Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 53 (2007) 1020–1028
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Reverse triple I method of fuzzy reasoning for the implication operator R_L ☆

Zhihong Zhao, Yongjin Li*

Institute of Logic and Cognition, Sun Yat-sen University, Guangzhou 510275, PR China
Basic Course of Department, Zhuhai Campus, Beijing Institute of Technology, Zhuhai 519085, PR China
Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, PR China

Received 13 September 2005; received in revised form 19 July 2006; accepted 21 July 2006

Abstract

First, the reverse triple I method with Lukasiewicz's implication operator is studied. The formulas of supremum for Fuzzy Modus Ponens (FMP) and infimum for Fuzzy Modus Tollens (FMT) of inverse triple I method are obtained respectively. Second, the reductivity of reverse triple I method is considered. Lastly, the theory of sustention degree is discussed and its properties are analysed. The generalized problem of reverse triple I method is solved and the corresponding formulas of α-reverse triple I method with sustention degree are also given.

Keywords: Fuzzy reasoning; Reverse triple I method; Implication operator R_L; Sustention degree; Reductivity

1. Preliminaries

Fuzzy reasoning is the mathematical and logical foundation of fuzzy control. Since the method of Composition Rule of Inference (CRI) was put forward in 1973 by Zadeh [1], the CRI method has been widely used and proved to be successful in dealing with many questions where the theory of fuzzy control are involved [2,3]. Moveover, very rigorous logical foundation of the method was provided by Hajek [4]. Using this idea of the CRI method, in 1999 Wang proposed first triple I method [5–7] with full inference rule that utilizes the implication operator in every step of the reasoning. Wang also offered full proofs for the triple I method in logic [8]. The method efficiently improves CRI method. It may be suitable to be used instead of Zadeh's CRI method in some circumstances. Its basic idea is as follows:

For known $A \in F(X)$, $B \in F(Y)$, and $A^* \in F(X)$ (or $B^* \in F(Y)$), seek the optimal $B^* \in F(Y)$ (or $A^* \in F(X)$) such that $A \to B$ sustains farthest $A^* \to B^*$, i.e.

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y))$$

☆ Supported by grant 10471155 from NNSF of China, by grant 031608 from NSF of Guangdong, and by the Foundation of Sun Yat-sen University Advanced Research Centre.

* Corresponding author at: Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, PR China.
E-mail addresses: stslyj@mail.sysu.edu.cn, stslyj@zsu.edu.cn (Y. Li).

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2006.07.008

takes the value as large as possible for any $x \in X$ and $y \in Y$, where $F(X)$ and $F(Y)$ denote the collections consisting of all fuzzy subsets of X and Y respectively. One can use the triple I method with diverse kinds of implication operators.

In addition, the theory of sustention degree for triple I method was also presented, and its generalized form should be expressed as the following optimal problem:

For any $\alpha \in [0, 1]$, $A \in F(X)$, $B \in F(Y)$, and $A^* \in F(X)$ (or $B^* \in F(Y)$), seek the optimal $B^* \in F(Y)$ (or $A^* \in F(X)$) satisfying

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \geq \alpha$$

for any $x \in X$ and $y \in Y$.

Based on this idea, for the implication operator $R_0 : [0, 1]^2 \to [0, 1]$

$$R_0(a, b) = \begin{cases} 1 & a \leq b \\ a' \bigvee b & a > b \end{cases}$$

where $a' = 1 - a$, reverse triple I method was proposed by Song and Wu in [9]. Its basic idea is as follows:

For known $A \in F(X)$, $B \in F(Y)$, and $A^* \in F(X)$ (or $B^* \in F(Y)$), seek the optimal $B^* \in F(Y)$ (or $A^* \in F(X)$) such that $A^* \to B^*$ sustains farthest $A \to B$, i.e.

$$(A^*(x) \to B^*(y)) \to (A(x) \to B(y)) \tag{1}$$

takes the value as large as possible for any $x \in X$ and $y \in Y$. Meanwhile, the authors discussed the generalized form of reverse triple I method expressed as the following optimal problem:

Under the hypothesis of (1), for any $\alpha \in [0, 1]$, seek the optimal $B^* \in F(Y)$ (or $A^* \in F(X)$) satisfying

$$(A^*(x) \to B^*(y)) \to (A(x) \to B(y)) \geq \alpha \tag{2}$$

for any $x \in X$ and $y \in Y$.

