Pseudo 2-factor isomorphic regular bipartite graphs

M. Abreu\textsuperscript{a}, A.A. Diwan\textsuperscript{b}, Bill Jackson\textsuperscript{c}, D. Labbate\textsuperscript{d}, J. Sheehan\textsuperscript{e}

\textsuperscript{a} Dipartimento di Matematica, Università della Basilicata, C. da Macchia Romana, 85100 Potenza, Italy
\textsuperscript{b} Department of Computer Science and Engineering, Indian Institute of Technology, Mumbai 400076, India
\textsuperscript{c} School of Mathematical Sciences, Queen Mary College, London E1 4NS, UK
\textsuperscript{d} Dipartimento di Matematica, Politecnico di Bari, I-70125 Bari, Italy
\textsuperscript{e} Department of Mathematical Sciences, King’s College, Old Aberdeen AB24 3UE, UK

Received 20 September 2006
Available online 27 September 2007

Abstract

A graph $G$ is pseudo 2-factor isomorphic if the parity of the number of circuits in a 2-factor is the same for all 2-factors of $G$. We prove that there exist no pseudo 2-factor isomorphic $k$-regular bipartite graphs for $k \geq 4$. We also propose a characterization for 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graphs and obtain some partial results towards our conjecture.

© 2007 Elsevier Inc. All rights reserved.

Keywords: 2-factor; Bipartite; Circuits; Connectivity

1. Introduction

All graphs considered are finite and simple (without loops or multiple edges). We shall use the term multigraph when multiple edges are permitted.

A graph with a 2-factor is said to be 2-factor hamiltonian if all its 2-factors are Hamilton circuits, and, more generally, 2-factor isomorphic if all its 2-factors are isomorphic. Examples of such graphs are $K_4$, $K_5$, $K_{3,3}$, the Heawood graph (which are all 2-factor hamiltonian) and the Petersen graph (which is 2-factor isomorphic).

Several recent papers have addressed the problem of characterizing families of graphs (particularly regular graphs) which have these properties. It is shown in [1,8] that $k$-regular 2-factor isomorphic bipartite graphs exist only when $k \in \{2, 3\}$ and an infinite family of 3-regular 2-factor...
hamiltonian bipartite graphs, based on $K_{3,3}$ and the Heawood graph, is constructed in [8]. It is conjectured in [8] that every 3-regular 2-factor hamiltonian bipartite graph belongs to this family, and, in [1], that every connected 3-regular 2-factor isomorphic bipartite graph is 2-factor hamiltonian. (We shall see in Section 3.2.4 of this paper that the latter conjecture is false.) Faudree, Gould and Jacobsen [7] determine the maximum number of edges in both 2-factor hamiltonian graphs and 2-factor hamiltonian bipartite graphs. In addition, Diwan [6] has shown that $K_4$ is the only 3-regular 2-factor hamiltonian planar graph.

In this paper, we extend the above mentioned results on regular 2-factor isomorphic bipartite graphs to the more general family of pseudo 2-factor isomorphic graphs i.e. graphs $G$ with the property that the parity of the number of circuits in a 2-factor is the same for all 2-factors of $G$. We prove in Theorem 3.2 that pseudo 2-factor isomorphic $k$-regular bipartite graphs exist only when $k \in \{2, 3\}$. We conjecture a characterization of 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graphs in Conjecture 3.5, and obtain some partial results towards our conjecture. We show in particular in Theorem 3.19 that there are no planar pseudo 2-factor isomorphic cubic bipartite graphs.

2. Preliminaries

An $r$-factor of a graph $G$ is an $r$-regular spanning subgraph of $G$. A 1-factorization of $G$ is a partition of the edge set of $G$ into 1-factors.

Let $G$ be a bipartite graph with bipartition $(X, Y)$ such that $|X| = |Y|$, and $A$ be its bipartite adjacency matrix. In general $0 \leq |\det(A)| \leq \text{per}(A)$. We say that $G$ is det-extremal if $G$ has a 1-factor and $|\det(A)| = \text{per}(A)$. Let $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. For $F$ a 1-factor of $G$, define the sign of $F$, $\text{sgn}(F)$, to be the sign of the permutation of $\{1, 2, \ldots, n\}$ corresponding to $F$. Then $G$ is det-extremal if and only if $G$ has a 1-factor and all its 1-factors have the same sign.

We shall need the following results. The first is elementary (and is a special case of [9, Lemma 8.3.1]).

**Lemma 2.1.** Let $F_1, F_2$ be 1-factors in a bipartite graph $G$ and $t$ be the number of circuits in $F_1 \cup F_2$ of length congruent to zero modulo four. Then $\text{sgn}(F_1) \text{sgn}(F_2) = (-1)^t$.

