Abstract

For a singular random matrix $X$, we find the Jacobians associated to the following decompositions: $QR$, Polar, Singular Value (SVD), $L'U$, $L'DM$ and modified $QR$ ($QDR$). Similarly, for the cross-product matrix $S = XX'$ we find the Jacobians of the Spectral, Cholesky's, $L'DL$ and symmetric nonnegative definite square root decompositions.

1. Introduction

During the past decades, several problems on the distribution theory for random matrices have found solutions. For example, the noncentral distributions were found using zonal polynomials or the hypergeometric function with matrix argument, [15,17]. Double noncentral distributions and distributions associated with eigenvalues of some specific matrices, were solved through the application of a generalization of zonal polynomials called invariant polynomials with matrix
A problem that has not been solved completely is the one related to the distribution of random singular matrices, which are not unusual to find in practical and theoretical problems. It is well known for example that a given sample matrix $Y \in \mathbb{R}^{N \times m}$, with $N$ subjects and $m$ variables, is singular when there is some linear dependence among variables or subjects. This case is usually solved by eliminating individuals or variables, accordingly. The solutions obtained this way were forced because the theory was not developed enough so as to deal with singular matrices. As a matter of fact, the distributions of such $Y$'s do not exist with respect to the Lebesgue measure in $\mathbb{R}^{Nm}$. Recently, some distributions for singular random matrices have been established; see [6–9,28,30]. As it can be noticed from these references, the main problem to determine the distributions for random matrices has been the search for a new basis and the corresponding coordinates for the rows and the columns of $Y$, since as a function of the new coordinates it is possible to define the measure for which the density function of $Y$ will exist. To do this, it is necessary to give a factorization of $Y$ (which is not unique) and to calculate the corresponding jacobian (which, might be not unique, either); not an easy task, due to the natural complications of working with this kind of distributions; see [1, Section 19] and [27, Chapter 5].

Formally, if $Y$ has a distribution with respect to the Lebesgue measure, and $\mathcal{K}$ and $\mathcal{N}$, are two given subspaces, what we do when we factorize $Y$ is to rewrite $Y$ as a product of at least two new matrices, e.g. $Y = KN$, such that $K \in \mathcal{K}$ and $N \in \mathcal{N}$. The main problem is then to find the image of the Lebesgue measure $(dY)$ defined on $\mathbb{R}^{Nm}$ under the mapping $(K \in \mathcal{K}) \times (N \in \mathcal{N})$. In other words, we have to find the jacobian of the transformation or equivalently, the volume element. To work out this problem, we can find different approaches: taking derivatives element by element, [5,26,29]; calculating the Gram determinant on Riemannian manifolds, which is the square of the Jacobian, [2]; making use of matrix differential calculus taking into account the linear structures of the transformations, [21]; or using the exterior product of the differential forms, [16] and [23, Chapter 2]. This last method has proved to be a very powerful technique when we are dealing with the factorization of singular random matrices and therefore we will use it here, [9,30].

We extend in this work the above ideas and introduce some new ones, to deal with the case when $Y$ has a distribution with respect to the Hausdorff measure; that is, when $Y$ is a singular random matrix. In particular, we extend to singular matrices the Jacobians associated with the $QR$, $SV$ and Polar decompositions; also, for singular and nonsingular matrices, we obtain the Jacobians associated with the modified $QR$, called $(QDR)$, the $L'U$ and $L'DM$ decompositions, as well as some other decompositions closely related to these, namely: the spectral, Cholesky's, $L'DL$ and symmetric positive square root decompositions, and some of their modifications, [6].

2. New Jacobians

We consider the Stiefel manifold whose elements are $N \times m$ ($N \geq m$) matrices $H_1$ (semiorthogonal matrices) such that $H_1'H_1 = I_m$, the $m \times m$
identity matrix. For \( N = m \), the Stiefel manifold is the group of orthogonal matrices.

The invariant measure on a Stiefel manifold is given by the differential form

\[
(H_1' dH_1) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{N} h_{ij}' dh_{ij}
\]

written in terms of the exterior product \( (\wedge) \), where we choose an \( N \times (N - m) \) matrix \( H_2 \) such that \( H \) is an \( N \times N \) orthogonal matrix, with \( H = (H_1; H_2) \) and \( dh \) is an \( N \times 1 \) vector of differentials, see \([11,16,22]\).

Some other authors have proposed alternative expressions to that given in (1) in terms of a standard differential denoted by \( g_{N,m}(H_1) \) and defined as

\[
(H_1' dH_1) \equiv J(H_1' dH_1 \to dH_1) dH_1 = g_{N,m}(H_1) dH_1,
\]

where \( J(\cdot) \) denotes the jacobian. An explicit form for \( g_{N,m}(H_1) \) is given in \([29, \text{Problem 1.33, p.38}]\).

Observe that, if \( X \) is a \( N \times m \) matrix with rank \( q \leq \min(N,m) \), with \( q \) distinct singular values, we can write \( X \) as

\[
X_1 = \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{11}
\end{pmatrix}
\]

in such a way that \( r(X_{11}) = q \). This is equivalent to the right product of the matrix \( X \) with a permutation matrix \( \Pi \), that is \( X_1 = X \Pi \). Note that the exterior product of the elements from the differential matrix \( dX \) are not affected by the fact that we multiply \( X \) (from right or left) by a permutation matrix; that is, \((dX_1) = (d(X \Pi)) = (dX)\), since \( \Pi \) is an orthogonal matrix. We use this fact through the different factorizations proposed in this section, i.e. the \( X \) matrix for which the factorization is sought, is assumed to be pre- or post- (or both) multiplied by the corresponding permutation matrix, \( \Pi \). Then, without loss of generality, \((dX)\) will be defined as the exterior product for the differentials \( dx_{ij} \), such that \( x_{ij} \) are functionally independent. It is important to note that we will have \( Nq + mq - q^2 \) functionally independent elements in the matrix \( X \), corresponding to the elements of \( X_{11}, X_{12} \) and \( X_{21} \). Explicitly,

\[
(dX) \equiv (dX_{11}) \wedge (dX_{12}) \wedge (dX_{21}) = \bigwedge_{i=1}^{N} \bigwedge_{j=1}^{q} dx_{ij} \bigwedge_{i=1}^{q} \bigwedge_{j=q+1}^{m} dx_{ij}.
\]

