On Cohen–Macaulay Rings of Invariants

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We investigate the transfer of the Cohen–Macaulay property from a commutative ring to a subring of invariants under the action of a finite group. Our point of view is ring theoretic and not a priori tailored to a particular type of group action. As an illustration, we discuss the special case of multiplicative actions, that is, actions on group algebras $k[Z^n]$ via an action on $Z^n$.

Key Words: finite group action; ring of invariants; invariant theory; height; depth; Cohen–Macaulay ring; cohomology; Sylow subgroup.

INTRODUCTION

This article addresses the following question: to what extent does the Cohen–Macaulay property pass from a commutative ring $R$ to a subring $R^G$ of invariants under the action of a finite group $G$ on $R$? As is well known, the Cohen–Macaulay property is indeed inherited by $R^G$ whenever the trace map $tr_G: R \to R^G$, $r \mapsto \sum_{g \in G} g(r)$, is surjective [HE]; see also Corollary 3.2 below. In general, however, the property does not transfer, even in the particular case of linear actions, that is, $G$-actions on polynomial algebras $R = k[X_1, \ldots, X_n]$ by linear substitutions of the variables. The Cohen–Macaulay problem for linear invariants has been rather thoroughly explored without, at present, being anywhere near a final solution.

Our focus in this article will not be on linear $G$-actions on polynomial algebras nor, for the most part, on any other kind of group action on affine algebras over a field. Rather, in Sections 1–4, we work entirely in the setting of commutative noetherian rings. Besides being more general, this approach has resulted in a number of simplifications of results previously obtained by Kemper [Ke$_1$, Ke$_2$] in a geometric setting using geometric

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methods. Nevertheless, the article owes a great deal to Kemper’s insights and originated from a study of his work.

A short outline of the contents is as follows. Section 1 is devoted to relative trace maps. We determine the height of their image, an ideal of $R^G$, and use this result to give a lower bound for the height of annihilators in $R^G$ of certain cohomology classes. Section 2 reviews basic material on Cohen–Macaulay rings and local cohomology. In particular, we describe a pair of spectral sequences constructed by Ellingsrud and Skjelbred [ES]. These quickly yield certain depth estimates. Section 3 develops the main technical tools of this article. We use the Ellingsrud–Skjelbred spectral sequences to derive a depth formula for modules of invariants. This formula underlies virtually all our subsequent applications to the Cohen–Macaulay property of rings of invariants $R^G$. These applications, in the main, concern the case where $R$ has characteristic $p > 0$ and focus on the role played by the Sylow $p$-subgroup of $G$. For the precise statements of our results, we refer the reader to Section 4 where they are presented. The final Section 5 initiates the study of the Cohen–Macaulay property in the special setting of multiplicative actions. These are defined to be $G$-actions on Laurent polynomial algebras $R = k[X_1^\pm, \ldots, X_n^\pm]$ stabilizing the lattice of monomials $\langle X_1, \ldots, X_n \rangle \cong \mathbb{Z}^n$; so we may think of $G$ as a subgroup of $\text{GL}_n(\mathbb{Z})$. We show that if $G$ maps onto some nontrivial $p$-group and has a cyclic Sylow $p$-subgroup, $P$, then $R^G$ is Cohen–Macaulay if and only if $P$ is generated by a bireflection, that is, a matrix $g \in \text{GL}_n(\mathbb{Z})$ so that $g - 1_{n \times n}$ has rank at most 2. In this case, $P$ must have order 2, 3, or 4. A more detailed study of the Cohen–Macaulay property for multiplicative invariants will form the subject of the second author’s Ph.D. thesis.

**Notations and Conventions.** Throughout, $G$ will denote a finite group and $R$ will be a commutative ring (with 1) on which $G$ acts by ring automorphisms, $r \mapsto g(r)$. The subring of $G$-invariant elements of $R$ will be denoted by $R^G$ and the skew group ring of $G$ over $R$ by $RG$. Thus, $RG$ is the free left $R$-module with basis the elements of $G$, made into a ring by means of the multiplication rule $rg \cdot r'g' = rg(r')gg'$ for $r, r' \in R, g, g' \in G$. The ring $R$ is a module over $RG$ via $rg \cdot r' = rg(r')$. All modules are understood to be left modules.

1. THE RELATIVE TRACE MAP

1.1. Throughout this section, $H$ denotes a subgroup of $G$. The relative trace map $\text{tr}_{G/H} : R^H \to R^G$ is defined by

$$\text{tr}_{G/H}(r) = \sum_{g \in G/H} g(r) \quad (r \in R^H).$$
Here, \( g \) runs over any transversal for the cosets \( gH \) of \( H \) in \( G \). Since \( \text{tr}_{G/H} \) is \( R^G \)-linear, the image of \( \text{tr}_{G/H} \) is an ideal of \( R^G \) which we shall denote by \( R^G_H \).

1.2. Covering Primes. The proof of the following lemma was communicated to us by Don Passman. The special case where \( R \) is an affine algebra over a field is covered by [Ke2, Satz 4.7]. As usual, we will write \( ^sH = gh^{-1} \) \((g \in G) \) and \( I_G(\mathfrak{L}) = \{ g \in G | (g - 1)(R) \subseteq \mathfrak{L} \} \) denotes the inertia group of an ideal \( \mathfrak{L} \) of \( R \). We mention that in the geometric context (i.e., when \( R \) is the coordinate ring of an affine variety \( X \) and \( \mathfrak{L} \) comes from a point \( x \in X \)), the inertia group \( I_G(\mathfrak{L}) \) is just the isotropy group \( G_x \).

**Lemma 1.1.** For any prime ideal \( \mathfrak{L} \) of \( R \),

\[
\mathfrak{L} \supseteq R^G_H \Leftrightarrow \left[ I_G(\mathfrak{L}) : I_{H}(\mathfrak{L}) \right] \in \mathfrak{L} \quad \text{for all } g \in G.
\]

**Proof.** The implication \( \Leftarrow \) follows from the straightforward formula

\[
\text{tr}_{G/H}(r) = \sum_{g \in I_G(\mathfrak{L}) \cap G/H} \left[ I_G(\mathfrak{L}) : I_{H}(\mathfrak{L}) \right] g(r) \mod \mathfrak{L}
\]

for all \( r \in R^H \). For \( \Rightarrow \), assume that \( \mathfrak{L} \supseteq R^G_H \). It suffices to show that

\[
\left[ I_G(\mathfrak{L}) : I_{H}(\mathfrak{L}) \right] \in \mathfrak{L}.
\]

Indeed, \( R^G_H = R^G_H \), since \( \text{tr}_{G/H}(r) = \text{tr}_{G/H}(g(r)) \) holds for all \( r \in R^H \) and \( g \in G \).