A $k$-circuit is a circuit of length $k$. A central circuit of a graph $G$ is a circuit $C$ such that $G - V(C)$ has a 1-factor. Lemma 2.1 easily implies:

**Lemma 2.2.** Let $G$ be a bipartite graph. Then $G$ is det-extremal if and only if $G$ has a 1-factor and every central circuit of $G$ has length congruent to two modulo four.

The next result follows from a more general theorem of Thomassen [12].

**Theorem 2.3.** Let $G$ be a det-extremal bipartite graph. If each edge of $G$ is contained in a 1-factor then $G$ has a vertex of degree at most three.

We next describe a result of Asratian and Mirumyan [4], see also [3], concerning transformations between 1-factorizations of a regular bipartite graph. Let $G$ be a $t$-regular bipartite graph, $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ be a 1-factorization of $G$, and $C$ be a circuit of $G$. 
Proposition 3.1. Suppose $E(C) \subseteq F_i \cup F_j$ for some $1 \leq i < j \leq t$. Then we may obtain a new 1-factorization $\mathcal{F}'$ of $G$ by putting $F'_i = F_i \Delta E(C)$, $F'_j = F_j \Delta E(C)$ and $\mathcal{F}' = (\mathcal{F} - \{F_i, F_j\}) \cup \{F'_i, F'_j\}$, where $\Delta$ denotes symmetric difference. We say that $\mathcal{F}'$ is obtained from $\mathcal{F}$ by a 2-transformation.

Suppose $E(C) \subseteq F_i \cup F_j \cup F_k$ for some $1 \leq i < j < k \leq t$, and that $F_i \cap E(C)$ is a 1-factor of $C$. Let $X = (F_j \cup F_k) \Delta E(C)$. Since the edges of $C$ alternate with respect to $F_j \cup F_k$, $X$ is a 2-factor of $G$. Let $\{F'_i, F'_k\}$ be a 1-factorization of $X$. We may obtain a new 1-factorization $\mathcal{F}'$ of $G$ by putting $F'_i = F_i \Delta E(C)$, and $\mathcal{F}' = (\mathcal{F} - \{F_i, F_j, F_k\}) \cup \{F'_i, F'_j, F'_k\}$. We say that $\mathcal{F}'$ is obtained from $\mathcal{F}$ by a 3-transformation.

Theorem 3.2. Let $G$ be a 1-factor in $F \cup F'$ of $G$. Then every 1-factorization of $G$ can be obtained from a given 1-factorization by a sequence of 2- and 3-transformations.

3. Pseudo 2-factor isomorphic regular bipartite graphs

Let $G$ be a bipartite graph. For each 2-factor $F$ of $G$ let $t^*(F)$ be the number of circuits of $F$ of length congruent to 0 modulo 4, and let

$$ t(F) = \begin{cases} 
0 & \text{if } t^*(F) \text{ is even}, \\
1 & \text{if } t^*(F) \text{ is odd}.
\end{cases} $$

We say that a bipartite graph $G$ is pseudo 2-factor isomorphic if $G$ has at least one 2-factor, and $t$ has the same value on all 2-factors of $G$. In this case, we denote this constant value of $t$ by $t(G)$. Equivalently, $G$ is pseudo 2-factor isomorphic if the parity of the number of circuits in a 2-factor is the same for all the 2-factors of $G$.

3.1. Regular graphs of degree at least four

We show that there are no pseudo 2-factor isomorphic $k$-regular bipartite graphs for $k \geq 4$. Our proof uses the results of Thomassen, and Asratian and Mirumyan described in Section 2. We also use the fact that there is a close relationship between pseudo 2-factor isomorphic bipartite graphs and det-extremal bipartite graphs. This is illustrated by the following proposition.

Proposition 3.1. Suppose $G$ is a pseudo 2-factor isomorphic bipartite graph.

(a) $G - F$ is det-extremal for all 1-factors $F$ of $G$.
(b) If $G$ is $k$-regular and $k \geq 3$ then $t^*(X) = 0$ for all 2-factors $X$ of $G$. In particular, $t(G) = 0$.

Proof. (a) Let $F$ be a 1-factor of $G$ and $H = G - F$. Let $F'$ be a 1-factor in $H$. Then $F \cup F'$ is a 2-factor of $G$, and hence the number of circuits of length congruent to 0 modulo 4 in $F \cup F'$ is congruent to $t(G)$ modulo 2. By Lemma 2.1, $\text{sgn}(F) \text{sgn}(F') = (-1)^{t(G)}$. Since the choice of $F'$ is arbitrary, all 1-factors of $H$ have the same sign. Thus $H$ is det-extremal.