Similarly, given an \( m \times m \) positive semidefinite real matrix \( S \) of rank \( q \), with \( q \) distinct eigenvalues, we define \((dS)\) as

\[
(dS) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=1}^{m} ds_{ij}.
\]
Note that $S$ can be written as

$$S \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

with $r(S_{11}) = q$ (6)

and, the number of functionally independent elements in $S$ are, $mq - q(q - 1)/2$ corresponding to the functionally independent elements of $S_{12}$ and $S_{11}$. Recall that $S_{11}$ is a $q \times q$ positive definite real matrix, in such a way that $S_{11}$ has $q(q + 1)/2$ functionally independent elements, therefore, (5) can be written as

$$(dS) \equiv (dS_{11}) \wedge (dS_{12}).$$

We shall include here some results concerning the jacobians for the singular value and spectral decompositions, as the results are going to be applied in the sequel.

The explicit form for measures (4) and (5) were used by Srivastava [28] to define the measures $(dX)$ and $(dS)$, respectively. Given these measures, Srivastava [28] found the jacobians of certain transformations of $X$ and $S$, like the $QR$, Cholesky and spectral decompositions, among others. These results are useful to study the singular Wishart and beta distributions.

Srivastava [28] considers the canonical basis for the subspace of $N \times m$ random matrices of rank $q$, with $q$ distinct singular values and their corresponding coordinates $X_{11}, X_{12}, X_{21}$ to define the measure $(dX)$. In a similar way, he takes the usual basis of the $(mq - q(q - 1)/2)$-dimensional manifold of rank $q$ positive semidefinite $m \times m$ real symmetric matrices with $q$ distinct positive eigenvalues with their corresponding coordinates $S_{11}, S_{21}$, to define the measure $(dS)$. As it was mentioned in the Introduction, there is no unique way to define the measures $(dX)$ and $(dS)$, see first paragraph Section 3 in [19]. In the following, we propose different bases and coordinate systems to define, in an explicit way, the measure $(dX)$ in terms of the factorizations $QR, QDR, L'U, L'DU$ and $SVD$, as well as for the measure $(dS)$, in terms of the Cholesky, spectral, $L'DL$ and the nonnegative definitive square root decomposition of $S$.

Finally, note that the jacobian associated to a specific transformation on $X$ or $S$, does not have to be the same for all the measures $(dX)$ and $(dS)$, v.g compare Theorems 2.3 and 2 in [28,30] respectively, (see also the equation after Eq. (1.4) in [28]); in the same spirit, compare our Theorem 4 below, with Theorem 2.1 in [28].

The following result establishes the jacobian for the nonsingular part of the singular value decomposition of a matrix, which is defined in [25, p. 42] and [10, p. 58]. Observe that, when $N \geq m = q$, the Jacobian given in Lemma 1 has been studied by James [16, Section 8.1]; Roy [26, A.6.3, p. 183]; Le and Kendall [20, Section 4] and by [30, Theorem 5]. One important consequence from Lemma 1, is that $W_1 = V_1$, when $X = X'$ and we then get the nonsingular part of the spectral decomposition of $X$.

**Lemma 1** (Singular value decomposition, $SVD$). Let $X$ be an $N \times m$ matrix of rank $q \leq \min(N, m)$, with $q$ distinct singular values. Then there exist an $N \times q$ semiorthogonal matrix $V_1$, an $m \times q$ semiorthogonal matrix $W_1$ and a diagonal
matrix $D = \text{diag}(D_{11}, \ldots, D_{qq})$, with $D_{11} > \cdots > D_{qq} > 0$, such that $X = V_1DW_1^t$, the nonsingular part of the SVD, and

\[(dX) = 2^{-q}|D|^{N-m-2q} \prod_{i<j}^{q} (D_{ii}^2 - D_{jj}^2)(dD)(V_1'dV_1)(W_1'dW_1)\]

where $(dD) \equiv \bigwedge_{i=1}^{q} dD_{ii}$, and $(V_1'dV_1)$ and $(W_1'dW_1)$ are given by (1) or (2).

For a proof see [9].

Now, observe that the Jacobian in Lemma 2(1) is a particular case of Lemma 1, considering the symmetry of $S$, [8]. This Jacobian was established by Uhlig [30]. When $m = q$, the Jacobian has been studied by James [16, Section 8.2]; James [18, Eq. (93)] (when $S$ is Hermitian); Srivastava and Khatri [29, p. 31] and by Muirhead [23, pp. 104–105]. Srivastava [28] propose an alternative expression for 2(1) based on the measure given in (5). Proof for Lemma 2 part(2) is given in [9].