To simplify notation, put \( I = I_G(\mathfrak{L}) \) and let \( P \) denote a Sylow \( p \)-subgroup of \( I \cap H = I_H(\mathfrak{L}) \), where \( p \geq 0 \) is the characteristic of the commutative domain \( R/\mathfrak{L} \). (Here \( P = \{1\} \) if \( p = 0 \).) Then our desired conclusion, \( [I : I \cap H] \in \mathfrak{L} \), is equivalent to

\[
[I : P] \in \mathfrak{L}.
\]

Furthermore, our assumption \( \mathfrak{L} \supseteq R^G_H \) entails that \( \mathfrak{L} \supseteq R^G_P \), because \( \text{tr}_{G/P} = \text{tr}_{G/H} \circ \text{tr}_{H/P} \). Thus, leaving \( H \) for \( P \), we may assume that \( H = P \) is a \( p \)-subgroup of \( I \). Let \( D = \{ g \in G | g(\mathfrak{L}) = \mathfrak{L} \} \) denote the decomposition group of \( \mathfrak{L} \); so \( I \subseteq D \). We claim that

\[
\mathfrak{L} \supseteq R^P.
\]

To see this, choose \( r \in R \) so that \( r \in g(\mathfrak{L}) \) for all \( g \in G \setminus D \) but \( r \notin \mathfrak{L} \). Then \( s = \prod_{g \in D} g(r) \) also belongs to \( \bigcap_{g \in G \setminus D} g(\mathfrak{L}) \) but not to \( \mathfrak{L} \) and, in addition, \( s \in R^P \). Now assume that, contrary to our claim, there exists an element \( f \in R^P \) so that \( \text{tr}_{D/P}(f) \notin \mathfrak{L} \). Then \( \text{tr}_{D/P}(sf) = s \text{tr}_{D/P}(f) \in \bigcap_{g \in G \setminus D} g(\mathfrak{L}) \setminus \mathfrak{L} \), and hence \( \text{tr}_{G/P}(sf) \notin \mathfrak{L} \), a contradiction.
By the claim, we may replace $G$ by $D$, thereby reducing to the case where $\mathfrak{C}$ is $G$-stable. (Note that $I$ is unaffected by this replacement.) So $G$ acts on $R/\mathfrak{C}$ with kernel $I$, $P$ is a $p$-subgroup of $I$, and $R_G^P \subseteq \mathfrak{C}$. Thus, $0 \equiv \text{tr}_{G/P}(r) \equiv [I:P] \cdot \sum_{g \in G/P} g(r) \mod \mathfrak{C}$ holds for all $r \in R_P$. Our desired conclusion, $[I:P] \subseteq \mathfrak{C}$, will follow if we can show that $0 \equiv \text{tr}_{G/P}(r) \equiv [I:P] \cdot \sum_{g \in G/P} g(s) \mod \mathfrak{C}$ holds for some $r \in R_P$. But $\sum_{g \in G/P} g$ induces a nonzero $G/P$-endomorphism on $R/\mathfrak{C}$, by linear independence of automorphisms of $K = \text{Fract}(R/\mathfrak{C})$; so $0 \equiv \sum_{g \in G/P} g(s) \mod \mathfrak{C}$ holds for some $s \in R$. Putting $r = \prod_{h \in P} h(s)$, we have $r \in R_P$ and $r \equiv s^{[P]} \mod \mathfrak{C}$. Since $|P|$ is 1 or a power of $p = \text{char } K$, we obtain $\sum_{g \in G/P} g(r) = \sum_{g \in G/P} g(s^{[P]}) = (\sum_{g \in G/P} g(s))^{[P]} \neq 0 \mod \mathfrak{C}$, as required. \hfill \[ \square \]

1.3. Height Formula. For any collection $\mathcal{X}$ of subgroups of $G$, we define the ideal $R_{\mathcal{X}}^G$ of $R^G$ by

$$R_{\mathcal{X}}^G = \sum_{H \in \mathcal{X}} R_H^G.$$ 

Inasmuch as $R_D^G \subseteq R_H^G = R_{DH}^G$ holds for all $D \leq H \leq G$ and $g \in G$, there is no loss in assuming that $\mathcal{X}$ is closed under $G$-conjugation and under taking subgroups.

Moreover, for any subgroup $H \leq G$, we define

$$I_R(H) = \sum_{h \in H} (h-1)(R)R.$$ 

Thus, $I_R(H)$ is an ideal of $R$, and $\mathfrak{C} \subseteq I_R(H)$ is equivalent with $H \leq I_{R_0}(\mathfrak{C})$. When $R$ is the coordinate ring of an affine variety $X$ the ideal $I_R(H)$ defines the subvariety $X^H$ of $H$-fixed points in $X$.

**Lemma 1.2.** Assume that $R$ has characteristic $p$, a positive prime, and let $\mathcal{X}$ be a collection of subgroups of $G$ that is closed under $G$-conjugation and under taking subgroups. Then

$$\text{height } R_{\mathcal{X}}^G = \inf \{ \text{height } I_R(P) \mid P \text{ is a } p\text{-subgroup of } G, P \notin \mathcal{X} \}.$$ 

**Proof.** One has

$$\text{height } R_{\mathcal{X}}^G = \inf_{\varpi} \text{height } \varpi = \inf_{\mathfrak{C}} \text{height } \mathfrak{C},$$

where $\varpi$ runs over the prime ideals of $R^G$ containing $R_{\mathcal{X}}^G$ and $\mathfrak{C}$ runs over the primes of $R$ containing $R_{\mathcal{X}}^G$. Here, the first equality is just the definition of height, while the second equality is a consequence of the standard relations between the primes of $R$ and $R^G$; see, e.g., [Bou, Théorème 2, p. 42].
By Lemma 1.1,

\[ \mathfrak{C} \supseteq R^G_{\mathfrak{C}} \iff p \left| \left[ I_G(\mathfrak{C}) : I_H(\mathfrak{C}) \right] \right. \quad \text{for all } H \in \mathfrak{X}. \]

Since \( I_H(\mathfrak{C}) = I_G(\mathfrak{C}) \cap H \) belongs to \( \mathfrak{X} \) for \( H \in \mathfrak{X} \), the latter condition just says that the Sylow \( p \)-subgroups of \( I_G(\mathfrak{C}) \) do not belong to \( \mathfrak{X} \) or, equivalently, some \( p \)-subgroup \( P \leq I_G(\mathfrak{C}) \) does not belong to \( \mathfrak{X} \). Therefore,

\[ \mathfrak{C} \supseteq R^G_{\mathfrak{C}} \iff \mathfrak{C} \supseteq \bigcap_{P \leq G \text{ a } p\text{-subgroup, } P \notin \mathfrak{X}} I_R(P), \]

which implies the asserted height formula.

1.4. Annihilators of Cohomology Classes. Let \( M \) be a module over the skew group ring \( RG \). Then, for each \( r \in R^G \), the map \( \rho: M \to M, m \mapsto rm \), is \( G \)-equivariant, and hence \( \rho \) induces a map on cohomology \( \rho_*: H^*(G, M) \to H^*(G, M) \). Letting \( r \) act on \( H^*(G, M) \) via \( \rho_* \) we make \( H^*(G, M) \) into an \( R^G \)-module.

The following lemma generalizes [Ke, Corollary 2.4].

**Lemma 1.3.** The ideal \( R^G_H \) of \( R^G \) annihilates the kernel of the restriction map \( \text{res}_H^G: H^*(G, M) \to H^*(H, M) \).

