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THE TUNNEL NUMBER OF THE SUM OF n KNOTS IS AT LEAST n

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We prove that the tunnel number of the sum of n knots is at least n.

1. INTRODUCTION

In [5], Norwood showed that tunnel number 1 knots are prime. This led to the more general conjecture, see for instance [4, Problem 1.70B], that the tunnel number of a sum of n knots is at least n. Here we prove this conjecture. The idea is to show that the splittig surface of a Heegard splitting corresponding to a *tunnel* system realizing the tunnel number of the sum of n knots intersects each individual knot complement essentially. Then a sophisticated Euler characteristic argument, based on the idea of untelescoping the Heegaard splitting, yields the result.

2. PRELIMINARIES

For standard definitions concerning knots, see [1] or [6] and for those concerning 3-manifolds, see [2] or [3].

Definition 1. Let N be a submanifold of M, we denote an open regular neighborhood of N in M by $\eta(N)$.

Definition 2. Let K be a knot in S^3. Denote the complement of K, $S^3 - \eta(K)$, by $C(K)$.

Remark 1. Let $K = K_1 \# K_2$ be the sum of two knots. Then the decomposing sphere gives rise to a decomposing annulus A properly embedded in $C(K)$ such that $C(K) = C(K_1) \cup_A C(K_2)$. If $K = K_1 \# \ldots \# K_n$, then we may assume that the decomposing spheres are nested, so that $C(K) = C(K_1) \cup_{A_1} \ldots \cup_{A_{n-1}} C(K_n)$.

Definition 3. A *Tunnel system* for a knot K is a collection of disjoint arcs $\mathcal{T} = t_1 \cup \ldots \cup t_n$ properly embedded in $C(K)$ such that $C(K) - \eta(\mathcal{T})$ is a handlebody. The *tunnel number of* K, denoted by $t(K)$, is the least number of arcs required in a tunnel system for K.

Definition 4. A *compression body* is a 3-manifold W obtained from a connected closed orientable surface S by attaching 2-handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2-sphere boundary components. We denote $S \times \{1\}$ by $\partial_+ W$ and $\partial W - \partial_+ W$ by $\partial_- W$.

Definition 5. A *set of defining disks* for a compression body W is a set of disks $\{D_1, \cdots, D_n\}$ properly embedded in W with $\partial D_i \subset \partial_+ W$ for $i = 1, \cdots, n$ such that the result of cutting W along $D_1 \cup \ldots \cup D_n$ is homeomorphic to $\partial_- W \times I$.

Definition 6. A *Heegaard splitting* of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V, W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$ and $M = V \cup W$. We call S the *splitting surface* or *Heegaard surface*.

Definition 7. Let $M = V \cup_S W$ be an irreducible Heegaard splitting. We may think of M as being obtained from $\partial_- V \times I$ by attaching all 1-handles dual to 2-handles in V followed by all 2-handles in W, followed, perhaps, by 3-handles. An *untelescoping* of $M = V \cup_S W$ is a rearrangement of the order in which the 1-handles (of V) and the 2-handles (dual to the 1-handles of W) are attached. This rearrangement is chosen so that M is decomposed into submanifolds $M_1, \cdots, M_m$, such that $M_1 \cap M_{i+1} = F_i$ and F_i is an incompressible surface in M, and such that the M_i inherit, from a subcollection of the original 1-handles and 2-handles, strongly irreducible Heegaard splittings $M_1 = V_1 \cup_{S_1} W_1, \cdots, M_m = V_m \cup_{S_m} W_m$. Unless M is a lens space or $S^1 \times S^2$, no $S_1, \cdots, S_m$ is a torus. For details see [8,7]. We denote the untelescoping of $M = V \cup_S W$ by $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$. For convenience, we will occasionally denote $\partial_- V = \partial_- V_1$ by F_0.

Lemma 2. $\chi(S) = \sum_{i=1}^{m} \chi(S_i) - \Sigma_{i=1}^{m-1} \chi(F_i)$.