**Lemma 2** (Spectral decomposition). Let $S$ be an $m \times m$ positive semidefinite real matrix of rank $q$, with $q$ distinct eigenvalues. Then, the nonsingular part of the spectral decomposition can be written as $S = W_1LW_1^t$, where $W_1$ is an $m \times q$ semiorthogonal matrix and $L = \text{diag}(L_{11}, \ldots, L_{qq})$, with $L_{11} > \cdots > L_{qq} > 0$. Also, let $X$ be as in Lemma 1, and write $X = V_1DW_1^t$ (SVD) and $S = X'X$. Then

1. \[(dS) = 2^{-q}|L|^{m-q} \prod_{i<j}^{q} (L_{ii} - L_{jj})(dL)(W_1'dW_1),\]
2. \[(dX) = 2^{-q}|L|^{(N-m-1)/2}(dS)(V_1'dV_1).\]

The following result, becomes very handy in establishing some other important results in this section. It provides us with the Jacobian associated with a quasi-triangular matrix when it is written as the product of a diagonal matrix and a quasi-triangular unit matrix. Its proof and the all the other proofs associated with the results given in the paper are provided in Appendix A.

**Theorem 1.** Let $J$ be a $q \times m$ upper quasi-triangular matrix with $r(J) = q$, that is, $J = (J_1; J_2)$ such that $J_1$ is a $q \times q$ upper triangular matrix with $j_{11} > \cdots > j_{qq} > 0$. Then $J = BG$ where $B = \text{diag}(b_{11}, \ldots, b_{qq})$, with $b_{11} > \cdots > b_{qq} > 0$, $G$ is a $q \times m$ upper quasi-triangular matrix with $g_{11} > \cdots > g_{qq} > 0$, and

1. \[(dJ) = \prod_{i=1}^{q} b_{ii}^{m-i} \prod_{i=1}^{q} g_{ii}(dB)(dG).\]
2. if $g_{ii} = 1$ for all $i = 1, \ldots, q$ then

\[(dJ) = \prod_{i=1}^{q} b_{ii}^{m-i}(dB)(dG),\]

where $(dJ) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{m} dj_{ij}$, $(dG) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{m} dg_{ij}$ and $(dB) \equiv \bigwedge_{i=1}^{q} db_{ii}$. 
For full rank square matrices $D$ and $G$, the Jacobian is given in Theorem 1(1) in [21, Theorem 810, p. 141] and it is derived using linear structures.

We present the jacobian for a nonfull rank rectangular matrix $X$, when we consider its decomposition as the product of two triangular matrices, that is, its $L'U$ decomposition.

**Theorem 2** ($L'U$ decomposition, Doolittle’s version). Let $X$ be an $N \times m$ matrix of rank $q \leq \min(N,m)$, with $q$ distinct singular values, and write $X = \Delta' Y$, where $\Delta$ is a $q \times N$ upper quasi-triangular matrix, with $\delta_{ii} = 1$, $i = 1, \ldots, q$ and $Y$ is a $q \times m$ upper quasi-triangular matrix with $v_{11} > \cdots > v_{qq} > 0$. Then,

\[ (dX) = \prod_{i=1}^{q} v_{ii}^{N-i} (dY)(d\Delta). \quad (9) \]

\[ (dX) = \prod_{i=1}^{q} v_{ii}^{N-i} \prod_{i=1}^{q} \delta_{ii}^{m-i+1} (dY)(d\Delta), \quad (10) \]

where $(d\Delta) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{m} d\delta_{jj}$, $(dY) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i}^{m} dY_{ij}$.

This theorem gives a variant of the decomposition $L'U$ known as Crout’s decomposition, see [13, p. 228], by letting $\Delta$ be a $q \times N$ upper quasi-triangular matrix with $\delta_{11} > \cdots > \delta_{qq} > 0$ and $Y$ a $q \times m$ upper quasi-triangular matrix, such that $v_{ii} = 1$ for all $i = 1, \ldots, q$.

**Corollary 1** ($L'U$ decomposition, Crout’s version). Let $X$ be an $N \times m$ matrix of rank $q \leq \min(N,m)$, with $q$ distinct singular values, such that $X = \Delta' Y$, where $\Delta$ is a $q \times N$ upper quasi-triangular matrix with $\delta_{11} > \cdots > \delta_{qq} > 0$ and $Y$ is a $q \times m$ upper quasi-triangular matrix, such that $v_{ii} = 1$ for all $i = 1, \ldots, q$. Then,

\[ (dX) = \prod_{i=1}^{q} \delta_{ii}^{m-i} (dY)(d\Delta). \quad (11) \]

\[ (dX) = \prod_{i=1}^{q} v_{ii}^{N-i+1} \prod_{i=1}^{q} \delta_{ii}^{m-i} (dY)(d\Delta), \quad (12) \]

where $(d\Delta) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{m} d\delta_{jj}$, $(dY) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i}^{m} dY_{ij}$.

**Proof.** The proof follows the steps of the one given in Theorem 2. □
Another variant of the $L’U$ decomposition is the so-called $L’DU$ decomposition. Theorem 3 provides the jacobian for this particular factorization.

**Theorem 3** ($L’DM$ decomposition). Let $X$ be an $N \times m$ matrix of rank $q \leq \min(N,m)$, with $q$ distinct singular values, such that $X = \Psi' \Pi \Xi$, where $\Psi$ is a $q \times N$ upper quasi-triangular matrix with $\psi_{ii} = 1$ for all $i = 1, \ldots, q$, $\Pi = \text{diag}(\pi_{11}, \ldots, \pi_{qq})$, $\pi_{11} > \cdots > \pi_{qq} > 0$ and $\Xi$ is a $q \times m$ upper quasi-triangular matrix with $\xi_{ii} = 1$ for all $i = 1, \ldots, q$. Then,

$$(dX) = \prod_{i=1}^{q} n_{ii}^{N+m-2i}(d\Psi)(d\Pi)(d\Xi).$$

A factorization that has been widely used in trying to establish the Wishart distribution as well as the distribution for rectangular coordinates, among many other applications, is the $QR$ decomposition. When $N \geq m = q$, this result is given in [26, A.6.1, p. 170], [29, Problem 1.33, p. 38]. In the same context, Muirhead [23, pp. 63–66] gives the proof under the same guidelines as the one given in [16, Section 8], for the $SVD$ case. Also, in the general case, an alternative expression is proposed by Srivastava [28], with respect to measure (4). Finally, Goodall and Maridia [12] establish, without proof, that the result is true when $q = \min(N,m)$.