**Proof.** The action of \( R^G = H^0(G, R) \) on \( H^*(G, M) \) can also be interpreted as coming from the cup product

\[ H^0(G, R) \times H^*(G, M) \ni (s, x) \mapsto H^*(G, R \otimes_x M) \to H^*(G, M), \]

where the map denoted by \( \cdot \) comes from the \( G \)-equivariant map \( R \otimes_x M \to M, r \otimes m \mapsto rm \); see e.g., [Br, Exercise 1, p. 114]. Furthermore, the relative trace map \( \text{tr}_{G/H}^H: R^H \to R^G \) is identical with the corestriction map \( \text{cor}_{H}^G: H^0(H, R) \to H^0(G, R) \); cf. [Br, p. 81]. Thus, the transfer formula for cup products [Br, (3.8), p. 112] gives, for \( s \in R^H \) and \( x \in H^*(G, M) \),

\[ \text{tr}_{G/H}^H(s) x = \cdot \left( \text{tr}_{G/H}^H(s) \cup x \right) = \cdot \left( \text{cor}_{H}^G(s \cup \text{res}_{H}^G(x)) \right). \]

Therefore, if \( \text{res}_{H}^G(x) = 0 \) then \( \text{tr}_{G/H}^H(s) x = 0 \).

We summarize the material of this section in the following proposition. For convenience, we write \( \text{res}_{p}^G(\cdot) = \cdot_{|_p} \).

**Proposition 1.4.** Assume that \( R \) has characteristic \( p \), and let \( M \) be an \( RG \)-module. Then, for any \( x \in H^*(G, M) \),

\[ \text{height } \text{ann}_{R^G}(x) \geq \inf \{ \text{height } I_R(P) \mid P \text{ a } p\text{-subgroup of } G, x|_p \neq 0 \}. \]
Proof. Let $\mathcal{S}$ denote the splitting data of $x$, that is, $\mathcal{S} = \{H \leq G \mid x|_H = 0\}$; cf. [CoR]. By Lemma 1.3, $\text{ann}_{R^G}(x) \supseteq R^G_x$, and by Lemma 1.2, $\text{height } R^G_x = \inf\{|\text{height } I_R(P)| \mid \text{P is a } p\text{-subgroup of } G, \ x|_P \neq 0\}$. The proposition follows.

2. DEPTH

2.1. In this section, $A$ denotes a commutative noetherian ring, $\alpha$ is an ideal of $A$, and $M$ is a finitely generated module over the group ring $A[G]$.

2.2. Depth and Local Cohomology. Let $H^i_\alpha$ denote the $i$th local cohomology functor with respect to $\alpha$, that is, the $i$th right derived functor of the $\alpha$-torsion functor

$$\Gamma_\alpha(M) = H^0_\alpha(M) = \{m \in M \mid m \text{ is annihilated by some power of } \alpha\}.$$ 

Then

$$\text{depth}(\alpha, M) = \inf\{i \mid H^i_\alpha(M) \neq 0\}$$

(where $\inf \emptyset = \infty$); see [BS, Theorem 6.2.7].

Recall from Section 1.4 (with $A = R^G$) that $H^*(G, M)$ is a module over $A$. Our hypotheses on $A$ and $M$ entail that $M$ is a noetherian $A$-module, and hence so are all $H^*(G, M)$. Therefore,

$$\text{depth}(\alpha, H^*(G, M)) = \inf\{i \mid H^i_\alpha(H^q(G, M)) \neq 0\}.$$


2.3. The Ellingsrud–Skjelbred Spectral Sequences. The above $A$-modules $H^i_\alpha(H^q(G, M))$ feature as the $E^{i,q}_{\alpha}$-terms of a certain spectral sequence due to Ellingsrud and Skjelbred [ES]. In fact, two related spectral sequences are constructed in [ES] in the following manner.

The $\alpha$-torsion functor $\Gamma_\alpha$ and the $G$-fixed point functor $(\cdot)^G = H^0(G, \cdot)$ clearly commute: $\Gamma_\alpha(M^G) = (\Gamma_\alpha(M))^G$. Moreover, if the $A[G]$-module $M$ is injective, then one checks that $\Gamma_\alpha(M)$ is also injective as $A[G]$-module (as in [BS, Proposition 2.1.4]) and $M^G$ is injective as an $A$-module. Therefore, $H^i(G, \Gamma_\alpha(M)) = 0$ and $H^i_\alpha(M^G) = 0$ holds for all $i > 0$ if $M$ is injective. We obtain two Grothendieck spectral sequences converging to

$$H^*_\alpha(G, M) := R^*(\Gamma_\alpha(\cdot)^G)(M) = R^*(\cdot)^G \Gamma_\alpha(M),$$

Furthermore, the spectral sequences (2.1) yield the following estimates for depth \(\alpha, M^G\).

**Lemma 2.1.** (a) **Lower bound:** \(\text{depth}(\alpha, M^G) \geq \min\{\text{depth}(\alpha, M), h_\alpha + 1\}\), where \(h_\alpha = \inf_{q > 0}(q + \text{depth}(\alpha, H^q(G, M)))\).

(b) **Upper bound:** Assume that \(H^p_a(H^q(G, M)) \neq 0\) for some \(p_0 \geq 0, q_0 > 0\) with \(s := p_0 + q_0 < \text{depth}(\alpha, M)\). Assume further that \(H^{p_0 + 1 - j}(H^q(G, M)) = 0\) holds for \(j = 1, \ldots, q_0 - 1\) and \(H^{p_0 + 1 - j}(H^q(G, M)) = 0\) holds for \(j > q_0\). Then \(\text{depth}(\alpha, M^G) \leq s + 1\).

**Proof.** Put \(m = \text{depth}(\alpha, M)\). Then \(H^q_2(M) = 0\) for \(q < m\), and so the \(\delta\)-sequence in (2.1) implies that \(H^p_a(G, M) = 0\) for \(n < m\). Therefore, the \(E\)-sequence satisfies

\[
E^{p,q}_2 = 0 \quad \text{if } p + q < m. 
\]

Furthermore, \(E^{p,0}_2 = H^p_2(M^G)\); so

\[
\text{depth}(\alpha, M^G) = \inf\{p \mid E^{p,0}_2 \neq 0\}. 
\]

Finally,

\[
h_\alpha = \inf\{p + q \mid q > 0, E^{p,q}_2 \neq 0\}. 
\]

To prove (a), assume that \(p < \min(m, h_\alpha + 1)\). Then \(E^{p,0}_2 = 0\), by (2.2), and \(E^{i,j}_2 = 0\) for \(j > 0, i + j < p, r \geq 2\). Recall that the differential \(d_r\) of \(E_r\) has bidegree \((r, 1 - r)\). Thus, \(E^{p,0}_2\) has no nontrivial boundaries and consists entirely of cycles. This shows that \(E^{p,0}_2 = E^{p,0}_1 = \cdots = E^{p,0}_0\), and hence \(E^{p,0}_2 = 0\). Thus, (a) is proved.