Proof. Let $M = V \cup_S W$ be a Heegaard splitting, then

$$\chi(S) = \chi(\partial_- V) - 2(\#(\text{1-handles attached in } V) - \#(\text{0-handles attached in } V))$$

and in an untelescoping,

$$\chi(S_i) = \chi(\partial_- V_i) - 2(\#(\text{1-handles attached in } V_i) - \#\ (\text{0-handles attached in } V_i))$$
$$= \chi(F_{i-1}) - 2(\#(\text{1-handles attached in } V_i) - \#\ (\text{0-handles attached in } V_i)).$$

So, since 1-handles are merely reordered in an untelescoping,

$$\chi(S) = \chi(\partial_- V_i) - 2 \sum_{i=1}^{m} \#((\text{1-handles attached in } V_i) - \#\ (\text{0-handles attached in } V_i))$$

$$= \chi(\partial_- V) - \sum_{i=1}^{m} \chi(F_{i-1}) + \sum_{i=1}^{m} \chi(S_i).$$ □

Lemma 3. *Let P be a properly embedded incompressible surface in an irreducible* 3-*manifold M and let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ be an untelescoping of a Heegaard splitting $M = V \cup_S W$. Then $(\bigcup_{i=1}^{m-1} F_i) \cup (\bigcup_{i=1}^{m} S_i)$ can be isotoped to intersect P only in curves essential in P.*

Remark 4. This lemma demonstrates the advantage of working with untelescopings of Heegaard splittings rather than Heegaard splittings. It is a deep fact that the splitting surface of a strongly irreducible Heegaard splitting can be isotoped to intersect a properly embedded incompressible surface only in curves essential in this surface. This fact is proven for instance in [9, Lemma 6].

Proof. Here $(\bigcup_{i=1}^{m-1} F_i)$ may be isotoped to intersect P only in curves essential in P by a standard innermost disk argument, since both are incompressible. Then $P_i = P \cap M_i$ is a properly embedded incompressible surface in M_i. It follows that each S_i may be isotoped in M_i to intersect P_i only in curves essential in P_i, by [9, Lemma 6]. Note that the latter isotopies fix $(\bigcup_{i=1}^{m-1} F_i)$. □

LEMMA 5. *Let K be a prime knot and let A be an annulus properly embedded in $C(K)$ such that the components of ∂A are meridians. Then A is boundary parallel.*

Proof. In S^3, A can be extended to a sphere by adding two meridian disks. This sphere intersects K in two points. Since K is prime, one side of the sphere contains a single unknotted arc. □

LEMMA 6. *Let P be an incompressible surface in a compression body W. Then the result of cutting W along P is a collection of compression bodies.*

Proof. This is [9, Lemma 2]. □

Remark 7. In the above lemma, P need not be connected.

LEMMA 8. *Let if $\mathscr{A}$ is a collection of incompressible annuli in a compression body W, then in any component X of $W - \mathscr{A}$, $\chi(\partial_+ W \cap X) \leqslant \chi(\partial_- W \cap X)$.*

Proof. Let $\mathscr{D}$ be a set of defining disks for W. We argue by induction on the pair $(|\chi(\partial_- W) - \chi(\partial_+ W)|, |\mathscr{A} \cap \mathscr{D}|)$. If $|\chi(\partial_- W) - \chi(\partial_+ W)| = 0$, then ($\mathscr{D} = \emptyset$ and) all annuli are spanning annuli and the result follows.

To complete the inductive step, suppose there is a disk D in $\mathscr{D}$ such that $D \cap \mathscr{A} = \emptyset$. The result of cutting W along D is a compression body W' with $|\chi(\partial_- W') - \chi(\partial_+ W')| < |\chi(\partial_- W) - \chi(\partial_+ W)|$, or two compression bodies W' and W'' with $|\chi(\partial_- W') - \chi(\partial_+ W')| < |\chi(\partial_- W) - \chi(\partial_+ W)|$ and $|\chi(\partial_- W) - \chi(\partial_+ W'')| < |\chi(\partial_+ W)|$. The components of $W - \mathscr{A}$ can be obtained from the components of $W' - \mathscr{A}$ or of $W' - \mathscr{A}$ and $W'' - \mathscr{A}$ by attaching a 1-*handle* either to a single component or so as to connect two components. In both cases, the result follows from the inductive hypotheses.