**Theorem 4** ($QR$ decomposition). Let $X$ be an $N \times m$ matrix of rank $q \leq \min(N,m)$, with $q$ distinct singular values, then there exist an $N \times q$ semiorthogonal matrix $H_1$ and a $q \times m$ upper quasi-triangular matrix $T$ with $t_{ii} \geq 0$, $i = 1, 2, \ldots, q$ such that $X = H_1T$ and

$$(dX) = \prod_{i=1}^{q} n_{ii}^{N-I}(H_1' dH_1)(dT).$$

In Theorem 5 we establish a similar result for another well-known variant of the $QR$ decomposition called, modified $QR$ decomposition. We will see later that this particular decomposition is related to the $L’DL$ decomposition.

**Theorem 5** (Modified $QR$ decomposition ($QDR$)). Let $X$ be an $N \times m$ matrix of rank $q \leq \min(N,m)$, with $q$ distinct singular values, then there exist an $N \times q$ semiorthogonal matrix $H_1$, a diagonal matrix $N = \text{diag}(n_{11}, \ldots, n_{qq})$, with $n_{11} > \cdots > n_{qq} > 0$ and a $q \times m$ upper quasi-triangular matrix $\Omega$ with $\omega_{ii} = 1$, $i = 1, 2, \ldots, q$ such that $X = H_1N\Omega$, and

$$(dX) = 2^{-q} \prod_{i=1}^{q} n_{ii}^{N+m-2i}(H_1' dH_1)(dN)(d\Omega).$$

In the following we provide the jacobians associated to a positive semidefinite real $m \times m$ matrix $S$ of rank $q$, and their relationships with the factorizations given in Theorems 2–5. The matrix $S$ will be factorized in general as $S = X'X$, where $X$ is an $N \times m$ matrix of rank $q \leq \min(N,m)$, with $q$ distinct singular values.
The first factorization presented here is the well-known Cholesky decomposition. When \( S \) is a positive definite real matrix, that is, \( q = m \), the Jacobian has been given by Muirhead [23, p. 60] and Srivastava and Khatri [29, Problem 1.29, p. 38], among many others. We extend the result for positive semidefinite matrices in the following way.

Theorem 6 (Cholesky’s decomposition). Let \( S \) be an \( m \times m \) positive semidefinite real matrix of rank \( q \), with \( q \) distinct eigenvalues. Then \( S = T' T \), where \( T \) is a \( q \times m \) upper quasi-triangular matrix with \( t_{ii} > 0 \), \( i = 1, 2, \ldots, q \). Also, let \( X \) be an \( N \times m \) matrix of rank \( q \leq \min(N,m) \), with \( q \) distinct singular values, with \( X = H_1 T \) (QR Decomposition) and \( S = X'X = T' T \) such that

\[
S = \begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix}_{q \times q} \begin{pmatrix}
S_{11} & 0 \\
0 & S_{22}
\end{pmatrix}_{m-q \times m-q}
\]

with \( r(S_{11}) = q \).

Then,

1. \( (dS) = 2^q \prod_{i=1}^q t_{ii}^{m-i+1} (dT) \),
2. \( (dX) = 2^{-q} |S_{11}|^{(N-m-1)/2} (dS)(H_1'dH_1) \).

We now give the Jacobian for the \( L' D L \) decomposition of a positive semidefinite real random matrix. For the case where \( S \) is a positive definite matrix, the Jacobian can be found in [22, p. 94].

Theorem 7 (\( L' D L \) decomposition). Let \( S \) be an \( m \times m \) positive semidefinite real matrix of rank \( q \), with \( q \) distinct eigenvalues, and \( S = \Omega' \Omega \), where \( \Omega \) is a \( q \times m \) upper quasi-triangular matrix with \( \omega_{ii} = 1 \), \( i = 1, 2, \ldots, q \) and a diagonal matrix \( \Omega = \text{diag}(o_{11}, \ldots, o_{qq}) \), with \( o_{11} > \cdots > o_{qq} > 0 \). Write \( S = X'X = \Omega' \Omega X \), with \( X = H_1' N \Omega \) being an \( N \times m \) matrix of rank \( q \leq \min(N,m) \), with \( q \) distinct singular values. Then,

1. \( (dS) = \prod_{i=1}^q o_{ii}^{m-i} (d\Omega)(d\Omega) \),
2. \( (dX) = 2^{-q}|\Omega|^{(N-m-1)/2} (H_1'dH_1)(dS) \).

The Jacobian of symmetric nonnegative definite square root factorization for the case where \( S \) is a positive definite real matrix, i.e., \( q = m \), was studied by Olkin and Rubin [24], Henderson and Searle [14] and Cadet [2]. The following theorem extends the result for positive semidefinite real matrices.

Theorem 8 (Symmetric nonnegative definite square root). Let \( S \) and \( R \) be \( m \times m \) positive semidefinite real matrices of rank \( q \), with \( q \) distinct eigenvalues, such that \( S = R^2 \). Then,

\[
(dS) = 2^q |D|^{m-q+1} \prod_{i < j}^q (D_{ij} + D_{ji})(dR) = |D|^{m-q} \prod_{i < j}^q (D_{ij} + D_{ji})(dR),
\]

(14)
where $R = Q_1 D Q_1'$ is the spectral decomposition of $R$, $Q_1$ an $m \times q$ semiorthogonal matrix and $D = \text{diag}(D_{11}, \ldots, D_{qq})$ with $D_{11} > \cdots > D_{qq}$.