For (b), we check that \(E^{s+1,0}_2 \neq 0\). Our hypotheses imply that, at position \((p_0, q_0)\), all incoming differentials \(d_r\) \((r \geq 2)\) are 0 as well as all outgoing \(d_r\) \((r \geq 2, r \neq q_0 + 1)\). Therefore, \(E^{p_0,q_0}_2 = E^{p_0,q_0}_1\) and \(E^{p_0,q_0}_2 = E^{p_0,q_2}_2 = \text{Ker}(d^{p_0,q_0}_2)\). The former implies that \(E^{p_0,q_0_1}_0 \neq 0\), by hypothesis on \((p_0, q_0)\), and the latter shows that \(d^{p_0,q_0+1}_2\) is injective, because \(E^{p_0,q_0}_2 = 0\) by (2.2). Thus, \(d^{p_0,q_0+1}_2\) embeds \(E^{p_0,q_0+1}_2\) into \(E^{s+1,0}_2\), forcing the latter to be nonzero. Hence, \(E^{s+1,0}_2\) is nonzero as well, as desired.

In this article, we will only apply the above estimates in a very limited way, namely with \(p_0 = 0\) in the notation of part (b). This case yields
estimates that could also be derived by other means, e.g., by using Koszul complexes as done by Kemper in [Ke1, Ke2].

2.5. Cohen–Macaulay Rings. For any finitely generated $A$-module $V$, one defines $\dim V = \dim(A/\text{ann}_AV)$ and

$$\text{height}(\alpha, V) = \text{height}(\alpha + \text{ann}_AV/\text{ann}_AV);$$

so $\dim V = \sup_{\alpha \supset \text{ann}_AV} \text{height}(\alpha, V)$. Always,

$$\text{depth}(\alpha, V) \leq \text{height}(\alpha, V);$$

see [BH, Exercise 1.2.22(a)]. The $A$-module $V$ is called Cohen–Macaulay if equality holds for all ideals $\alpha$ of $A$. In order to show that $V$ is Cohen–Macaulay, it suffices to check that $\text{depth}(\alpha, V) \geq \text{height}(\alpha, V)$ holds for all maximal ideals $\alpha$ of $A$ with $\alpha \supset \text{ann}_AV$.

3. MODULES OF INVARIANTS

3.1. Throughout this section, $R^G$ is assumed noetherian and $\alpha$ denotes an ideal of $R^G$. Moreover, $M$ denotes an $RG$-module that is finitely generated as an $RG$-module. Our finiteness assumptions hold, for example, whenever $R$ is an affine algebra over some noetherian subring $k \subset R^G$ and $M$ is a finitely generated $RG$-module; see [Bou, Théorème 2, p. 33].

3.2. The Problem and a Sufficient Condition. Assuming $_RM$ to be Cohen–Macaulay, we are interested in the question under what circumstances $R^GM$ will be Cohen–Macaulay as well. We remark that $_RM$ is Cohen–Macaulay if and only if $_RM^G$ is; see [Ke2, Proposition 1.17].

For future reference, we record the following simple lemma.

**Lemma 3.1.** Assume that $_RM$ is Cohen–Macaulay and that $\sqrt{\alpha} \supset \text{ann}_RM^G$. Then $\text{depth}(\alpha, M) = \text{height}(\alpha, M) \geq \text{height}(\alpha, M^G)$.

**Proof.** Note that $\sqrt{\alpha} \supset \text{ann}_RM^G \supset \text{ann}_RM$ entails that $\text{height}(\alpha, M) \geq \text{height}(\alpha, M^G)$. Further, $\text{height}(\alpha, M) = \text{depth}(\alpha, M)$, because $_RM^G$ is Cohen–Macaulay. The lemma follows.

We now give a sufficient condition for $_RM^G$ to be Cohen–Macaulay. We note that $\dim _RM = \dim _RM^G$, by the familiar relations between the primes of $R$ and of $R^G$.

**Corollary 3.2.** Assume that $_RM$ is Cohen–Macaulay. If $H^q(G, M) = 0$ holds for $0 < q < \dim _RM - 1$ then $_RM^G$ is Cohen–Macaulay as well. In particular, this holds whenever the trace map $\text{tr}_G = \text{tr}_{G/(1)}: R \to R^G$ is surjective.
Proof. Let \( \alpha \) be an ideal of \( R^G \) with \( \alpha \supseteq \text{ann}_{R^G} M^G \). Our hypothesis on \( H^q(G, M) \) entails that the value of \( h_q \) in Lemma 2.1 satisfies \( h_q \geq \dim_R M - 1 \). Also, \( \dim_R M = \dim_{R'} M \geq \text{height}(\alpha, M) \geq \text{height}(\alpha, M^G) \), by Lemma 3.1. Thus, Lemma 2.1(a) gives \( \text{depth}(\alpha, M^G) \geq \text{height}(\alpha, M^G) \), which proves that \( R_{\alpha :} M^G \) is Cohen–Macaulay.

For the last assertion, just note that Lemma 1.3 implies that \( H^q(G, M) \) = 0 holds for all \( q > 0 \) when \( R_{(i)}^G = R^G \). □

Note that the condition \( H^q(G, M) = 0 \) for \( 0 < q < \dim_R M - 1 \) is vacuous for \( \dim_R M \leq 2 \). For \( \dim_R M = 3 \) it becomes \( H^4(G, M) = 0 \). The latter holds, for example, whenever \( M \) is a \( G \)-permutation module without \( |G| \)-torsion; explicitly, as a \( G \)-module, \( M \cong \bigoplus_H (\mathbb{Z}/G) \otimes_{\mathbb{Z}(H)} M(H) \), where \( H \) runs over certain subgroups of \( G \) and each \( M(H) \) is an \( H \)-submodule of \( M^H \) so that \( |H| m = 0 \), \( m \in M(H) \) implies \( m = 0 \).

3.3. Example: Multiplicative Invariants over Cohen–Macaulay Rings. Let \( M = R = k[A] \) be the group ring of a free abelian group \( A \cong \mathbb{Z}^n \) over a Cohen–Macaulay ring \( k \); so \( R \) is Cohen–Macaulay as well. Let \( G \) act on \( A \) by group automorphisms. By \( k \)-linear extension to \( R \) we obtain a \( G \)-action on \( R \) by algebra automorphisms which is indeed a permutation action. Thus, the foregoing implies immediately that \( R^G \) is Cohen–Macaulay when \( n + \dim k \leq 3 \) and \( k \) has no \( |G| \)-torsion. We will return to this type of group action, called multiplicative, in greater detail in Section 5, focusing on the case where \( k \) is a field whose characteristic divides \( |G| \).

3.4. Depth Formula. In view of Corollary 3.2, we may concentrate on the case where \( M \) has non-vanishing positive \( G \)-cohomology. The following proposition is a version of results of Kemper, see [Ke1, Corollary 1.6; Ke2, Kor. 1.18].