(b) Let $X$ be a 2-factor of $G$ and $F$ be a 1-factor of $G - X$. By (a), $H = G - F$ is det-extremal. Since every circuit of $X$ is a central circuit of $H$, Lemma 2.2 implies that $t^*(X) = 0$. \square

Theorem 3.2. Let $G$ be a pseudo 2-factor isomorphic $k$-regular bipartite graph. Then $k \in \{2, 3\}$.

Proof. Suppose the theorem is false. Let $G$ be a pseudo 2-factor isomorphic $k$-regular bipartite graph with $k \geq 4$. By Proposition 3.1(a), all 1-factors in any 1-factorization of $G$ have the same
sign. By Theorem 2.3, $G$ contains two 1-factors with different signs. Since every 1-factor is contained in a 1-factorization of $G$, there are two 1-factorizations $F_0, F_1$ of $G$ such that all 1-factors in $F_0$ have positive sign and all 1-factors in $F_1$ have negative sign. However, by Theorem 2.4, $F_1$ can be obtained from $F_0$ by a sequence of 2- and 3-transformations. Since $k \geq 4$, at least one 1-factor is preserved in every transformation, and hence the signs of all 1-factors in the resulting 1-factorization must be the same as those of the 1-factors in the original 1-factorization. This gives a contradiction.

Theorem 3.2 generalizes the analogous results for 2-factor hamiltonian bipartite graphs [8] and 2-factor isomorphic bipartite graphs [1]. Its proof is substantially simpler than the proofs given for the latter two results.

### 3.2. Cubic graphs

It is straightforward to show that $K_{3,3}$ and the Heawood graph $H_0$, shown in Fig. 1(a), are 2-factor hamiltonian and hence pseudo 2-factor isomorphic, see [8]. We first show that the Pappus graph $P_0$, shown in Fig. 1(b), is pseudo 2-factor isomorphic but not 2-factor isomorphic.

**Proposition 3.3.** The Pappus graph $P_0$ is pseudo 2-factor isomorphic but not 2-factor isomorphic.

**Proof.** We adopt the labeling of the Pappus graph $P_0$ given in Fig. 1(b). Let $F$ be a 2-factor of $P_0$ and $C$ be a shortest circuit in $F$. Since $P_0$ is 3-arc-transitive, see [5], we may assume that the path $P = v_1v_2v_3v_4$ is contained in $C$. Since $P_0$ is bipartite, has 18 vertices, and has girth six, we have $|C| \in \{6, 8, 18\}$.

Suppose $|C| = 6$. By inspection, $P$ is contained in exactly one 6-circuit $v_1v_2v_3v_4v_5v_6v_1$. This implies that edges $v_1v_8, v_6v_7, v_2v_9, v_3v_{14}, v_4v_{11}$ do not belong to $F$, which in turn implies that $F$ contains the 6-circuits $v_{13}v_{14}v_{15}v_{16}v_{17}v_{18}v_1$, and $v_7v_8v_9v_{10}v_{11}v_{12}v_7$. Thus $F$ consists of exactly three 6-circuits.

Now, suppose that $|C| = 8$. Then, by inspection, $C$ is either: $v_1v_2v_3v_4v_5v_{16}v_{17}v_{18}v_1$, $v_1v_2v_3v_4v_{11}v_{10}v_{17}v_{18}v_1$, $v_1v_2v_3v_4v_{11}v_{12}v_{13}v_{18}v_1$, or $v_1v_2v_3v_4v_{11}v_{12}v_7v_6v_1$. These in turn, re-

Fig. 1. (a) Heawood $H_0$. (b) Pappus $P_0$. 
isomorphic if and only if

**Proof.**

\[ x \]

respectively, imply that \( v_6, v_9, v_{14}, v_5 \) have degree 1 in \( F \) which is impossible. Thus we cannot have \( |C| = 8 \).

The remaining case, when \( |C| = 18 \), occurs when \( C \) is a hamiltonian circuit of \( P_0 \), which clearly can occur.

In both the cases \( |C| = 6 \) and \( |C| = 18 \), we have \( t(F) = 0 \). Thus \( P_0 \) is pseudo 2-factor isomorphic. It is not 2-factor isomorphic since, by the above, it has two non-isomorphic 2-factors. \( \square \)

### 3.2.1. Star products

We show that \( K_{3,3}, H_0 \) and \( P_0 \) can be used to construct an infinite family of 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graphs.

Let \( G, G_1, G_2 \) be graphs such that \( G_1 \cap G_2 = \emptyset \). Let \( y \in V(G_1) \) and \( x \in V(G_2) \) such that \( d_{G_1}(y) = 3 = d_{G_2}(x) \). Let \( x_1, x_2, x_3 \) be the neighbors of \( y \) in \( G_1 \) and \( y_1, y_2, y_3 \) be the neighbors of \( x \) in \( G_2 \). If \( G = (G_1 - y) \cup (G_2 - x) \cup \{y_1x_1, y_2x_2, y_3x_3\} \), then we say that \( G \) is a *star product* of \( G_1 \) and \( G_2 \) and write \( G = (G_1, y) * (G_2, x) \), or more simply as \( G = G_1 * G_2 \) when we are not concerned which vertices are used in the star product. The set \( \{x_1y_1, x_2y_2, x_3y_3\} \) is a 3-edge cut of \( G \) and we shall also say that \( G_1 \) and \( G_2 \) are *3-cut reductions* of \( G \).