Finally, we give the jacobian for the polar decomposition and make some important remarks after we have established the result.

**Theorem 9** (Polar decomposition). Let $X$ be an $N \times m$ matrix of rank $q \leq \min(N, m)$, with $q$ distinct singular values, $N \geq m$, and write $X = P_1 R$, with $P_1$ an $N \times m$ semiorthogonal matrix, and $R$ an $m \times m$ positive semidefinite real matrix of rank $q$, with $q$ distinct eigenvalues. Also, let $S = X' X = R^2$ be the nonnegative definite square root of $S$. Then,

(1) \[ (dX) = \frac{\text{Vol}([P]^{N-q} \prod_{i < j} (D_{ii} + D_{jj})(dR)(P_1' dP_1), \]

(2) \[ (dX) = \frac{2^{-q} \text{Vol}([P]^{N-q} \prod_{i < j} L)^{(N-m-1)/2} (dS)(P_1' dP_1), \]

where $L = D^2$ and \( \text{Vol}(\mathcal{J}_{m-q,N-q}) = \int_{K_1 \in \mathcal{J}_{m-q,N-q}} (K_1' dK_1) = \frac{2^{(m-q)(m-q)} \Gamma_{m-q}(N-q)/2}{\Gamma_{m-q} \Gamma_{N-q}/2}. \]

**Remark 1.**

(1) The Jacobian in Theorem 9(1) was studied by Cadet [2] when $q = m$, by computing Grams' determinant on a Riemannian manifold. In Cadet's notation, $ds$ denotes the Riemannian measure on the Stiefel manifold (the invariant measure on the Stiefel manifold), which has the normalizing constant

\[ \int_{\mathcal{J}_{m,q}} ds = \frac{2^{(p+3)/2} p^{m/2}}{\Gamma_q \left[ \frac{1}{2} m \right]}, \]

which differs from the normalizing constant proposed by James [16], for $(P_1' dP_1)$, see also [29, p. 75] and Muirhead [23, p. 70]. But it is known that the invariant measure on a Stiefel manifold is unique, in the sense that if there are two invariant measures on a Stiefel manifold, one is a scalar multiple of the other, (see [16] and [11, p. 43]). In particular

\[ ds = 2^{(p-1)/4} (P_1' dP_1). \] (15)

From expression (15) the Jacobian in Theorem 9(2) is found, when $q = m$, with respect to the measure $(P_1' dP_1)$, any of the jacobians studied here may be expressed as a function of the $ds$ measure proposed by Cadet, considering the different normalizing constants see [2, Remark (4)]. The result given in Theorem 9(2), and also the assumption of $q = m$, was proposed (without proof) by Hertz [15].

(2) On the other hand, observe that for any of the factorizations $X = KN$, with $X$ some $N \times m$ matrix of rank $q \leq \min(N, m)$, with $q$ distinct singular values, the number of functionally independent elements in $X \ (Nq + mq - q^2)$, must be equal to the number of functionally independent elements in $K$, plus the number of functionally independent elements in $N$. For example, in the $QR$
decomposition, \( X = H_1 T \), the number of functionally independent elements in the \( n \times q \) semiorthogonal matrix \( H_1 \), is \( Nq - q(q + 1)/2 \) (see [23, p. 67]) and the number of functionally independent elements in the \( q \times m \) upper quasi-triangular matrix \( T \), is \( mq - q(q - 1)/2 \) with the sum being \( Nq + mq - q^2 \) the number of functionally independent elements. On a first look, it would seem like this rule does not hold for the Polar decomposition of a singular matrix, since, if \( X = P_1 R \), with \( P_1 \) an \( N \times q \) semiorthogonal matrix with \( Nm - m(m + 1)/2 \) elements functionally independent and \( R \) an \( m \times m \) positive semidefinite real matrix of rank \( q \), with \( q \) distinct eigenvalues, and \( mq - q(q - 1)/2 \) elements functionally independent, we would have that the total sum equals \( Nm - m(m + 1)/2 + mq - q(q - 1)/2 \neq Nq + mq - q^2 \) elements functionally independent. This is due to the conditions on the dimensionality of the matrices \( P_1 \) and \( R \), in the definition of the Polar decomposition. Note, that the Polar decomposition of \( X \) can be written as

\[
X = P_1 R
\]

\[
= \begin{pmatrix}
 P_q & P_r
 \end{pmatrix}_{N \times q} \begin{pmatrix}
 R_1 & 0
 \end{pmatrix}_{q \times m}
\]

\[
= P_1 R_1 + P_2 R_2,
\]  

where \( P_1 = (P_q; P_r) \), \( R' = (R_1'; R_2') \) are such that \( R_1 \) contains the \( mq - q(q - 1)/2 \) functionally independent elements in \( R \); the \( m - q \) columns of \( P_r \) are arbitrary and they are functions of the columns of \( P_q \), that is, the functionally independent elements in the decomposition of \( X = P_1 R \), are contained on the first summand of (16). If we proceed as in the case of the \( QR \) decomposition, dropping the second summand on Eq. (16) (since \( T_2 = 0 \)), is easy to see that the Jacobian for the Polar decomposition will be proportional to \( (P_q'dP_q') \wedge (dR_1) \). However, since the Jacobian has to be a function of \( P_1 \) and \( R \), it is enough to write \( (P_q'dP_q') \), as a function of \( P_1 \), since by definition \( (dR) = (dR_1) \), see Eq. (6). In this way, we get the proportionality constants for Theorems 9(1) and 9(2). To see this, let \( P_2 \) and \( P_{N-q} \) such that \( P = (P_1; P_2) \) and \( P = (P_q; P_{N-q}) \) are \( N \times N \) orthogonal matrices, then by Lemma 9.5.3 in Muirhead [23, p. 397],