In this paper, for the implication operator $R_L : [0, 1]^2 \to [0, 1]$

$$R_L(a, b) = (1 - a + b) \wedge 1 = \begin{cases} 1 & a \leq b \\ a' + b & a > b \end{cases}$$

where $a' = 1 - a$, we will discuss the reverse triple I method and its generalized form. The formulas of supremum for Fuzzy Modus Ponens (FMP) and infimum for Fuzzy Modus Tollens (FMT) of inverse triple I method are obtained respectively. Moreover, the reductivity of reverse triple I method is considered. In addition, the corresponding formulas of α-reverse triple I method with sustention degree are also studied.

2. Reverse triple I method for the implication operator R_L

First of all, we want to establish the reverse triple I principle.

Reverse triple I FMP principle with sustention degree. Suppose that X and Y are non-empty sets, $A, A^* \in F(X)$, $B \in F(Y)$. Then B^* satisfying this principle is the maximal fuzzy set in $F(Y)$ that maximizes (1).

Reverse triple I FMT principle with sustention degree. Suppose that X and Y are non-empty sets, $A \in F(X)$, $B, B^* \in F(Y)$. Then A^* satisfying this principle is the minimal fuzzy set in $F(X)$ that maximizes (1).

For the implication operator R_L, if $R_L(A(x), B(y)) = 1$, according to the definition of it, then the maximum 1 of (1) will be always taken, so $B^*(y) = 1$.

If $R_L(A(x), B(y)) \leq 1$, then we have the following reverse triple I method with sustention degree:

Theorem 2.1 (*Reverse Triple I FMP Formula of Supremum*). *Suppose that X and Y are non-empty sets, $A, A^* \in F(X)$, $B \in F(Y)$. Then the supremum B^* consisting of fuzzy subsets in $F(Y)$ that maximizes (1) is determined by*

$$B^*(y) = \inf_{x \in X} \{A^*(x) + R_L(A(x), B(y)) - 1\} \vee 0. \tag{3}$$

Proof. For any $y \in Y$ and $C(y) < B^*(y)$ with $C(y) \in F(Y)$, we will first prove: $C(y)$ must make (1) take its maximum 1, where $B^*(y)$ is determined by (3). It is clear that

$$C(y) < B^*(y) \leq A^*(x) + R_L(A(x), B(y)) - 1. \tag{4}$$

Note that $R_L(A(x), B(y)) - 1 \leq 0$ and from (4), we get $C(y) < A^*(x)$. Using the sense of the operator R_L, we have

$$A^*(x) \to C(y) = (A^*(x))' + C(y) = 1 - A^*(x) + C(y)$$

and from (4), we know

$$1 - A^*(x) + C(y) < R_L(A(x), B(y)).$$

That means $(A^*(x) \to C(y)) < (A(x) \to B(y))$. So

$$M_{xy} = (A^*(x) \to C(y)) \to (A(x) \to B(y)) \equiv 1$$

i.e. $C(y)$ makes (1) take its maximum 1.

On the other hand, if there exists $y_0 \in Y$ such that $D(y_0) > B^*(y_0)$, then we will prove: $D(y_0)$ cannot maximize (1). By (3), we have

$$D(y_0) > \beta = A^*(x_0) + R_L(A(x_0), B(y_0)) - 1. \tag{5}$$

We will discuss in the cases as follows:

Case 1. If $D(y_0) < A^*(x_0)$, then

$$A^*(x_0) \to D(y_0) = (A^*(x_0))' \to D(y_0) = 1 - A^*(x_0) + D(y_0).$$

By (5), we get

$$1 - A^*(x_0) + D(y_0) > R_L(A(x_0), B(y_0)).$$

That means

$$\begin{aligned} M_{x_0y_0} &= (A^*(x_0) \to D(y_0)) \to (A(x_0) \to B(y_0)) \\ &= A^*(x_0) - D(y_0) + R_L(A(x_0), B(y_0)) < 1. \end{aligned}$$

Case 2. If $D(y_0) \geq A^*(x_0)$, then $A^*(x_0) \to D(y_0) = 1$. That means

$$\begin{aligned} M_{x_0y_0} &= (A^*(x_0) \to D(y_0)) \to (A(x_0) \to B(y_0)) \\ &= 1 \to R_L(A(x_0), B(y_0)) \\ &= R_L(A(x_0), B(y_0)) < 1. \end{aligned}$$

That is, $D(y_0)$ cannot maximize (1).