We next show that star products preserve the property of being pseudo 2-factor isomorphic in the family of cubic bipartite graphs.

**Lemma 3.4.** Let \( G \) be a star product of two pseudo 2-factor isomorphic cubic bipartite graphs \( G_1 \) and \( G_2 \). Then \( G \) is also pseudo 2-factor isomorphic.

**Proof.** Suppose \( G = (G_1, y) * (G_2, x) \) with \( x_1, x_2, x_3 \) the neighbors of \( y \) in \( G_1 \) and \( y_1, y_2, y_3 \) the neighbors of \( x \) in \( G_2 \). Suppose further that \( G \) is not pseudo 2-factor isomorphic. Then \( G \) has a 2-factor \( F \) with \( t(F) = 1 \). Since \( G \) is bipartite \( F \) contains exactly two edges of the 3-edge-cut \( S = \{x_1y_1, x_2y_2, x_3y_3\} \). Let \( C \) be the circuit of \( F \) which intersects \( S \) and \( C_i \) be the circuit of \( G_i \) corresponding to \( C \), \( i = 1, 2 \). Let \( F_i \) be the 2-factor of \( G_i \) consisting of the circuits of \( F \) which are contained in \( G_i \) together with \( C_i \). Since \( |C| = |C_1| + |C_2| - 2 \), we have \( 1 = t(F) \equiv t(F_1) + t(F_2) \mod 2 \). Hence \( t(F_i) = 1 \) for some \( i \in \{1, 2\} \). Applying Proposition 3.1, we contradict the hypothesis that \( G_i \) is pseudo 2-factor isomorphic. \( \square \)

Given a set \( \{G_1, G_2, \ldots, G_k\} \) of 3-edge-connected cubic bipartite graphs let \( SP(G_1, G_2, \ldots, G_k) \) be the set of cubic bipartite graphs which can be obtained from \( G_1, G_2, \ldots, G_k \) by repeated star products. Lemma 3.4 implies that all graphs in \( SP(K_{3,3}, H_0, P_0) \) are pseudo 2-factor isomorphic. We conjecture that these are the only 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graphs.

**Conjecture 3.5.** Let \( G \) be a 3-edge-connected cubic bipartite graph. Then \( G \) is pseudo 2-factor isomorphic if and only if \( G \) belongs to \( SP(K_{3,3}, H_0, P_0) \).

Note that McCuaig [10] has shown that a 3-edge-connected cubic bipartite graph \( G \) is det-extremal if and only if \( G \in SP(H_0) \).

Let \( G \) be a graph and \( E_1 \) be an edge-cut of \( G \). We say that \( E_1 \) is a *non-trivial edge-cut* if all components of \( G - E_1 \) have at least two vertices. The graph \( G \) is *essentially 4-edge-connected* if \( G \) is 3-edge-connected and has no non-trivial 3-edge-cuts. It is easy to see that Conjecture 3.5 holds if and only if Conjectures 3.6 and 3.7 below are both valid.
Conjecture 3.6. Let $G$ be an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph. Then $G \in \{K_{3,3}, H_0, P_0\}$.

Conjecture 3.7. Let $G$ be a 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graph and suppose that $G = G_1 \ast G_2$. Then $G_1$ and $G_2$ are both pseudo 2-factor isomorphic.

We will obtain partial results on Conjectures 3.6 and 3.7 in the following two subsections.

3.2.2. Essentially 4-edge-connected cubic bipartite graphs

We show that if $G$ is an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph and $G$ has a 4-circuit then $G = K_{3,3}$. We need the following result of Plummer [11].

Proposition 3.8. (See [11].) Let $G$ be an essentially 4-edge-connected cubic bipartite graph and $e, f$ be independent edges of $G$. Then $\{e, f\}$ is contained in a 1-factor of $G$.

Proposition 3.9. Let $G$ be an essentially 4-edge-connected cubic bipartite graph distinct from $K_{3,3}$, and $C$ be a 4-circuit in $G$. Then $C$ is contained in a 2-factor of $G$.