**Proposition 3.3.** Assume that \( R M \) is Cohen–Macaulay and that \( \sqrt{\alpha} \supseteq \text{ann}_{R^G} M^G \). Furthermore, assume that, for some \( r \geq 0 \), \( H^q(G, M) = 0 \) holds for \( 0 < q < r \) but \( \alpha x = 0 \) for some \( 0 \neq x \in H^q(G, M) \). Then

\[
\text{depth}(\alpha, M^G) = \min\{r + 1, \text{depth}(\alpha, M)\}.
\]

**Remark.** \( \text{height}(\alpha, M) = \text{depth}(\alpha, M) \) holds in the above formula; see Lemma 3.1.

**Proof of Proposition 3.3.** Our hypothesis \( \alpha x = 0 \) for some \( 0 \neq x \in H^q(G, M) \) is equivalent with \( H^0_{\alpha}(H^q(G, M)) \neq 0 \); so \( \text{depth}(\alpha, H^q(G, M)) = 0 \). The asserted equality is trivial for \( r = 0 \), since \( \text{depth}(\alpha, M^G) = \text{depth}(\alpha, M) = 0 \) holds in this case. Thus we assume that \( r > 0 \). Then, in
the notation of Lemma 2.1, we have \( r = h \), and part (a) of the lemma gives the inequality \( \geq \).

To prove the reverse inequality, note that Lemma 3.1 gives depth(\( \alpha, M \)) \( \geq \) depth(\( \alpha, M^G \)). Therefore, it suffices to show that depth(\( \alpha, M^G \)) \( \leq r + 1 \) if depth(\( \alpha, M \)) \( > r + 1 \). For this, we quote Lemma 2.1(b) with \( p_0 = 0 \) and \( q_0 = r \) (so \( s = r \)).

4. THE SYLOW SUBGROUP OF \( G \)

4.1. In this section, we focus on rings of invariants \( R^G \). If not explicitly mentioned otherwise, \( R \) is assumed to be noetherian as a \( R^G \)-module and to have characteristic \( p \), a positive prime. We let \( P \) denote a Sylow \( p \)-subgroup of \( G \).

4.2. A Necessary Condition. Put

\[
\mu = \mu(G, R) = \inf \{ r > 0 | H^r(G, R) \neq 0 \}.
\]

**Proposition 4.1.** Put \( \mathcal{P} = \{ P' \leq P | \text{height } I_R(P') \leq \mu + 1 \} \). If \( R \) and \( R^G \) are both Cohen–Macaulay and \( \mu < \infty \) then the restriction map

\[
\text{res}^G_{\mathcal{P}}: H^\mu(G, R) \to \prod_{P' \in \mathcal{P}} H^\mu(P', R)
\]

is injective.

**Proof.** Let \( 0 \neq x \in H^\mu(G, R) \) be given and put \( \alpha = \text{ann}_{R^G}(x) \). Then, by Proposition 1.4,

\[
\text{height } \alpha \geq \inf \{ \text{height } I_R(P') | P' \text{ a } p \text{-subgroup of } G, \ x|_{P'} \neq 0 \}.
\]

Since \( R^G \) is Cohen–Macaulay, height \( \alpha = \text{depth } \alpha \). Finally, Proposition 3.3 with \( M = R \) gives depth \( \alpha \leq \mu + 1 \). Thus, there exists a \( p \)-subgroup \( P' \) of \( G \) with \( x|_{P'} \neq 0 \) and height \( I_R(P') \leq \mu + 1 \). Note that both the condition \( x|_{P'} \neq 0 \) and the value of height \( I_R(P') \) are preserved upon replacing \( P' \) by a conjugate \( gP' \) with \( g \in G \). Therefore, we may assume that \( P' \leq P \), which proves the proposition.

4.3. Galois and Almost Galois Actions. Recall that the \( G \)-action on a commutative ring \( R \) is Galois, in the sense of Auslander and Goldman [AG], if every maximal ideal of \( R \) has trivial inertia group in \( G \) or, equivalently, \( I_R(H) = R \) holds for all subgroups \( 1 \neq H \leq G \). Thus, a \( G \)-action on the coordinate ring of an affine variety \( X \) is Galois precisely if the fixed point subvarieties \( X^H \) are empty.
We will say that a $G$-action on $R$ is almost Galois if height $I_R(H) \geq \dim R$ holds for all $1 \neq H \leq G$ (where height $R = \infty$). In the geometric setting, this means that the fixed point subvarieties $X^H$ are finite. For linear actions, $X$ is a nonzero vector space on which $G$ acts linearly; so each $X^H$ is a subspace. Thus, linear actions are never Galois (when $G \neq 1$), and they are not even almost Galois in the modular case (i.e., when $P \neq 1$). Similarly, multiplicative actions (cf. Section 3.3) are never Galois but they are often almost Galois; see Section 5.2. Part (a) of the following proposition holds for any commutative ring $R$; it is included for the sake of completeness.

**Proposition 4.2.** (a) If the $G$-action on $R$ is Galois then $R^G$ is Cohen–Macaulay if and only if $R$ is.

(b) Assume that $R$ is Cohen–Macaulay and that the action of the Sylow $p$-subgroup $P$ on $R$ is almost Galois. Then $R^G$ is Cohen–Macaulay if and only if $\dim R \leq \mu + 1$.

**Proof.** (a) By [CHR, Lemma 1.6 and Theorem 1.3], the trace map $\text{tr}_G : R \to R^G$ is surjective for Galois actions and $R$ is finitely generated projective as an $R^G$-module. Thus, $R$ is faithfully flat as an $R^G$-module. Moreover, for any prime $\mathfrak{p}$ of $R$, the fibre $R_{\mathfrak{p}}/(\mathfrak{p} \cap R^G)R_{\mathfrak{p}}$ has dimension 0. Therefore, by [BH, 2.1.23], $R^G$ is Cohen–Macaulay if and only if $R$ is.

(b) The implication $\Leftarrow$ follows from Corollary 3.2 with $M = R$. For the converse, let $R^G$ be Cohen–Macaulay and assume, without loss, that $\mu < \infty$. Then Proposition 4.1 implies that there is a subgroup $1 \neq P' \leq P$ with height $I_R(P') \leq \mu + 1$. On the other hand, by hypothesis on the $P$-action, height $I_R(P') \geq \dim R$; so $\dim R \leq \mu + 1$.

**4.4. Example: Affine Actions on Polynomial Algebras.** A $G$-action on the polynomial algebra $R = k[X_1, \ldots, X_n]$ over a field $k$ is called affine if $G$ stabilizes the subspace $L = k + \sum_kX_j$ of polynomials of degree at most 1. Since $G$ acts trivially on $k$, the quotient $V = L/k$ inherits a $k[G]$-module structure. Let $\gamma \in \text{Ext}_{k[G]}(V, k)$ denote the extension class of $0 \to k \to L \to V \to 0$. We claim

the $G$-action on $R$ is Galois if and only if the restrictions

$\gamma|_H \in \text{Ext}_{k[H]}(V, k)$ are nonzero for all subgroups

$1 \neq H \leq G$.

Indeed, if $\gamma|_H = 0$ for some $H$ then $H$ acts linearly on $R$, and hence the action won’t be Galois. Conversely, assume that all $\gamma|_H$ are nonzero. Then $G$ is a $p$-group, where $p = \text{char } k$. In order to show that all $I_R(H) = R$, it suffices to consider cyclic subgroups $1 \neq H = \langle h \rangle$. But then the restriction
map $\text{Ext}_{k[H]}(V, K) \to \text{Ext}_{k[H]}(V^H, k)$ can be identified with the canonical map $\text{ann}_{V^*} \sum h/(h - 1)V^* \to (V^H)^* \cong V^*/(h - 1)V^*$, and hence it is injective. Thus, $\gamma|_H$ has a nonzero image in $\text{Ext}_{k[H]}(V^H, k)$, that is, $0 \neq h(l) - l \in k$ holds for some $l \in L$. Since $h(l) - l \in I_R(H)$, we conclude that $I_R(H) = R$, as desired.