Summing up the above proof, $B^*(y)$ is the supremum consisting of fuzzy subsets in $F(Y)$ that maximizes (1).

Likewise, if $R_L(A(x), B(y)) = 1$, then $A^*(x_0) = 0$ is minimal fuzzy set in $F(X)$ that maximizes (1).

For the case where $R_L(A(x), B(y)) < 1$, we get the following reverse triple I method with sustention degree:

Theorem 2.2 (*Reverse Triple I FMT Formula of Infimum*). *Suppose that X and Y are non-empty sets, $A \in F(X)$, $B, B^* \in F(Y)$. Then the infimum A^* consisting of fuzzy subsets in $F(X)$ that maximizes (1) is determined by*

$$A^*(x) = \sup_{y \in Y}\{B^*(y) - R_L(A(x), B(y)) + 1\}. \tag{6}$$

Proof. For any $x \in X$ and $C(x) > A^*(x)$ with $C(x) \in F(X)$, we will first prove: $C(x)$ must maximize (1), where $A^*(x)$ is determined by (6). It is clear that

$$C(x) > A^*(x) \geq B^*(y) - R_L(A(x), B(y)) + 1. \tag{7}$$

Note that $1 - R_L(A(x), B(y)) > 0$ and from (7), we have $C(x) > B^*(y)$. Using the sense of R_L, we have

$$C(x) \to B^*(y) = (C(x))' + B^*(y) = 1 - C(x) + B^*(y)$$

and from (7), we know

$$1 - C(x) + B^*(y) < R_L(A(x), B(y)).$$

That means

$$N_{xy} = (C(x) \to B^*(y)) \to (A(x) \to B(y)) = 1$$

i.e. $C(x)$ makes (1) take its maximum 1.

On the other hand, if there exists $x_0 \in X$ such that $D(x_0) < A^*(x_0)$, then we will prove: $D(x_0)$ cannot maximize (1). By (6), we have

$$D(x_0) < \beta = B^*(y_0) - R_L(A(x_0), B(y_0)) + 1. \tag{8}$$

We will discuss in the cases as follows:

Case 1. If $D(x_0) > B^*(y_0)$, then

$$D(x_0) \to B^*(y_0) = (D(x_0))' + B^*(y_0) = 1 - D(x_0) + B^*(y_0).$$

By (8), we get

$$1 - D(x_0) + B^*(y_0) > R_L(A(x_0), B(y_0)).$$

That means

$$\begin{aligned} N_{x_0y_0} &= (D(x_0) \to B^*(y_0)) \to (A(x_0) \to B(y_0)) \\ &= D(x_0) - B^*(y_0) + R_L(A(x_0), B(y_0)). \end{aligned}$$

And from (8), we have

$$N_{x_0y_0} < 1.$$

Case 2. If $D(x_0) \le B^*(y_0)$, then $D(x_0) \to B^*(y_0) = 1$. That means

$$\begin{aligned} N_{x_0y_0} &= (D(x_0) \to B^*(y_0)) \to (A(x_0) \to B(y_0)) \\ &= 1 \to R_L(A(x_0), B(y_0)) \\ &= R_L(A(x_0), B(y_0)) < 1. \end{aligned}$$

That is, $D(x_0)$ cannot maximize (1). Combining the above proof, $A^*(x)$ is the infimum consisting of fuzzy sets in $F(X)$ that maximizes (1).

3. The reductivity of reverse triple I method

We first give two definitions related to this section.

Definition 3.1 (*[8,10,11]*). For the algorithm of reverse triple I FMP, if A is a regular fuzzy subset (i.e. there exists $x_0 \in X$ such that $A(x_0) = 1$), and $A^* = A$ implies $B^* = B$, then this algorithm is called reductive algorithm.

Definition 3.2 (*[8,10,11]*). For the algorithm of reverse triple I FMT, if B' is a regular fuzzy subset (i.e. there exists $y_0 \in Y$ such that $B(y_0) = 0$), and $B^* = B$ implies $A^* = A$, then this algorithm is called reductive algorithm.