Proof. Suppose the theorem is false and let $G$ be a counterexample. Let $C = x_1y_2x_3y_4x_1$ and let $y_1, x_2, y_3, x_4$ be the neighbors in $V(G) - V(C)$ of $x_1, y_2, x_3, y_4$ respectively. If $y_1, x_2, y_3, x_4$ were not distinct then the essential 4-edge-connectivity of $G$ would imply that $G = K_{3,3}$. Thus $y_1, x_2, y_3, x_4$ are distinct. By Proposition 3.8, $G$ has a 1-factor $F$ with $\{x_1y_1, x_3y_3\} \subseteq F$. This implies that we must also have $\{x_2y_2, x_4y_4\} \subseteq F$. Thus $G - F$ is a 2-factor of $G$ containing $C$. □

Propositions 3.1(b) and 3.9 immediately imply:

Theorem 3.10. Let $G$ be an essentially 4-edge-connected pseudo 2-factor isomorphic cubic bipartite graph. Suppose $G$ contains a 4-circuit. Then $G = K_{3,3}$.

3.2.3. Cubic bipartite graphs of edge-connectivity three

We present a partial converse of Lemma 3.4. We need the following definition.

Let $G$ be a connected cubic bipartite graph. We say that $G$ is badly behaved if there is an edge $f$ of $G$ with the property that, for every 2-factor $F$ of $G$:

(i) $t(F) = 1$ if and only if $f \in F$;
(ii) if $t(F) = 0$ then each circuit of $F$ has length congruent to two modulo four;
(iii) if $t(F) = 1$ then $F$ has exactly one circuit $C$ of length congruent to zero modulo 4 and $f \in E(C)$.

In this case $f$ is said to be a bad edge of $G$. Note that a badly behaved graph cannot be pseudo 2-factor isomorphic by (i).

We next introduce some additional notation for working with 2-factors. Given a 2-factor $F$ of a graph $G$ containing a vertex $x$ and edge $e$, we use $C_x$ and $C_e$ to denote the circuits of $F$ to which $x$ and $e$ belong. Let $G = (G_1, y) \ast (G_2, x)$ be a cubic bipartite graph with bipartition $(X, Y)$. Let $F_i$ be a 2-factor of $G_i$, $i = 1, 2$. We say that $F_1$ and $F_2$ are compatible 2-factors if for each $j \in \{1, 2, 3\}$, $yx_j \in C_y$ if and only if $yx_j \in C_x$. In this case we define a circuit $C_x \ast C_y$ in $G$ by setting $C_x \ast C_y = (C_y - y) \cup (C_x - x) \cup \{x_jy_j : yx_j \in C_y, j = 1, 2, 3\}$, and a 2-factor $F_1 \ast F_2$
of \( G \) by setting \( F_1 \ast F_2 = (F_1 - C_x) \cup (F_2 - C_x) \cup \{C_x \ast C_y\} \). The 2-factor \( F_1 \ast F_2 \) is said to be the join 2-factor of \( F_1 \) and \( F_2 \). Note that the circuit \( C \) has length \( |C| = |C_x| + |C_y| - 2 \). Using this notation we have the following lemma.

**Lemma 3.11.** Let \( F_i \) be a 2-factor of \( G_i \), \( i = 1, 2 \), such that \( F_1, F_2 \) are compatible. Then \( t(F_1 \ast F_2) = 1 \) if and only if \( t(F_1) \neq t(F_2) \).

**Proof.** It follows from the above definition that \( |C_x \ast C_y| = |C_x| + |C_y| - 2 \). Thus, \( t^*(F_1 \ast F_2) = t^*(F_1) + t^*(F_2) \mod 2 \). Hence, \( t(F_1 \ast F_2) = 1 \) if and only if \( t(F_1) \neq t(F_2) \). \( \square \)

**Theorem 3.12.** Let \( G = (G_1, y) \ast (G_2, x) \) be a cubic bipartite graph with \( x_1, x_2, x_3 \) the neighbors of \( y \) in \( G_1 \) and \( y_1, y_2, y_3 \) the neighbors of \( x \) in \( G_2 \). Then \( G \) is pseudo 2-factor isomorphic if and only if either:

(a) \( G_1, G_2 \) are both pseudo 2-factor isomorphic, or
(b) \( G_1, G_2 \) are both badly behaved and, for some \( i \in \{1, 2, 3\} \), \( xy_i \) is a bad edge of \( G_1 \) and \( xy_i \) is a bad edge of \( G_2 \).

**Proof.** We first assume that (a) or (b) holds. If (a) holds, \( G \) is pseudo 2-factor isomorphic by Lemma 3.4. Hence we may suppose that (b) holds and, relabeling if necessary, that \( yx_1 \) and \( xy_3 \) are bad edges of \( G_1 \) and \( G_2 \), respectively. Let \( F \) be a 2-factor of \( G \). Then \( F = F_1 \ast F_2 \) for 2-factors \( F_1 \) of \( G_1 \) and \( F_2 \) of \( G_2 \). If \( x_3y_3 \notin F \) then \( x_3y \notin F_1 \) and \( xy_3 \notin F_2 \). This implies that \( t(F_1) = 0 = t(F_2) \). Otherwise, if \( x_3y_3 \in F \) then \( x_3y \in F_1 \) and \( xy_3 \in F_2 \). This implies that \( t(F_1) = 1 = t(F_2) \). In both cases \( t(F) = 0 \) by Lemma 3.11. Since the choice of \( F \) was arbitrary, \( G \) is pseudo 2-factor isomorphic.