In the special case where the $G$-action on $V$ is trivial, $\text{Ext}_{k[G]}(V, k)$ can be identified with $\text{Hom}(G, V^*)$ and the above Galois condition just says that $\gamma$ is an injection $G \to V^*$. Geometrically, this case corresponds to an action of $G$ on $\text{Hom}(R, k) \cong V^*$ by translations, $g(f) = f - \gamma(g)$.

Returning to general affine actions on $R$, we remark that if the action of the Sylow $p$-subgroup $P$ is almost Galois then it is actually Galois. For, as we have remarked above, no nonidentity subgroup of $P$ can act linearly, and so $\gamma$ must restrict nontrivially to all these subgroups. In this case, the $P$-trace $\text{tr}_P$ is surjective, and hence so is $\text{tr}_G$ which in turn entails that $\mu = \infty$. Hence Proposition 4.2(b) says

$$\gamma|_H \neq 0 \text{ for all subgroups } 1 \neq H \leq P \Rightarrow R^G \text{ is Cohen-Macaulay.}$$

4.5. Bireflections. Following [Ke$_2$], we will call an element $g \in G$ a bireflection on $R$ if height $I_R(\langle g \rangle) \leq 2$.

**Corollary 4.3.** Assume that $R$ and $R^G$ are Cohen-Macaulay. Let $H$ denote the subgroup of $G$ that is generated by all $p'$-elements of $G$ and all bireflections in $P$. Then $R^G = R^G_H$.

**Proof.** First note that $H$ is a normal subgroup of $G$ and $G/H$ is a $p$-group. Thus, if $R^G \neq R^G_H$ or, equivalently, $\ Henri(G/H, R^H) \neq 0$ then also $H^1(G/H, R^H) \neq 0$; see [Br, Theorem VI.8.5]. In view of the exact sequence

$$0 \to H^1(G/H, R^H) \to H^1(G, R) \xrightarrow{\text{res}_H^G} H^1(H, R)$$

(see [Ba, 35.3]) we further obtain $H^1(G, R) \neq 0$. Thus, $\mu = 1$ holds in Proposition 4.1 and every $P' \in \mathcal{P}$ consists of bireflections. Therefore $P' \subseteq H$ and Proposition 4.1 implies that $\text{res}_H^G : H^1(G, R) \to H^1(H, R)$ is injective, contradicting the above exact sequence. Therefore, we must have $R^G = R^G_H$. $\blacksquare$

In case $\mathbb{F}_p = \mathbb{Z} \cdot 1_R$ is a $G$-module direct summand of $R$, the conclusion $R^G = R^G_H$ of the above corollary is equivalent with $G = H$. In this form, the corollary is essentially [Ke$_2$, Korollar 4.10] (at least for affine $R$).

4.6. The Case $|P| = p$. Put

$$\mu_p(G) = \mu(G, \mathbb{F}_p) = \inf\{r > 0 \mid H^r(G, \mathbb{F}_p) \neq 0\}.$$
We will determine this number in the case where the order of \( G \) is
divisible by \( p \) but not by \( p^2 \); in other words, \(|P| = p\). As usual \( \mathbb{N}_G(P) \) and
\( C_G(P) \) will denote the normalizer and the centralizer, respectively, of \( P \) in
\( G \). Thus, \( \mathbb{N}_G(P)/C_G(P) \) is a subgroup of \( \text{Aut}(P) = \text{Aut}(\mathbb{Z}/p) \cong \mathbb{F}_p^* \), and hence it is cyclic of order dividing \( p - 1 \).

**Corollary 4.4.** Assume that \(|P| = p\). Then \( \mu_p(G) = 2[\mathbb{N}_G(P) : C_G(P)] - 1 \). Moreover, if \( \mathbb{F}_p \) is a \( G \)-module direct summand of \( R \) and \( R \) and \( R^G \) are both Cohen–Macaulay then height \( I_{\mathbb{F}}(P) \leq 2[\mathbb{N}_G(P) : C_G(P)] \).

**Proof.** Put \( N = \mathbb{N}_G(P), C = C_G(P), \) and \( r = 2[\mathbb{N} : C] - 1 \). In order to prove that \( \mu_p(G) = r \), we use the fact that \( H^*(G, \mathbb{F}_p) \cong H^*(P, \mathbb{F}_p)^{N/C} \) holds for \( \ast > 0 \); see [Be, Corollary 3.6.19]. If \( p = 2 \) then \( N = C \) and so \( r = 1 \). Moreover, \( H^*(P, \mathbb{F}_p)^{N/C} \cong H^*(\mathbb{Z}/2, \mathbb{F}_2) \) equals \( \mathbb{F}_2 \) in all degrees.

This proves the assertion for \( p = 2 \); so we assume \( p \) odd from now on. In
this case, \( H^*(\mathbb{Z}/p, \mathbb{F}_p) \cong \mathbb{F}_p[v_1, b_2]/(v_1^2, v_1b_2 - b_2v_1) \) with \( \deg v_1 = 1 \) and
\( \deg b_2 = 2 \); see [AM, Corollary II.4.2]. Moreover, identifying \( \text{Aut}(\mathbb{Z}/p) \) with \( \mathbb{F}_p^* \), the action of \( \text{Aut}(\mathbb{Z}/p) \) on \( H^*(\mathbb{Z}/p, \mathbb{F}_p) \) becomes scalar multiplication, \( v_1 \mapsto \phi v_1, b_2 \mapsto \phi b_2 \), where \( \phi \in \mathbb{F}_p^* \). Taking \( \phi \) to be a generator for the subgroup of \( \mathbb{F}_p^* \) corresponding to \( N/C \), we see that

\[
H^*(P, \mathbb{F}_p)^{N/C} \cong \bigoplus_{i \geq 0} \mathbb{F}_p[v_1 b_2^{[N/C]}] \oplus \bigoplus_{i > 0} \mathbb{F}_p v_1 b_2^{[N/C]} - 1;
\]

see [AM, pp. 104–105]. The smallest positive degree where \( H^*(P, \mathbb{F}_p)^{N/C} \) does not vanish is therefore indeed \( 2([N : C] - 1) + 1 = r \).

Now assume that \( \mathbb{F}_p \) is a \( G \)-module direct summand of \( R \) and \( R \) and \( R^G \) are both Cohen–Macaulay. The former hypothesis implies that \( H^*(G, R) \neq 0 \) and hence \( \mu \leq r \). Moreover, our hypothesis on \(|P|\) implies that \( \mathcal{P} \ni P \) holds in Proposition 4.1, because otherwise \( \mathcal{P} \) would consist of the identity subgroup alone. Therefore, height \( I_{\mathbb{F}}(P) \leq \mu + 1 \leq r + 1 \), as desired.