Theorem 3.1. *(The Reductivity of Reverse Triple I FMP Algorithm) Suppose that X and Y are non-empty sets, $A, A^* \in F(X)$, $B \in F(Y)$, if $R_L(A(x), B(y)) < 1$ and A is a regular fuzzy subset (i.e. there exists $x_0 \in X$ such that $A(x_0) = 1$), then for the algorithm of reverse triple I FMP, $A^* = A$ implies $B^* = B$.*

Proof. Suppose $A^* = A$, A is a regular fuzzy set, then

$$\begin{aligned} B^*(y) &= \inf_{x \in X} \{A^*(x) + R_L(A(x), B(y)) - 1\} \vee 0 \\ &\le \{A(x_0) + R_L(A(x_0), B(y)) - 1\} \vee 0 \\ &= \{1 + R_L(1, B(y)) - 1\} \vee 0 = B(y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} B(y) \to B^*(y) &= B(y) \to \inf_{x\in X}\{A^*(x) + R_L(A(x), B(y)) - 1\} \vee 0 \\ &= \inf_{x\in X} B(y) \to (A^*(x) + R_L(A(x), B(y)) - 1) \\ &= \inf_{x\in X} B(y) \to (A^*(x) + 1 - A(x) + B(y) - 1) \\ &= \inf_{x\in X} B(y) \to B(y) = 1. \end{aligned}$$

Then, we get $B(y) \le B^*(y)$.

So, we deduce $B(y) = B^*(y)$.

Theorem 3.2 (*The Reductivity of Reverse Triple I FMT Algorithm*). *Suppose that X and Y are non-empty sets, $A \in F(X)$, B, $B^* \in F(Y)$, if $R_L(A(x), B(y)) < 1$ and B' is a regular fuzzy subset (i.e. there exists $y_0 \in Y$ such that $B(y_0) = 0$), then for the algorithm of reverse triple I FMT, $B^* = B$ implies $A^* = A$.*

Proof. Suppose $B^* = B$, B' is a regular fuzzy set, then

$$\begin{aligned} A^*(x) &= \sup_{y\in Y}\{B^*(y) - R_L(A(x), B(y)) + 1\} \\ &\ge B(y_0) - R_L(A(x), B(y_0)) + 1 \\ &= 0 - R_L(A(x), 0) + 1 = A(x). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} A^*(x) \to A(x) &= \sup_{y\in Y}\{B^*(y) - R_L(A(x), B(y)) + 1\} \to A(x) \\ &= \inf_{y\in Y}\{(B^*(y) - R_L(A(x), B(y)) + 1) \to A(x)\} \\ &= \inf_{y\in Y}\{(B^*(y) - (1 - A(x) + B(y)) + 1) \to A(x)\} \\ &= \inf_{y\in Y} A(x) \to A(x) = 1. \end{aligned}$$

Then, we get $A(x) \ge A^*(x)$. Hence, we deduce $A(x) = A^*(x)$.

Note. For the case $R_L(A(x), B(y)) = 1$, we know that reverse triple I FMP and FMT algorithm are not reductive algorithms.

Example. (1) Suppose that $A(x) = A^*(x) = 0$, $B(y) = 1 - y$, for the implication operator R_L,

$$\begin{aligned} (A^*(x) \to B^*(y)) &\to (A(x) \to B(y)) \\ &= (0 \to B^*(y)) \to (0 \to 1 - y) \\ &= (0 \to B^*(y)) \to 1. \end{aligned}$$

$B^*(y) = 1$ is the maximal fuzzy set in $F(Y)$ that maximizes (1), it is clear that $B^*(y) \ne B(y)$.

(2) Suppose that $A(x) = \frac{1}{2}$, $B^*(y) = B(y) = 1$, for the implication operator R_L,

$$\begin{aligned} (A^*(x) \to B^*(y)) &\to (A(x) \to B(y)) \\ &= (A^*(x) \to 1) \to \left(\frac{1}{2} \to 1\right) \\ &= (A^*(x) \to 1) \to 1. \end{aligned}$$

$A^*(x) = 0$ is the minimum fuzzy set in $F(X)$ that maximizes (1), it is clear that $A^*(x) \ne A(x)$.

4. The theory of sustention degree

At first, we give the following reverse triple I FMP principle and FMT principle with sustention degree.