We next assume that \( G \) is pseudo 2-factor isomorphic. Choose \( j \in \{1, 2, 3\} \) and let \( F_j \), respectively \( F'_j \), be a 2-factor of \( G_j \), respectively \( G'_j \), avoiding \( jy \), respectively \( yj \). Then \( F_j \) and \( F'_j \) are compatible 2-factors and \( F = F_j \ast F'_j \) is a 2-factor of \( G \) avoiding \( jy \). Since \( G \) is pseudo 2-factor isomorphic, Proposition 3.1(b) and Lemma 3.11 imply that \( t(F_j) = t(F'_j) = t_j \), say. It follows that every 2-factor \( X_j \) of \( G_j \) which avoids \( jy \) satisfies \( t(X_j) = t_j \) and every 2-factor \( X'_j \) of \( G'_j \) which avoids \( yj \) satisfies \( t(X'_j) = t_j \). If \( t_1 = t_2 = t_3 \) then \( G_1 \) and \( G_2 \) are both pseudo 2-factor isomorphic and (a) holds. Hence we suppose without loss of generality that \( 1 = t_1 \geq t_2 \geq t_3 = 0 \).

Suppose \( t_2 = 0 \). Let \( L_1, L_2, L_3 \) be a 1-factorization of \( G_1 \), labeled so that \( yx_j \in L_j \) for all \( 1 \leq j \leq 3 \). By Lemma 2.1, \( \text{sgn}(L_1) \text{sgn}(L_2) = (-1)^{t_3} = 1 \), \( \text{sgn}(L_1) \text{sgn}(L_3) = (-1)^{t_2} = 1 \), and \( \text{sgn}(L_2) \text{sgn}(L_3) = (-1)^{t_1} = -1 \). Clearly this is impossible. Hence \( t_2 = 1 \), and thus \( t_3 = 0 \).

Let \( F_j \), respectively \( F'_j \), be a 2-factor of \( G_1 \), respectively \( G'_2 \), avoiding \( jy \), respectively \( jy \). Then \( F = F_j \ast F'_j \) is a 2-factor of \( G \). Since \( G \) is pseudo 2-factor isomorphic, Proposition 3.1(b) implies that all circuits of \( F \) have length congruent to two modulo four. This implies that all circuits of \( F_j \cup F'_j \) other than \( C_y, C_x \) have length congruent to two modulo four. Furthermore, the facts that \( |C_y \cup C_x| = |C_x| + |C_y| - 2 \) have length congruent to two modulo four, \( t_1 = 1 = t_2 \) and \( t_3 = 0 \), imply that \( |C_y| \equiv |C_x| \equiv 0 \mod 4 \) if \( j \in \{1, 2\} \) and \( |C_y| \equiv |C_x| \equiv 2 \mod 4 \) if \( j = 3 \). Thus \( G_1 \) and \( G_2 \) are both badly behaved, \( yx_3 \) is a bad edge of \( G_1 \) and \( xy_3 \) is a bad edge of \( G_2 \). \( \square \)
Theorem 3.12 implies that Conjecture 3.7 is equivalent to the statement that there are no 3-edge-connected badly behaved cubic bipartite graphs. We will see in the next subsection that 2-edge-connected badly behaved cubic bipartite graphs can exist. We close this subsection by showing that a 3-edge-connected badly behaved cubic bipartite graph can have at most one bad edge. This will follow easily from the following lemma, which is a special case of a result of Aldred, Holton, Porteous and Plummer [2, Theorem 3.1].

**Lemma 3.13.** Let $G$ be a 3-edge-connected cubic bipartite graph and $e, f \in E(G)$. Then $G$ has a 1-factor containing $e$ and avoiding $f$.

**Corollary 3.14.** Suppose that $G$ is a badly behaved 3-edge-connected cubic bipartite graph. Then $G$ contains exactly one bad edge.