If there is no such disk, consider $\mathscr{D} \cap \mathscr{A}$. If there is an arc α in $\mathscr{D} \cap \mathscr{A}$ that is inessential in $\mathscr{A}$, then we may assume that α is outermost in $\mathscr{A}$, and we may cut the disk D in $\mathscr{D}$ containing α along α and paste on two copies of the disk cut off by α in $\mathscr{A}$ to obtain a disk D'. Replacing D by D' in $\mathscr{D}$ produces a new set of defining disks $\mathscr{D}'$ with $|\mathscr{A} \cap \mathscr{D}'| < |\mathscr{A} \cap \mathscr{D}|$.

If all arcs in $\mathscr{D} \cap \mathscr{A}$ are essential in $\mathscr{A}$, let β be an arc in $\mathscr{D} \cap \mathscr{A}$ that is outermost in $\mathscr{D}$. Let A be the annulus in $\mathscr{A}$ that gives rise to β. Cutting and pasting A along β and the outermost disk cut off in $\mathscr{D}$ yields a disk D' disjoint from $\mathscr{A}$. If D' is inessential, then A is inessential and can be ignored. (Since cutting along A does not alter any components or their Euler characteristics.) If D' is essential, the result follows as above. This completes the inductive step. □

3. THE COMBINATORICS

In the following, we consider a tunnel system $\mathscr{T}$, realizing the tunnel number of $K_1 \# \ldots \# K_n$. We also consider the Heegaard splitting $C(K_1 \# \ldots \# K_n) = V \cup_S W$

corresponding to $\mathscr{T}$ and an untelescoping $C(K_1\# \ldots \# K_n) = (V_1 \cup_{S_1} W_1)\cup_{F_1} \ldots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ of $C(K_1\# \ldots \# K_n) = V\cup_S W$. Set $M_i = V_i \cup W_i$. By Remark 1, $C(K_1\# \ldots \# K_n) = C(K_1)\cup_{A_1}\cup \ldots \cup_{A_{n-1}} C(K_n)$. We will always assume that $\partial_- V_1 = \partial C(K_1\# \ldots \# K_n)$ and that $\bigcup_{i=1}^{m-1} F_i$ and $\bigcup_{i=1}^{m-1} S_i$ intersect $\bigcup_{j=1}^{n-1} A_i$ only in curves essential in $\bigcup_{j=1}^{n-1} A_i$. We will, furthermore, assume that, subject to these constraints, the number of intersections of $\bigcup_{i=1}^{m-1} F_i$ and $\bigcup_{i=1}^{m} S_i$ with $\bigcup_{j=1}^{n-1} A_j$ is minimal.

Definition 8. Set $S_{ij} = S_i \cap C(K_j)$, $F_{ij} = F_i \cap C(K_j)$ and $A_{ij} = M_i \cap A_j$.

LEMMA 9. *For all i, j, $\chi(S_{ij})$ and $\chi(F_{ij})$ are even.*

Proof. Here F_i is separating, so $F_i \cap A_{j-1}$ is separating. Since $\partial A_{j-1} \subset \partial C(K_1\# \ldots \# K_n)$ which is a torus, hence connected, both components lie on one side of F_i, hence $|F_i \cap A_{j-1}|$ is even. The same is true for $|F_i \cap A_j|$. Thus $\chi(F_i \cap C(K_j)) = 2 - 2(genus(F_i \cap C(K_j))) - |F_i \cap (A_{j-1} \cup A_j)|$ is even. Similarly for S_i. □

Definition 9. Set $x_{ij} = -1/2\chi(F_{ij})$ and $y_{ij} = -1/2\chi(S_{ij})$.

LEMMA 10. *Under the assumptions above, $y_{ij} \geqslant \max\{x_{i-1j}, x_{ij}\}$.*

Proof. This follows from Lemma 8. □

LEMMA 11. *For all j, there is an i, such that $y_{ij} > 0$.*

Proof. Suppose $y_{ij} = 0$ for $i = 1, \cdots, m$. Then $x_{ij} = 0$ for $i = 1, \cdots, m-1$. So

$$G_j = (\textstyle\bigcup_{i=1}^{m-1} F_{ij}) \cup (\bigcup_{i=1}^{m} S_{ij}) \subset C(K_j)$$

is a collection of annuli and tori. Since the tori arise only in $\bigcup_{i=1}^{m-1} F_{ij}$, they are incompressible separating tori. Thus if a torus component T of F_i is in $C(K_j)$, then so is a component of $S_{i'}$, which cannot be a torus, for some i'. But this would contradict $y_{i'j} = 0$. Hence G_j consists entirely of annuli. By Lemma 5, the annuli are all boundary parallel. Hence cutting $C(K_j)$ along the annular components of G_j yields a copy of $C(K_j)$. By Lemma 6, all components of $C(K_j)$ cut along G are compression bodies, a contradiction. □