α-reverse triple I FMP principle with sustention degree. Suppose that X and Y are non-empty sets, $A, A^* \in F(X)$, $B \in F(Y)$. Then B^* satisfying this principle is the maximal fuzzy set in $F(Y)$ that holds (2).

α-reverse triple I FMT principle with sustention degree. Suppose that X and Y are non-empty sets, $A \in F(X)$, $B, B^* \in F(Y)$. Then A^* satisfying this principle is the minimal fuzzy set in $F(X)$ that holds (2).

Definition 4.1 (*Sustaining Degree [7]*). Suppose that X is a non-empty set, $A, B \in F(X)$, $\alpha \in [0, 1]$, if

$$\inf_{y \in Y}\{A(x) \to B(y) | x \in X\} = \alpha,$$

then we say that sustention degree from A to B is α, and denoted this by $\text{sust}(A, B) = \alpha$.

It is clear that (2) holds if and only if the sustention degree from $A^* \to B^*$ to $A \to B$ is no less than α. For the implication operator R_L, $\text{sust}(A, B) \geq \alpha$ if and only if $(A(x))' + B(x) \geq \alpha$, for any $x \in X$. The properties is as follows:

Theorem 4.1. *If* $\text{sust}(A, B) = \alpha$, $\text{sust}(B, C) = \beta$, *then* $\text{sust}(A, C) \geq \alpha + \beta - 1$. *Especially, if* $\text{sust}(A, B) = 1$, *and* $\text{sust}(B, C) = 1$, *then* $\text{sust}(A, C) = 1$. *i.e. at this point, transitivity holds.*

Theorem 4.2. *If* $A, B, A_i, B_i \in F(X)$, $(i \in I)$, *then*

$$\text{(a) sust}\left(\bigvee_{i \in I} A_i, B\right) = \bigwedge_{i \in I}(A_i, B);$$

$$\text{(b) sust}\left(A, \bigwedge_{i \in I} B\right) = \bigwedge_{i \in I}(A, B_i).$$

Theorem 4.3. *If* $A, B, C, B_i, C_i \in F(X)$, $(i \in I)$, *then*

$$\text{(a) sust}\left(A, \bigvee_{i \in I} B_i \to C\right) = \bigwedge_{i \in I}(A, B_i \to C);$$

$$\text{(b) sust}\left(A, B \to \bigwedge_{i \in I} C_i\right) = \bigwedge_{i \in I}(A, B \to C_i).$$

5. Supremum (infimum) for FMP (FMT) of α-rerverse triple I method with sustention degree

Now, we will consider the generalized problem of reverse triple I method with sustention degree. That is, for given $\alpha \in [0, 1]$, our purpose is how to seek the optimal solution satisfying (2).

For the implication operator R_L, if $R_L(A(x), B(y)) \geq \alpha$, according to the sense of the operator R_L, it is clear that $B^*(y) = 1$ is the maximal fuzzy set in $F(Y)$ that holds (2).

If $R_L(A(x), B(y)) < \alpha$, then we have the following α-reverse triple I FMP formula of supremum with sustention degree:

Theorem 5.1 (*α-Reverse Triple I FMP Formula of Supremum*). *Suppose that X and Y are non-empty sets, $A, A^* \in F(X)$, $B \in F(Y)$, $\alpha \in (0, 1]$. Then the supremum B^* consisting of fuzzy subsets in $F(Y)$ that holds (2) is determined by*

$$B^*(y) = \inf_{x \in E_y}\{A^*(x) + R_L(A(x), B(y)) - \alpha\} \vee 0. \tag{9}$$

where $E_y = \{x \in X | R_L(A(x), B(y)) < \alpha\}$.

Proof. For any $y \in Y$ and $C(y) < B^*(y)$ with $C(y) \in F(Y)$, $C(y)$ must hold (2). It is clear that

$$C(y) < B^*(y) \leq A^*(x) + R_L(A(x), B(y)) - \alpha. \tag{10}$$

Note that $R_L(A(x), B(y)) - \alpha < 0$ and from (10), we get $C(y) < A^*(x)$. Using the sense of the operator R_L, we have

$$A^*(x) \to C(y) = (A^*(x))' + C(y) = 1 - A^*(x) + C(y)$$

we will discuss in the cases as follows:

Case 1. If $(A^*(x) \to C(y)) > (A(x) \to B(y))$, then

$$M_{xy} = (A^*(x) \to C(y)) \to (A(x) \to B(y)) = A^*(x) - C(y) + R_L(A(x), B(y)).$$

From (10), we have $M_{xy} > \alpha$.