**Proof.** Suppose $f$ and $f^*$ are distinct bad edges of $G$. By Lemma 3.13, $G$ has a 1-factor $F$ containing $f$ and avoiding $f^*$. Let $X = G - F$. Since $f^* \in X$ we must have $t(X) = 1$ and since $f \notin X$ we must have $t(X) = 0$, a contradiction. □

3.2.3.1. 3-cut reductions

Let $G$ be a cubic bipartite graph with bipartition $(X, Y)$ and $K$ be a non-trivial 3-edge-cut of $G$. Let $H_1, H_2$ be the components of $G - K$. We have seen that $G$ can be expressed as a star product $G = (G_1, y_K) \ast (G_2, x_K)$ where $G_1 - y_K = H_1$ and $G_2 - x_K = H_2$. We say that $y_K$, respectively $x_K$, is the marker vertex of $G_1$, respectively $G_2$, corresponding to the cut $K$. Each non-trivial 3-edge-cut of $G$ distinct from $K$ is a non-trivial 3-edge-cut of $G_1$ or $G_2$, and vice versa. If $G_i$ is not essentially 4-edge-connected for $i = 1, 2$, then we may reduce $G_i$ along another non-trivial 3-edge-cut. We can continue this process until all the graphs we obtain are essentially 4-edge-connected. We call these resulting graphs the constituents of $G$. It is easy to see that the constituents of $G$ are unique i.e. they are independent of the order we choose to reduce the non-trivial 3-edge-cuts of $G$. Furthermore, each vertex of $G$ and each marker vertex belong to a unique constituent of $G$. Let $T(G)$ be the graph whose vertex set is the set of constituents of $G$, in which two vertices are adjacent if the corresponding constituents contain two marker vertices $x_K, y_K$ corresponding to the same non-trivial 3-edge-cut $K$. It is straightforward to check that $T(G)$ is a tree, which we will call the 3-cut reduction tree of $G$. Conjecture 3.5 is equivalent to the statement that if $G$ is a 3-edge-connected pseudo 2-factor isomorphic cubic bipartite graph then every constituent of $G$ is isomorphic to $K_{3,3}, H_0$ or $P_0$.

We can use Theorem 3.10 to deduce some evidence in favor of this statement.

**Theorem 3.15.** Let $G$ be a 3-edge-connected pseudo 2-factor isomorphic bipartite graph. Suppose $G$ contains a 4-cycle $C$. Then $C$ is contained in a constituent of $G$ which is isomorphic to $K_{3,3}$.

**Proof.** It is easy to see that no edge of $C$ can be obtained in a non-trivial 3-edge-cut of $G$. Thus $C$ is contained in a unique constituent $G_1$ of $G$ and no vertex of $C$ is a marker vertex of $G_1$. Suppose $G_1 \neq K_{3,3}$. By Theorem 3.10, $C$ is contained in a 2-factor $F_1$ of $G_1$. It is straightforward to show, as in the proof of Theorem 3.12, that $F_1$ can be extended to a 2-factor $F$ of $G$ with $C \subseteq F$. This contradicts Proposition 3.1(b). □
3.2.4. Cubic bipartite graphs of edge-connectivity two

We shall construct infinite families of 2-edge-connected badly behaved cubic bipartite graphs and 2-edge-connected non-hamiltonian 2-factor isomorphic cubic bipartite graphs.

Let \( G, G_1, G_2 \) be graphs such that \( G_1 \cap G_2 = \emptyset \). Let \( e_i = u_i v_i \in E(G_i) \) for \( i = 1, 2 \). If \( G = (G_1 - e_1) \cup (G_2 - e_2) \cup \{u_1u_2, v_1v_2\} \), then we say that \( G \) is a 2-join of \( G_1 \) and \( G_2 \) and write \( G = (G_1, e_1) \circ (G_2, e_2) \), or more simply \( G = G_1 \circ G_2 \) when we are not concerned which edges are used in the 2-join. The set \( \{u_1u_2, v_1v_2\} \) is a 2-edge cut of \( G \) and we shall also say that \( G_1 \) and \( G_2 \) are 2-cut reductions of \( G \).

Lemma 3.16. Let \( G_i \) be a pseudo 2-factor isomorphic cubic bipartite graph and \( e_i = u_i v_i \in E(G_i) \) for \( i = 1, 2 \). Let \( G = (G_1, e_1) \circ (G_2, e_2) \). Then \( G \) is badly behaved and both \( u_1u_2 \) and \( v_1v_2 \) are bad edges of \( G \).

Proof. The lemma can be proved in a similar way to Lemma 3.4. \( \square \)

Lemma 3.16 can be used to construct an infinite family of badly behaved cubic bipartite graphs of edge-connectivity two, by choosing any \( G_1, G_2 \in SP(K_{3,3}, H_0, P_0) \). The badly behaved graphs \( G \) constructed in this way will all have the property that their bad edges belong to 2-edge-cuts. We can modify the construction to obtain badly behaved graphs without this property. Let \( G_1, G_2 \) be graphs and \( e_i = x_i y_i \in E(G_i) \) for \( i = 1, 2 \). Define \( (G_1, e_1) \circ (G_2, e_2) \) to be the graph consisting of the disjoint union of \( G_1 - e_1 \) and \( G_2 - e_2 \) and two new adjacent vertices \( u, v \) together with the new edges \( uv, x_1 u, y_1 v, x_2 u, y_2 v \). It is straightforward to show that if \( G_1, G_2 \) are pseudo 2-factor isomorphic cubic bipartite graphs then \( (G_1, e_1) \circ (G_2, e_2) \) is badly behaved with \( uv \) as its bad edge.