LEMMA 12. *For all j,*

$$\sum_{i=1}^{m} y_{ij} > \sum_{i=1}^{m-1} x_{ij}.$$

Proof. This follows by comparing the tables in Fig. 1. By Lemma 10, the largest value encountered in a given column of the table in Fig. 1(a) occurs one time more often in the corresponding columns of the table in Fig. 1(b). If the largest value encountered in a column in the table in Fig. 1(a) is zero, then by Lemma 11, there must be nonzero entries in the corresponding column of the table in Fig. 1b. □

Remark 13. Since all numbers involved are integers, it follows that $\sum_{i=1}^{m} y_{ij} \geqslant 1 + \sum_{i=1}^{m-1} x_{ij}$, for all j.

THEOREM 14. *$t(K_1\# \cdots \# K_n) \geqslant n$.*

	1	...	j	...	n
1					
...		...		...	
i			x_{ij}		
...		...		...	
m-1					

(a)

	1	...	j	...	n
1					
...		...		...	
i			y_{ij}		
...		...		...	
m					

(b)

Fig. 1.

Proof. Here

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} y_{ij}\right) \geqslant \sum_{j=1}^{n}\left(1+\sum_{i=1}^{m-1} x_{ij}\right) = n + \sum_{j=1}^{n}\sum_{i=1}^{m-1} x_{ij}$$

Hence,

$$\sum_{j=1}^{n}\sum_{i=1}^{m} y_{ij} - \sum_{j=1}^{n}\sum_{i=1}^{m-1} x_{ij} \geqslant n.$$

Thus,

$$\sum_{j=1}^{n}\sum_{i=1}^{m} -2(y_{ij}) - \sum_{j=1}^{n}\sum_{i=1}^{m-1} -2(x_{ij}) \leqslant -2n$$

and by definition

$$\sum_{j=1}^{n}\sum_{i=1}^{m} \chi(S_i \cap C(K_j)) - \sum_{j=1}^{n}\sum_{i=1}^{m-1} \chi(F_i \cap C(K_j)) \leqslant -2n.$$

So,

$$\chi(S) = \sum_{i=1}^{m} \chi(S_i) - \sum_{i=1}^{m-1} \chi(F_i) \leqslant -2n.$$

Whence

$$genus(S) \geqslant n+1$$

and

$$t(K_1 \# \dots \# K_n) \geqslant n.$$

□

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REFERENCES

1. Burde, G., Zieschang, H., Knots, Studies in Mathematics, Vol. 5. de Gruyter, Berlin, 1985.
2. Hempel, J., *Annals of Mathematial Studies*, 1976, **86**.
3. Jaco, W., Lectures on Three-manifold Topology, Regional Conference Series in Mathematics, Vol. 43. Amer. Math. Soc., Providence, RI, 1981.
4. Kirby, R., Problem's in low-dimensional topology. In *Geometric Topology—Proceedings of the* 1993 *Georgia International Topology Conference*, ed. W. H. Kazez, AMS/IP Studies in Advanced Mathematics, *International Press*, Cambridge, MA, 1997, Vol. 2, Part 2, pp. 35–473.
5. Norwood, F.H., *Proceedings of the American Mathematical Society*, 1982, **86**, 143–147.
6. Rolfsen, D., Knots and Links, Publish or Perish, Houston, Texas.
7. Scharlemann, M. and Thompson, A., *AMS Contemporary Mathematics*, 1994, **164**, 231–238.
8. Scharlemann, Heegaard splittings of compact 3-manifolds. *Handbook of Geometric Topology*, eds R. Daverman and R. Sherr. Elsevier, Amsterdam (to appear).
9. Schultens, J., Additivity of tunnel number. Preprint.

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