\[
(P'dP) = (P_1'dP_1) \wedge (M'dM),
\]

similarly,

\[
(P'dP) = (P_q'dP_q) \wedge (A'dA),
\]

where the \( (N - m) \times (N - m) \) matrix \( M \) and the \( (N - q) \times (N - q) \) matrix \( A \) are orthogonal matrices. Thus applying again Lemma 9.5.3 in [23, p. 397] to \( (A'dA) \) in Eq. (18) we have

\[
(P'dP) = (P_q'dP_q) \wedge (B_1'dB_1) \wedge (C'dC),
\]
with $B_1$ an $N - q \times m - q$ semiorthogonal matrix and $C$ an $(N - m) \times (N - m)$ orthogonal matrix. Now, equating (17) and (19), we get

\[(P_1' dP_1) \wedge (M' dM) = (P_q' dP_q) \wedge (B_1' dB_1) \wedge (C' dC).\] (20)

By the uniqueness of the Haar measure on the group of orthogonal $(N - m) \times (N - m)$ matrices, we have that $(M' dM) = (C' dC)$ therefore $(P_1' dP_1) = (P_q' dP_q) \wedge (B_1' dB_1)$, and the result is established.

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Appendix A. Proofs

We present here all the proofs for Theorems 1–9 given in Section 2.

A.1. Proof of Theorem 1

(1) Writing $J$ and $G$ by rows, $B$ through its diagonal elements and taking the product we get,

\[J = \begin{pmatrix} J_1' \\ \vdots \\ J_q' \end{pmatrix} = \begin{pmatrix} b_{11} G_1' \\ \vdots \\ b_{qq} G_q' \end{pmatrix}.\]

Therefore,

\[J_1 = b_{11} G_1 \quad \text{so} \quad dT_1 = db_{11} G_1 + b_{11} dG_1; \quad \text{and similarly,}\]

\[J_2 = b_{22} G_2 \quad \quad dT_2 = db_{22} G_2 + b_{22} dG_2 \]

\[\vdots \]

\[J_q = b_{qq} G_q \quad \quad dT_q = db_{qq} G_q + b_{qq} dG_q \]

taking the exterior product of the differential; recalling that $g_{ii}$ are fixed for all $i, i = 1, 2, \ldots, q$; and that the product of repeated differentials is zero, we get

\[(dJ_i) \equiv \bigwedge_{j=i}^{m} dt_{ij} = g_{ii} db_{ii} \wedge \ell_{ii}^{m-i}(dG_i)\]
with \((dG_i) \equiv \bigwedge_{j=i+1}^{m} dg_{ij}\). Finally,

\[
(dJ) \equiv \bigwedge_{i=1}^{q} \bigwedge_{j=i}^{m} dt_{ij} = \bigwedge_{i=1}^{q} (dJ_i) = \prod_{i=1}^{q} g_{ii} \prod_{i=1}^{q} b_{ii}^{m-i} \bigwedge_{i=1}^{q} db_{ii} \bigwedge_{i=1}^{q} (dG_i)
\]

(2) The proof follows immediately.

A.2. Proof of Theorem 2

We will only work the proof for part (2), since part (1) follows from (2) taking \(\delta_{ii} = 1\) for all \(i\). Let \(X\) and \(A\) be denoted by columns, \(X = (X_1X_2 \cdots X_q \cdots X_m), A = (A_1A_2 \cdots A_q)\). Write \(Y = (v_{ij})\). Then,

\[
\begin{align*}
X_1 &= v_{11}A_1 \\
X_2 &= v_{12}A_1 + v_{22}A_2 \\
X_3 &= v_{13}A_1 + v_{23}A_2 + v_{33}A_3 \\
& \quad \vdots \\
X_{q-1} &= v_{1q-1}A_1 + v_{2q-1}A_2 + v_{3q-1}A_3 + \cdots + v_{q-1q-1}A_{q-1} \\
X_q &= v_{1q}A_1 + v_{2q}A_2 + v_{3q}A_3 + \cdots + v_{qq}A_q \\
X_{q+1} &= v_{1q+1}A_1 + v_{2q+1}A_2 + v_{3q+1}A_3 + \cdots + v_{qq+1}A_q \\
& \quad \vdots \\
X_m &= v_{1m}A_1 + v_{2m}A_2 + v_{3m}A_3 + \cdots + v_{qm}A_q
\end{align*}
\]

taking differentials and then omitting those differentials that appeared previously, we get the following expressions:

\[
\begin{align*}
dX_1 &= dv_{11}A_1 + v_{11}dA_1 \\
dX_2 &= dv_{12}A_1 + dv_{22}A_2 + v_{22}dA_2 \\
dX_3 &= dv_{13}A_1 + dv_{23}A_2 + dv_{33}A_3 + v_{33}dA_3 \\
& \quad \vdots \\
dX_{q-1} &= dv_{1q-1}A_1 + dv_{2q-1}A_2 + dv_{3q-1}A_3 + \cdots + dv_{q-1q-1}A_{q-1} \\
& \quad + v_{q-1q-1}dA_{q-1} \\
dX_q &= dv_{1q}A_1 + dv_{2q}A_2 + dv_{3q}A_3 + \cdots + dv_{qq}A_q + v_{qq}dA_q \\
dX_{q+1} &= dv_{1q+1}A_1 + dv_{2q+1}A_2 + dv_{3q+1}A_3 + \cdots + dv_{qq+1}A_q \\
& \quad \vdots \\
dX_m &= dv_{1m}A_1 + dv_{2m}A_2 + dv_{3m}A_3 + \cdots + dv_{qm}A_q.
\end{align*}
\]