We next state a similar result to Proposition 3.12 for 2-edge-cuts, which we will use in the following subsection to show that there are no planar pseudo 2-factor isomorphic cubic bipartite graphs.

Lemma 3.17. Let \( G_i \) be a cubic bipartite graph and \( e_i = u_i v_i \in E(G_i) \) for \( i = 1, 2 \). Let \( G = (G_1, e_1) \circ (G_2, e_2) \) and suppose that \( G \) is pseudo 2-factor isomorphic. Then \( G_i \) is pseudo 2-factor isomorphic and \( G_{3-i} \) is badly behaved with \( e_{3-i} \) as a bad edge, for some \( i \in \{1, 2\} \).

Proof. The lemma can be proved in a similar way to Lemma 3.12. \( \square \)

We close this subsection by constructing an infinite family of non-hamiltonian connected 2-factor isomorphic cubic bipartite graphs.

Proposition 3.18. Let \( G_i \) be a 2-factor hamiltonian cubic bipartite graph with \( k \) vertices and \( e_i = u_i v_i \in E(G_i) \) for \( i = 1, 2, 3 \). Let \( G \) be the graph obtained from the disjoint union of the graphs \( G_i - e_i \) by adding two new vertices \( w \) and \( z \) and new edges \( wu_i \) and \( zv_i \) for \( i = 1, 2, 3 \). Then \( G \) is a non-hamiltonian connected 2-factor isomorphic cubic bipartite graph of edge-connectivity two.

Proof. The assertion that \( G \) has edge-connectivity two follows from the fact that connected cubic bipartite graphs are 2-edge-connected. The assertion that \( G \) is non-hamiltonian holds since \( G - \{w, z\} \) has three components.
Let $F$ be a 2-factor of $G$. By symmetry we may assume that $F = F' \cup F_3$, where $F_3$ is a 2-factor of $G_3$ avoiding $u_3v_3$ and $F' = (F_1 - e_1) \cup (F_2 - e_2) \cup \{wu_1, wu_2, zv_1, zv_2\}$ is a 2-factor of $G - G_3$, with $F_i$ a 2-factor of $G_i$ containing $u_iv_i$ for $i = 1, 2$. Since $G_i$ is 2-factor hamiltonian, $F_i$ is a $k$-circuit for $i = 1, 2, 3$. Thus $F$ has exactly two circuits, one of which has length $k$ and the other length $2k + 2$. Hence $G$ is 2-factor isomorphic.

It was shown in [8] that all graphs in $SP(K_{3,3}, H_0)$ are 2-factor hamiltonian. Thus we may apply Proposition 3.18 by taking $G_1 = G_2 = G_3$ to be any graph in $SP(K_{3,3}, H_0)$ to obtain an infinite family of 2-edge-connected non-hamiltonian 2-factor isomorphic cubic bipartite graphs. This family gives counterexamples to the conjecture [1, Conjecture 1.2] that all connected 2-factor isomorphic cubic bipartite graphs are 2-factor hamiltonian. Note, however, that Conjecture 3.5 would imply the truth of the modified conjecture that all 3-edge-connected 2-factor isomorphic cubic bipartite graphs are 2-factor hamiltonian.

### 3.2.5. Planar cubic bipartite graphs

We show that there are no planar pseudo 2-factor-isomorphic cubic bipartite graphs.

**Theorem 3.19.** Let $G$ be a pseudo 2-factor-isomorphic cubic bipartite graph. Then $G$ is non-planar.

**Proof.** Suppose the theorem is false and let $G$ be a counterexample with as few edges as possible. Clearly $G$ is connected, and hence 2-edge-connected. Since $G$ is a planar cubic bipartite graph Euler’s formula implies that $G$ has a face of size four. Thus $G$ contains a 4-circuit. If $G$ were 3-edge-connected then Theorem 3.15 would imply that some constituent of $G$ is isomorphic to $K_{3,3}$. This would contradict the planarity of $G$ since each constituent of $G$ can be obtained by edge-contractions (which preserve planarity). Hence $G$ has edge-connectivity two. Lemma 3.17 now implies that some 2-cut reduction of $G$ is a pseudo 2-factor-isomorphic planar cubic bipartite graph. This contradicts the minimality of $G$. □

**Acknowledgment**

The fifth named author would like to thank Claude Candat for his help while writing this paper.

**References**