In the case of a linear action, the upper bound for height \( I_{\mathbb{F}}(P) \) given in
the above corollary also follows from [Ke3, Theorem 3.1].

4.7. The Non-Cohen–Macaulay Locus. We finish this section by recording an elementary observation independent of the local cohomology methods used thus far and valid for any commutative ring \( R \).

By definition, the non-Cohen–Macaulay locus of \( R^G \) consists of those prime ideals \( \mathfrak{q} \) of \( R^G \) so that the localization \( (R^G)_\mathfrak{q} \) is not Cohen–Macaulay. Thus, \( R^G \) is Cohen–Macaulay if and only if its non-Cohen–Macaulay locus is empty. Here, we point out a general bound for the non-Cohen–Macaulay locus in terms of relative trace maps. More detailed results for affine algebras over a field can be found in [Ke2, Kapitel 5].

Recall the notation \( R^G_{\mathcal{P}} \) from Section 1.3.
**Proposition 4.5.** Let $\mathcal{H}$ denote the set of subgroups $H$ of $G$ so that $R^H$ is Cohen–Macaulay. Then, for every prime ideal $\mathfrak{p}$ of $R^G$ so that $\mathfrak{p} \not\supseteq R^G_{\mathfrak{p}^G}$, the localization $(R^G)_{\mathfrak{p}}$ is Cohen–Macaulay.

**Proof.** By hypothesis, $\mathfrak{p} \not\supseteq R^G_H$ for some $H \in \mathcal{H}$. Let $R_\mathfrak{p}$ denote the localization of $R$ at the multiplicative subset $R^G \setminus \mathfrak{p}$. Then the $G$-action on $R$ extends to $R_\mathfrak{p}$ and $(R_\mathfrak{p})^G = (R^G)_{\mathfrak{p}}$. Similarly, $(R_\mathfrak{p})^H = (R^H)_{\mathfrak{p}}$, so $(R_\mathfrak{p})^H$ is Cohen–Macaulay. By choice of $\mathfrak{p}$ the relative trace map $\text{tr}_{G/H}: (R_\mathfrak{p})^H \rightarrow (R_\mathfrak{p})^G$ is onto. Fix an element $c \in (R_\mathfrak{p})^H$ so that $\text{tr}_{G/H}(c) = 1$ and define $\rho: (R_\mathfrak{p})^H \rightarrow (R_\mathfrak{p})^G$ by $\rho(x) = \text{tr}_{G/H}(cx)$. This map is a “Reynolds operator,” i.e., $\rho$ is $(R_\mathfrak{p})^G$-linear and restricts to the identity on $(R_\mathfrak{p})^G$. Since $(R_\mathfrak{p})^H$ is integral over $(R_\mathfrak{p})^G$, a result of Hochster and Eagon [HE; BH, Theorem 6.4.5] implies that $(R_\mathfrak{p})^G$ is Cohen–Macaulay, which proves the proposition. 

As an application, we note that if $G$ has subgroups $H_i$ so that each $R^H_i$ is Cohen–Macaulay and the indices $[G : H_i]$ are coprime in $R^G$ then $R^G$ is Cohen–Macaulay as well. Indeed, writing $1 = \sum [G : H_i] r_i$ with $r_i \in R^G$, we obtain $1 = \sum \text{tr}_{G/H_i}(r_i) \in R^G_{\mathfrak{p}^G}$; so the non-Cohen–Macaulay locus of $R^G$ is empty.

## 5. Multiplicative Actions

5.1. In this section, we focus on a particular type of group action often called multiplicative actions. These arise from $G$-actions on lattices $A \cong \mathbb{Z}^n$ by extending the action $k$-linearly to the group algebra $R = k[A] \equiv k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$. Here, we assume $k$ to be a field such that $p = \text{char } k$ divides the order of $G$; otherwise the invariant subalgebra $R^G$ would certainly be Cohen–Macaulay because $R$ is; see Proposition 4.5. There is no loss in assuming $G$ to be faithfully embedded in $\text{GL}(A) \cong \text{GL}_n(\mathbb{Z})$, and we will do so. The above notations will remain valid throughout this section.

5.2. The action of $G$ on $R = k[A]$ is almost Galois (see Section 4.3) if and only if $G$ acts fixed-point-freely on $A$, that is, no $1 \neq g \in G$ has an eigenvalue 1 on $A$. Furthermore, an element $g \in G$ is a bireflection on $R$ (Section 4.5) if and only if the endomorphism $g - 1 \in \text{End}(A) \cong M_n(\mathbb{Z})$ has rank at most 2. Both these observations are consequences of the following

**Lemma 5.1.** For any subgroup $H \leq G$, height $I_R(H) = n - \text{rank } A^H$.

**Proof.** By definition, the ideal $I_R(H)$ of $R$ is generated by the elements $h(a) - a = (h(a)a^{-1} - 1)a$ for $h \in H$, $a \in A$. Thus, $R/I_R(H) \cong$
\( k[A/\langle H, A \rangle] \), where we have put \( [H, A] = \langle h(a)a^{-1} \mid h \in H, a \in A \rangle \leq A \).

Consequently, height \( I_R(H) = \dim R - \dim R/I_R(H) = n - \text{rank } A/[H, A] \). Finally, since the group algebra \( \mathbb{Q}[H] \) is semisimple, \( A \otimes \mathbb{Q} = (A^H \otimes \mathbb{Q}) \oplus ([H, A] \otimes \mathbb{Q}) \); so rank \( A/[H, A] = \text{rank } A^H \).

5.3. Since \( G \) permutes the \( k \)-basis of \( R \), the Eckmann–Shapiro Lemma [Br, VI 5.2] implies that

\[
H^*(G, R) \cong \bigoplus_{a \in A/G} H^*(G_a, k),
\]

where \( G_a \) denotes the isotropy group of \( a \) in \( G \). In particular, using the notations of Sections 4.2 and 4.6, we have

\[
\mu = \inf_{a \in A} \mu_p(G_a). \tag{5.1}
\]

5.4. Example: Inversion. Let \( G = \langle g = -1_{n \times n} \rangle \) act on \( R = k[X_1^\pm, \ldots, X_n^\pm] \) via \( g(X_i) = X_i^{-1} \). This action is fixed-point-free on \( A = \mathbb{Z}^n \).

Moreover, assuming \( p = 2 \), we have \( \mu = \mu_2(G) = 1 \) by (5.1). Therefore, Proposition 4.2(b) gives

\( R^G \) is Cohen–Macaulay if and only if \( n \leq 2 \).

5.5. Example: Reflection Groups. An element \( g \in G \) is called a reflection on \( R \) if height \( I_R(\langle g \rangle) \leq 1 \) or, equivalently, if the endomorphism \( g - 1 \in \text{End}(A) \cong M_n(\mathbb{Z}) \) has rank at most 1; see Lemma 5.1. If \( G \) is generated by reflections then \( R^G \) is an affine normal semigroup algebra over \( k \); see [Lo1]. Therefore, \( R^G \) is Cohen–Macaulay in this case, for any field \( k \); see [BH, Theorem 6.3.5]. This is in contrast with the situation for finite group actions on polynomial algebras by linear substitutions of the variables, where (modular) reflection groups need not lead to Cohen–Macaulay invariants [Nak].