Taking exterior products of the differentials, recalling that the product of repeated differentials are zero and noticing that the differentials that appeared before do not
have to be taken into account again, we get,

\[
\begin{align*}
(dX_1) &= \delta_{11}d
v_{11} \wedge v_{11}^{N-1}d\Delta_1 \\
(dX_2) &= \delta_{12}d
v_{12} \wedge v_{22}^{N-2}d\Delta_2 \\
(dX_3) &= \delta_{13}d
v_{13} \wedge v_{22} \wedge v_{33}^{N-3}d\Delta_3 \\
& \quad \vdots \\
(dX_{q-1}) &= \delta_{1q-1}d
v_{1q-1} \wedge v_{2q-1} \wedge v_{3q-1}^{N-(q-1)} \wedge \cdots \wedge v_{q-1q-1}^{N-(q-1)}d\Delta_{q-1} \\
(dX_q) &= \delta_{1q}d
v_{1q} \wedge v_{2q} \wedge v_{3q} \wedge \cdots \wedge v_{qq}^{N-q}d\Delta_q \\
(dX_{q+1}) &= \delta_{1q+1}d
v_{1q+1} \wedge v_{2q+1} \wedge v_{3q+1} \wedge \cdots \wedge v_{qq+1}d\Delta_{qq} \\
& \quad \vdots \\
(dX_m) &= \delta_{1m}d
v_{1m} \wedge v_{2m} \wedge v_{3m} \wedge \cdots \wedge v_{qm}d\Delta_{qm}
\end{align*}
\]

with \(d\Delta_j = \bigwedge_{i=j+1}^{N} d\delta_{ij}\). Therefore,

\[
(dX) \equiv \bigwedge_{i=1}^{N} \bigwedge_{j=1}^{q} dx_{ij} \bigwedge_{i=1}^{q} \bigwedge_{j=q+1}^{m} dx_{ij}
\]

\[
= \prod_{i=1}^{q} u_{ii}^{m-i+1} \prod_{i=1}^{q} v_{ii}^{N-i} \bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{m} d\delta_{ij} \bigwedge_{i=1}^{q} \bigwedge_{j=i}^{m} dv_{ij}.
\]

A.3. Proof of Theorem 3

Write \(X = \Psi'\Pi\Xi = \Psi'U\) with \(U = \Pi\Xi\) and observe that \(U\) is a \(q \times N\) upper quasi-triangular matrix with \(u_{11} > \cdots > u_{qq} > 0\). Then, by Theorem 2,

\[
(dX) = \prod_{i=1}^{q} u_{ii}^{N-i}(dU)(d\Psi).
\] (A.1)

Now, \(U = \Pi\Xi\), with \(\Pi = \text{diag}(\pi_{11}, \ldots, \pi_{qq})\) and \(\Xi\) is a \(q \times m\) upper quasi-triangular matrix with \(\xi_{ii} = 1, i = 1, 2, \ldots, q\). Therefore by Theorem 1,

\[
(dU) = \prod_{i=1}^{q} \pi_{ii}^{m-i}(d\Pi)(d\Xi).
\] (A.2)

Note that \(u_{ii} = \pi_{ii}\), since \(\chi_{ii} = 1\) for all \(i\). Therefore, substituting (A.2) into (A.1), the result follows.

A.4. Proof of Theorem 4

Given that \(X = H_1T\) we have \(dX = dH_1T + H_1dT\). Let \(H = (H_1'; H_2)\) (\(H_2\) a function of \(H_1\)), such that \(H\) is an \(N \times N\) orthogonal matrix, then

\[
H'dX = \begin{pmatrix} H_1'dH_1T + dT \\
H_2'dH_1T \end{pmatrix}.
\]
Now, observe that $T$ can be written as $T = (T_1; T_2)$, where $T_1$ is a $q \times q$ upper triangular matrix with $t_{ii} > 0$ for all $i = 1, \ldots, q$. Thus the proof reduces to the one given in [23, pp. 64–66], observing that

$$H_1' dH_1 T = [H_1' dH_1 T_1; H_1' dH_1 T_2],$$

$$H_2' dH_1 T = [H_2' dH_1 T_1; H_2' dH_1 T_2]$$

and computing the exterior product, column by column, $[H_1' dH_1 T_2]$, and noticing that $[H_2' dH_1 T_2]$ does not contribute at all to the exterior product, since its elements appear in previous columns.

A.5. Proof of Theorem 5

(1) The proof is analogous to the proof of Theorem 2. Alternatively, the Jacobian may be computed via patterned matrices, (see [14]).

(2) Observe that we can write $T = (T_1; T_2)$ and

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} T_1' \\ T_2' \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_1' & T_2' \end{pmatrix},$$

thus, $|S_{11}| = |T_1' T_1| = |T_1|^2 = \prod_{i=1}^q t_{ii}^2$ and from Theorem 3.2(1), $(dT) = 2^{-q} \prod_{i=1}^q t_{ii}^{-(m-i+1)}(dS)$. Then substituting in (13), we obtain the result.

A.6. Proof of Theorem 6

Write $X = H_1 Z$ were $Z = N \Omega$ and observe that $Z$ is a $q \times m$ upper quasi-triangular matrix with $z_{11} > \cdots > z_{qq} > 0$. Then, by Theorem 4,

$$(dX) = 2^{-q} \prod_{i=1}^q z_{ii}^{N-i}(H_1' dH_1)(dZ). \quad (A.3)$$

Now, $Z = N \Omega$, and $z_{ii} = \bar{N}_{ii}$, since $\omega_{ii} = 1$ for all $i$. By Theorem 1 we get,

$$(dZ) = \prod_{i=1}^q n_{ii}^{m-i}(d\Omega)(dN), \quad (A.4)$$

Therefore, by substituting (A.4) into (A.3), the result follows.

