C}^\times) \) the group of linear characters of \( Y \).

As an application, we will show that there is an \( r \)-extension morphism for the trivial source ring. Let \( F \) be a field of prime characteristic \( p \), and let \( X \) be a finite group. Recall that an \( FX \)-module \( V \) is called a trivial source module if \( V \) is a direct summand of a permutation \( FX \)-module. The images of the trivial source \( FX \)-modules in the Green ring of \( FX \) form a subring \( T_F(X) \), called the trivial source ring (of \( X \) over \( F \)). Clearly \( T_F \) is a Mackey functor with the usual conjugation, restriction and induction maps. A canonical induction formula for \( T_F \) has been given by Boltje in [3]. Here the subfunctor \( A \) is \( R_{ab}^F \), where \( R_{ab}^F(Y) \) is the free abelian group with basis \( \hat{Y} = \text{Hom}(\hat{Y}, \mathbb{C}^\times) \). Of course we identify \( \psi \in \hat{Y}(F) \) with the isomorphism class of the rank one \( FY \)-module \( F\psi \), on which \( y \in Y \) acts by multiplication with \( \psi(y) \).

Now condition 5.2 just requires that every \( \psi \in \hat{Y} \cap \hat{G}(F) \) can be extended to some \( \hat{\psi} \in \hat{Y}(F) \) in a distinguished way. In fact, we can always extend \( \psi \) such that the restriction of the extension to a complement of \( Y \cap G \) in \( Y \) is trivial. This obviously satisfies the conditions in 5.2.
For the morphism $\Psi$ in 5.3 we will use the Adams operators for the trivial source ring as defined by Boltje in [2]. We again choose $n$ to be an integer such that
\[ n \equiv 1 \mod |G| \quad \text{and} \quad n \equiv 0 \mod [X : G]. \]

By [2, Proposition 3.3], we have
\[ \Psi^{|G|}_n([F^G]) = [F^{G_n}], \]

hence condition 5.3 is clearly satisfied. We obtain

5.6. Proposition. There exists an $r$-extension morphism $\text{ext} : \tilde{T}_F \to T_F$.

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