Case 2. If $(A^*(x) \to C(y)) \leq (A(x) \to B(y))$, then

$$M_{xy} = (A^*(x) \to C(y)) \to (A(x) \to B(y)) = 1 \geq \alpha.$$

So, $C(y)$ holds (2).

On the other hand, if there exists $y_0 \in Y$ such that $D(y_0) > B^*(y_0)$, then we will prove: $D(y_0)$ cannot hold (2). By (10), we have

$$D(y_0) > \beta = A^*(x_0) + R_L(A(x_0), B(y_0)) - \alpha. \tag{11}$$

The discussion will be divided into two cases:

Case 1. If $D(y_0) < A^*(x_0)$, then

$$A^*(x_0) \to D(y_0) = (A^*(x_0))' \to D(y_0) = 1 - A^*(x_0) + D(y_0).$$

We will discuss:

(a) If $(A^*(x_0) \to D(y_0)) > (A(x_0) \to B(y_0))$, then

$$M_{x_0y_0} = (A^*(x_0) \to D(y_0)) \to (A(x_0) \to B(y_0)) = A^*(x_0) - D(y_0) + R_L(A(x_0), B(y_0))$$

from (11), we get

$$M_{x_0y_0} < \alpha.$$

(b) If $(A^*(x_0) \to D(y_0)) \leq (A(x_0) \to B(y_0))$, then $M_{x_0y_0} = 1$, that is

$$1 - A^*(x_0) + D(y_0) \leq R_L(A(x_0), B(y_0))$$

but by (11), we can deduce

$$R_L(A(x_0), B(y_0)) > 1 - A^*(x_0) + (A^*(x_0)) + R_L(A(x_0), B(y_0)) - \alpha$$

i.e. $1 < \alpha$, it's a contradiction.

Case 2. If $D(y_0) \geq A^*(x_0)$, then $A^*(x_0) \to D(y_0) = 1$

$$\begin{aligned} M_{x_0y_0} &= (A^*(x_0) \to D(y_0)) \to (A(x_0) \to B(y_0)) \\ &= 1 \to R_L(A(x_0), B(y_0)) \\ &= R_L(A(x_0), B(y_0)) < \alpha. \end{aligned}$$

That is, $D(y_0)$ cannot hold (2).

Summing up the above proof, $B^*(y)$ is the supremum consisting of fuzzy sets in $F(Y)$ that holds (2).

If $R_L(A(x), B(y)) \geq \alpha$, using the sense of the operator R_L, then $A^*(x) = 0$ is the minimal fuzzy set in $F(X)$ that holds (2).

If $R_L(A(x), B(y)) < \alpha$, then we have the following α-reverse triple I FMT formula of infimum with sustaining degree:

Theorem 5.2 (*α-Reverse Triple I FMT Formula of Infimum*). *Suppose that X and Y are non-empty sets, $A \in F(X)$, $B, B^* \in F(Y)$, $\alpha \in (0, 1]$. Then the infimum A^* consisting of fuzzy subsets in $F(X)$ that holds (2) is determined by*

$$A^*(x) = \sup_{y \in K_x} \{B^*(y) - R_L(A(x), B(y)) + \alpha\} \tag{12}$$

where $K_x = \{y \in Y | R_L(A(x), B(y)) < \alpha\}$.

Proof. For any $x \in X$ and $C(x) > A^*(x)$ with $C(x) \in F(X)$, we will first prove: $C(x)$ must hold (2). It is easy to see that

$$C(x) > A^*(x) \geq B^*(y) - R_L(A(x), B(y)) + \alpha. \tag{13}$$

Because of $R_L(A(x), B(y)) < \alpha$, we know $C(x) > B^*(y)$. Using the sense of R_L, we get

$$C(x) \to B^*(y) = (C(x))' + B^*(y) = 1 - C(x) + B^*(y)$$

we will discuss in the following cases:
Case 1. If $(C(x) \to B^*(y)) > (A(x) \to B(y))$, then

$$N_{xy} = (C(x) \to B^*(y)) \to (A(x) \to B(y)) = C(x) - B^*(y) + R_L(A(x), B(y))$$

from (13), we get

$$N_{xy} \geq \alpha.$$

Case 2. If $(C(x) \to B^*(y)) \leq (A(x) \to B(y))$, then

$$N_{xy} = (C(x) \to B^*(y)) \to (A(x) \to B(y)) = 1 \geq \alpha.$$

That is, $C(x)$ holds (2).