A.7. Proof of Theorem 7

(1) Write $S = \Omega' O \Omega = G' G$, where $G = C \Omega$, $C = O^{1/2}$ and $C$ is a $q \times N$ upper quasi-triangular matrix with $c_{11} > \cdots > c_{qq} > 0$. Then, by Theorem 6

$$(dS) = 2^q \prod_{i=1}^q g_{ii}^{m-i+1}(dG). \quad (A.5)$$
Now, \( G = C \Omega \), with \( g_{ii} = c_{ii} \), since \( \omega_{ii} = 1 \) for all \( i \). Then, by Theorem 1

\[
(dG) = \prod_{i=1}^{q} c_{ii}^{m-1}(dC)(d\Omega).
\]  

(A.6)

Substituting (A.6) into (A.5), we get,

\[
(dS) = 2^q \prod_{i=1}^{q} c_{ii}^{2m-2i+1}(dC)(d\Omega).
\]  

(A.7)

But \( C = O^{1/2} \) with \( (dC) = 2^{-q} |O|^{-1/2}(dO) \) and \( c_{ii} = o_{ii}^{1/2} \), from which we get the result.

(2) The proof follows from Theorems 5 and 7(1).

A.8. Proof of Theorem 8

Let \( R = Q_{1}DQ_{1}' \) with \( D = \text{diag}(D_{11}, \ldots, D_{qq}) \) and \( Q_{1} \) an \( m \times q \) semiorthogonal matrix. Applying Lemma 2

\[
(dR) = 2^{-q} |D|^{m-q} \prod_{i<j} (D_{ii} - D_{jj})(dD)(Q_{1}'dQ_{1}).
\]  

(A.8)

Now let \( S = R^2 = RR = Q_{1}DQ_{1}'Q_{1}DQ_{1}' = Q_{1}D^2Q_{1}' \), applying Lemma 2 once again, we have

\[
(dS) = 2^{-q} |D|^{2m-q} \prod_{i<j} (D_{ii}^2 - D_{jj}^2)(dD^2)(Q_{1}'dQ_{1}).
\]

Observing that \( (dD^2) = \prod_{i=1}^{q} 2D_{ii}(dD) = 2^q |D|(dD), \ (D_{ii}^2 - D_{jj}^2) = (D_{ii} + D_{jj})(D_{ii} - D_{jj}) \), and from (A.8), we get,

\[
(dS) = 2^q |D|^{m-q+1} \prod_{i<j} (D_{ii} + D_{jj}) \left[ 2^{-q} |D|^{m-q} \prod_{i<j} (D_{ii} - D_{jj})(Q_{1}'dQ_{1})(dD) \right]
\]

\[
= 2^q |D|^{m-q+1} \prod_{i<j} (D_{ii} + D_{jj})(dR).
\]

The second expression for \( (dS) \) is found by observing that

\[
\prod_{i<j} (D_{ii} + D_{jj}) = \prod_{i=1}^{q} 2D_{ii} \prod_{i<j} (D_{ii} + D_{jj}).
\]

A.9. Proof of Theorem 9

(1) From [9] we have that the density of \( S = X'X \) (central case) is

\[
\frac{\pi^{qN/2}|L|^{(N-m-1)/2}}{\Gamma_{q/2} N!(\prod_{i=1}^{r} N_i)} \hat{h}(\text{tr} \Sigma^{-1}S)(dS).
\]
Let \( S = \mathbb{R}^2 \), with 
\[
    (dS) = 2^q |D|^{m-q+1} \prod_{i<j}^q (D_{ii} + D_{jj})(dR) \quad \text{and} \quad L = D^2 \quad \text{(see Theorem 8)}.
\]
Then
\[
    \frac{\pi^{N/2} |L|^{(N-m-1)/2}}{\Gamma_q\left[\frac{1}{2} N\right]\left(\prod_{i=1}^r \lambda_i^{N/2}\right)} h(\text{tr} \Sigma^{-1} S)(dS)
\]
\[ = \frac{2^q \pi^{N/2} |D|^{(N-q)} \prod_{i<j}^q (D_{ii} + D_{jj})}{\Gamma_q\left[\frac{1}{2} N\right]\left(\prod_{i=1}^r \lambda_i^{N/2}\right)} h(\text{tr} \Sigma^{-1} R^2)(dR).
\]
Denote this function by \( f_R(R) \).

Now, the density of \( X (\mu_x = 0) \) is
\[
    \frac{1}{\prod_{i=1}^r \lambda_i^{N/2}} h(\text{tr} \Sigma^{-1} X'X)(dX).
\]
Let \( X = P_1 R \) with Jacobian, 
\[
    (dX) = \alpha(dR)(P_1 dP_1), \quad \alpha \text{ is independent of } P_1.
\]
Then the joint density of \( R \) and \( P_1 \) is
\[
    \frac{\alpha}{\prod_{i=1}^r \lambda_i^{N/2}} h(\text{tr} \Sigma^{-1} R^2)(dR)(P_1 dP_1).
\]
Integrating with respect to \( P_1 \) we get
\[
    \frac{\alpha 2^m \pi^{Nm/2}}{\Gamma_m\left[\frac{1}{2} N\right]\prod_{i=1}^r \lambda_i^{N/2}} h(\text{tr} \Sigma^{-1} R^2)(dR).
\]
Denote this function by \( g_R(R) \). By considering the ratio
\[
    f_R(R)/g_R(R) = 1
\]
and using the fact that \( \mathcal{V}_{m,N} / \mathcal{V}_{m-q,N-q} = \mathcal{V}_{r,s} \), where \( \mathcal{V}_{r,s} \) denotes the Stiefel manifold, the result follows.

(2) The result is obtained substituting \( (dR) \), from (14), in Theorem 9(1).

References