5.6. Cyclic Sylow Subgroups. As before, we let \( P \) denote a fixed Sylow \( p \)-subgroup of \( G \). Moreover, \( O^p(G) \) denotes the intersection of all normal subgroups \( N \) of \( G \) so that \( G/N \) is a \( p \)-group.

**THEOREM 5.2.** Assume that \( O^p(G) \neq G \) and that \( P \) is cyclic. Then \( R^G \) is Cohen–Macaulay if and only if \( P \) is generated by a bireflection. In this case, \( P \) has order 2, 3, or 4.

**Proof.** Our hypothesis \( O^p(G) \neq G \) is equivalent with \( \mu_p(G) = 1 \); so \( \mu = 1 \) holds as well, by (5.1). Assuming \( R^G \) to be Cohen–Macaulay, Corollary 4.3 and the subsequent remark imply that \( G = H \). Since all \( p \)-elements of \( G \) belong to \( O^p(G) \), it follows that \( G/O^p(G) = P/P \cap O^p(G) \) is generated by the images of the bireflections in \( P \). Since \( P \) is cyclic, it follows that \( P \) is generated by a bireflection. Now, \( P \) acts faithfully on the lattice \( A/A^p \) of rank at most 2. Thus, \( P \) is isomorphic to
a cyclic $p$-subgroup of $GL_2(\mathbb{Z})$, and these are easily seen to have orders 2, 3, or 4.

The converse follows from the more general lemma below which does not depend on cyclicity of $P$ or nontriviality of $G/O^p(G)$. 

**Lemma 5.3.** If $\text{rank } A/A^p \leq 2$ then $R^G$ is Cohen–Macaulay.

**Proof.** By Proposition 4.5, it suffices to show that $R^p$ is Cohen–Macaulay; so we may assume that $G = P$ is a $p$-group. Note that $G$ acts faithfully on $A/A^G$. If $G$ acts as a reflection group on $A$ then it does so on $A$ as well, and hence the invariants $R^G$ will be Cohen–Macaulay; see Section 5.5. Thus we may assume that $A$ has rank 2 and $G$ acts on $A \cong \mathbb{Z}^2$ as a non-reflection $p$-group. By the well-known classification of finite subgroups of $GL_2(\mathbb{Z})$ (e.g., [Lo, 2.7]), this leaves the cases $G \cong \mathbb{Z}/q$ with $q = 2, 3, or 4$ to consider.

The cases $q = 2 or 3$ can be dealt with along similar lines. Indeed, for both values of $q$, the only indecomposable $G$-lattices, up to isomorphism, are $\mathbb{Z}$, $\mathbb{Z}[G]$, and $\mathbb{Z}[G]/(\hat{G})$, where $\hat{G} = \sum_{g \in G} g$; see [CR, Exercise 4, p. 514/5]. Thus, $A \cong \mathbb{Z}^m \oplus (\mathbb{Z}[G]/(\hat{G}))/\mathbb{Z}[G]$, and $R^G \cong k[B]^G[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$, where we have put $B = (\mathbb{Z}[G]/(\hat{G}))/\mathbb{Z}[G]$. Since $R^G$ is Cohen–Macaulay if and only if $k[B]^G$ is, we may assume that $m = 0$.

Now, $A \cong (\mathbb{Z}[G]/(\hat{G}))^{r+s}$; so $2 = (r+s)|G| - 1$. When $n = 3$, this leads to either $r = 1, s = 0$ or $r = 0, s = 1$. In the former case, rank $A = 2$ and so $R^G$ is surely Cohen–Macaulay, being a normal domain of dimension 2. If $r = 0, s = 1$ then $A$ is a $G$-permutation lattice of rank 3. Hence, $R = k[A]$ is a localization of the symmetric algebra $S(A \otimes k)$, and likewise for the subalgebras of invariants. Since linear invariants of dimension $\leq 3$ are known to be Cohen–Macaulay (e.g., [Ke]), $R^G$ is Cohen–Macaulay in this case as well. For $n = 2$, there are three cases to consider, one of which ($r = 2, s = 0$) leads to an invariant algebra of dimension 2 which is clearly Cohen–Macaulay. Thus, we are left with the possibilities $r = 1, s = 1$, and $r = 0, s = 2$. Explicitly, after an obvious choice of basis, $G$ acts as one of the following groups on $A$:

**Case 1.**

$$G_1 = \langle g_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle.$$  

**Case 2.**

$$G_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
For $G$, $A \cong \mathbb{Z}^4$ is a permutation lattice. Hence, as above, it suffices to check that the linear invariant algebra $S(V)^G$ for $V = A \otimes k$ is Cohen–Macaulay which is indeed the case, by [ES], since $\dim V/V^G = 2$.

For $G_1$, one can proceed as follows: Embed $G_1$ into $\Gamma = \langle g_1, \text{diag}(-1, 1, 1) \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ and denote the corresponding basis of $A \cong \mathbb{Z}^3$ by $\{x, y, z\}$; so $g_1(x) = x^{-1}$, $g_1(y) = z$, and $g_1(z) = y$. One easily checks that $R^\Gamma = k[\xi, \alpha_1, \sigma_2]$, where $\xi = x + x^{-1}$, $\alpha_1 = y + z$, and $\sigma_2 = yz$. Furthermore, $R = k[A] = R^\Gamma \otimes xR^\Gamma \otimes yR^\Gamma \otimes xyR^\Gamma$. With this, the invariant subalgebra $R^{G_1}$ is easily determined; the result (for char $k = 2$) is $R^{G_1} = R^G \otimes (xy + x^{-1}z)R^\Gamma$ which is indeed Cohen–Macaulay. This completes the proof for $G \cong \mathbb{Z}/2$ or $\cong \mathbb{Z}/3$.

We now sketch the remaining case, $G = \langle g \rangle \cong \mathbb{Z}/4$. The action on $\mathcal{A} = A/A^G$ can then be described by $G \bar{x} = \langle (0, -1) \rangle$; so $\mathcal{A} \cong \mathbb{Z}[G]/(g^2 + 1)$. With this, one calculates $\text{Ext}_G(\mathcal{A}, \mathcal{A}) \cong \mathbb{Z}/2$. Thus, there is exactly one (up to isomorphism) non-split extension of $G$-modules $0 \to \mathcal{A} \to U \to \mathcal{A} \to 0$. A suitable module $U$ is $U = \mathcal{A}[G]/(g - 1)(g^2 + 1)$. Furthermore, one calculates $\text{Ext}_G(U, \mathcal{A}) = 0$. Consequently, either $A \cong A^G \otimes \mathcal{A}$ or $A \cong \mathbb{Z}^m \oplus U$, and hence either $R^G \cong k[\mathcal{A}][A^G]$ which is Cohen–Macaulay because $k[A]^G$ has dimension 2, or $R^G \cong k[U]^G[X_1^\pm 1, \ldots, X_n^\pm 1]$ which is Cohen–Macaulay precisely if $k[U]^G$ is. This reduces the problem to the case where $A = U$ which can be handled by direct calculation, taking advantage of the fact that a conjugate of group $G_1$ is contained in $G$. We leave the details to the reader.

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