On the other hand, if there exists $x_0 \in X$ such that $D(x_0) < A^*(x_0)$, then we will prove: $D(x_0)$ cannot hold (2). It is easy to see that:

$$D(x_0) < \beta = B^*(y_0) - R_L(A(x_0), B(y_0)) + \alpha. \tag{14}$$

The discussion will be divided into two cases.
Case 1. If $D(x_0) > B^*(y_0)$, then

$$D(x_0) \to B^*(y_0) = (D(x_0))' + B^*(y_0) = 1 - D(x_0) + B^*(y_0)$$

we will discuss in two subcases:

(a) If $(D(x_0) \to B^*(y_0)) > (A(x_0) \to B(y_0))$, then

$$N_{x_0 y_0} = (D(x_0) \to B^*(y_0)) \to (A(x_0) \to B(y_0)) = D(x_0) - B^*(y_0) + R_L(A(x_0), B(y_0))$$

from (14), we have

$$N_{x_0 y_0} < \alpha.$$

(b) If $(D(x_0) \to B^*(y_0)) \leq (A(x_0) \to B(y_0))$, then $N_{x_0 y_0} = 1$, that means

$$1 - D(x_0) + B^*(y_0) \leq R_L(A(x_0), B(y_0))$$

and by (14), we deduce

$$R_L(A(x_0), B(y_0)) > 1 - (B^*(y_0) - R_L(A(x_0), B(y_0)) + \alpha) + B^*(y_0)$$

i.e. $1 < \alpha$, it is a contradiction.
Case 2. If $D(x_0) \leq B^*(y_0)$, then $D(x_0) \to B^*(y_0) = 1$. At this point,

$$\begin{aligned} N_{x_0 y_0} &= (D(x_0) \to B^*(y_0)) \to (A(x_0) \to B(y_0)) \\ &= 1 \to R_L(A(x_0), B(y_0)) \\ &= R_L(A(x_0), B(y_0)) < \alpha. \end{aligned}$$

That is, $D(x_0)$ cannot hold (2).

All of the above proof shows that $A^*(x_0)$ is the infimum consisting of fuzzy sets in $F(X)$ that holds (2).

6. Conclusion

The theory of reverse triple I method with sustention degree is discussed with the implication operator R_L and the computation formulas of infimum for Fuzzy Modus Ponens and supremum for Fuzzy Modus Tollens are shown. In addition, its generalization problem is studied. Results that have been gained will enrich the theory of α-reverse triple I method. Furthermore, we hope them to build the important theoretical basis for designing fuzzy controllers of new type.

Acknowledgement

The authors would like to thank the anonymous referee for his or her suggestions and corrections.

References

[1] L.A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Transactions on System Man and Cybernetics 3 (1973) 28–44.
[2] D. Dubois, H. Prade, J. Lang, Fuzzy sets in approximate reasoning, Fuzzy Sets and Systems 40 (1991) 143–244.
[3] N.Y. Zhang, Structure analyses of typical fuzzy controllers, Fuzzy Systems and Mathematics 2 (1997) 10–21.
[4] P. Haiek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.
[5] G.J. Wang, On the logic foundations of fuzzy reasoning, Informations Sciences 117 (1999) 47–88.
[6] G.J. Wang, A new method for fuzzy reasoning, Fuzzy Systems and Mathematics 13 (3) (1999) 1–10.
[7] G.J. Wang, The full implicational triple I method for fuzzy reasoning, Science in China 29 (1999) 43–53.
[8] G.J. Wang, Nonclassical Mathematical Logic and Approximate Reasoning, Science Press, Beijing, 2000.
[9] S.J. Song, C. Wu, Reverse triple I method of fuzzy reasoning, Science in China 45 (5) (2000) 344–364.
[10] D.W. Pei, Two triple I method for FMT problem and their reductivity, Fuzzy Systems and Mathematics 15 (4) (2001) 1–7.
[11] D.W. Pei, Full implication triple I algorithms and their consistency in fuzzy reasoning, Journal of Mathematical Research and Exposition 24 (2) (2004) 359–368.