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ABSTRACT

In this note we study the constraints on F-theory GUTs with extra $U(1)$'s in the context of elliptic fibrations with rational sections. We consider the simplest case of one abelian factor (Mordell–Weil rank one) and investigate the conditions that are induced on the coefficients of its Tate form. Converting the equation representing the generic hypersurface P_{112} to this Tate's form we find that the presence of a $U(1)$, already in this local description, is consistent with the exceptional \mathcal{E}_6 and \mathcal{E}_7 non-abelian singularities. We briefly comment on a viable $\mathcal{E}_6 \times U(1)$ effective F-theory model.

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1. Introduction

It has been by now widely accepted that additional $U(1)$ or discrete symmetries constitute an important ingredient in GUT model building. Such symmetries are useful to prevent dangerous superpotential couplings of the effective field theory model, in particular those inducing proton decay operators and lepton number violating interactions at unacceptable rates. Model building in the context of string theory has shown that such symmetries are naturally incorporated in the emerging effective field theory model. In the context of F-theory [1] in particular, the last few years several GUT symmetries have been analysed with the presence of additional $U(1)$ factors [2].²

In F-theory models the non-abelian part of the gauge group is determined by specific geometric singularities of the internal manifold. The internal space is an elliptically fibred Calabi–Yau (CY) fourfold Y_4 , over a three-fold base B_3 . The fibration is determined by the Weierstraß model

$$y^2 = x^3 + f(\xi)xz^4 + g(\xi)z^6 \quad (1)$$

where the base of the fibration corresponds to the point of the torus $z \rightarrow 0$ and as such it defines a zero section at $[x : y : z] = [t^2 : t^3 : 0]$. For particularly restricted f, g functions the fiber degenerates over certain points of the base. The non-abelian singularities of the fiber are well known and have been systematically classified with respect to the vanishing order of the functions f, g and

the roots of the discriminant of (1), by Kodaira [15]. An equivalent description useful for local model building is also given by Tate [16,17]. There are $U(1)$ symmetries however which do not emerge from a non-abelian singularity and as such they do not fall into the category of a Cartan subalgebra. There is no classification for such $U(1)$ symmetries analogous to the non-abelian case and up to now they have not been fully explored. Abelian factors correspond to extra rational sections and as such they imply additional restrictions on the form of the functions f, g . Because sections are given in terms of divisors whose intersection points with the fiber should be distinct and not identifiable by any monodromy action, this can occur only for rational intersection points. Therefore, for such points of an elliptic curve fibred over B_3 , their corresponding degree line bundle has a section that vanishes at these points.

It is known that rational points on elliptic curves constitute a group, the so called Mordell–Weil group. The Mordell–Weil group is finitely generated in the sense that there exists a finite basis which generates all its elements [18]. A finitely generated group can be written as

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathcal{G}$$

where \mathcal{G} is the torsion subgroup, which in principle could be a source for useful discrete symmetries in the effective Lagrangian. Recent developments in F-theory have analysed some properties of the latter and its implications on effective field theory models. The rank of the abelian group is the rank of the Mordell–Weil group [19,20], however, the latter is not known. Up to now, studies with one, two and three extra sections have appeared and some general implications on the low energy models have been accounted for [21–33].

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² For an incomplete list see [3–10], the reviews [11–14] and references therein.

In this note we argue that the appearance of extra sections has significant implications on the engineering of non-abelian gauge symmetries based on the local Tate form of the model. In particular, in the case of local constructions based on the simple Tate's algorithm the rational sections impose certain restrictions on the defining equation. When the latter is converted to the familiar local Tate's form, in order to meet the requirements of the extra rational section, certain relations among Tate's form coefficients occur. We will see that such constraints make impossible the appearance of familiar groups such as $SU(5)$ in the local Tate form. To our knowledge, this issue has not been observed, and it might constitute another obstruction on the validity of simple Tate's algorithm similar to those observed in reference [32]. Such obstructions can be evaded in more general models based on the 'top' constructions of toric geometry [34]. Using the latter techniques, $SU(5)$ models with several Mordell–Weil $U(1)$'s have been built [21–33]. However, in this note we show that in the context of the familiar local Tate's algorithm, viable effective models based on the exceptional singularities can be still easily accommodated.

Therefore, it is the purpose of this note to examine the aforementioned constraints and discuss the implications in the effective theory. As a “test ground”, we consider in particular the simplest case of two sections, i.e., one extra section in addition to the universal one and since abelian factors are related to extra sections, this means that the GUT symmetry will be supported by an extra $U(1)$. Given the existence of one extra section, we re-derive the constraints on the Weierstraß model written in Tate's form. Investigating the relations of the coefficients we find that there are basically two viable GUT symmetries, namely E_6 and E_7 supplemented by the extra abelian factor. We briefly discuss the spectrum of the model $E_6 \times U(1)$.

2. Case of two rational points

To set the stage, we recapitulate in this section some relevant results derived in [20]. In fact, we re-consider thoroughly the derivation of the Weierstraß equation from the $P_{(1,1,2)}$ fibration with two rational sections. As a result, in the process of converting the initial form we find a second solution which is distinct from the first one with respect to the signs of the coefficients in Tate's model.

We consider an elliptic curve \mathcal{E} over a field \mathcal{K} , a point P associated to the holomorphic (zero) section, a rational point Q , and denote $\mathcal{M} = \mathcal{O}(P + Q)$ the corresponding line bundle of degree 2. From the Riemann–Roch theorem for genus one curves, we know that the number of global sections of a line bundle \mathcal{M} is equal to its degree, $h^0(\mathcal{M}) = d$. Because in our case $d = 2$, the group $H^0(\mathcal{M})$ must have two sections which we call them u and v with weights equal to 1. Considering now $H^0(2\mathcal{M})$, it can be seen that a new section w with weight 2 is required, so that the three weights are $[u, v, w] = [1, 1, 2]$. Further, from u, v, w one can form six sections of degree 6 which match exactly the number of independent sections of $H^0(3\mathcal{M})$, while all possible sections corresponding to $H^0(4\mathcal{M})$ that can be constructed are nine, exceeding the independent ones by one. Hence there has to be a constraint among them which defines a hyper-surface in the weighted projective space $P_{(1,1,2)}$ given by the equation which relates them

$$w^2 + a_0 u^2 w + a_1 u v w + a_2 v^2 w = b_0 u^4 + b_1 u^3 v + b_2 u^2 v^2 + b_3 u v^3 + b_4 v^4 \quad (2)$$

with a_i, b_j coefficients in \mathcal{K} .

One of the sections corresponds to the universal one so it vanishes at the two points P, Q . We can take this to be the u section and therefore Eq. (2) at these points becomes

$$w^2 + a_2 v^2 w = b_4 v^4 \quad (3)$$

The roots of the equation correspond to the points P, Q and since these are rational points the equation should split in two factors, with all coefficients in the field \mathcal{K} . To avoid square roots we may redefine $\tilde{w} = w + \zeta v^2$, $\tilde{a}_2^2 = a_2^2 + 4b_4$ with $2\zeta = a_2 - \tilde{a}_2$ and write this equation as $\tilde{w}^2 + \tilde{a}_2 \tilde{w} v^2 = 0$. Renaming $\tilde{w} \rightarrow w$ for simplicity, we get

$$w(w + a_2 v^2) = 0$$

whose roots are the points P, Q

$$[u : v : w] = [0 : 1 : 0] \quad \text{and} \quad [u : v : w] = [0 : 1 : -a_2]$$

With this redefinition, we can eliminate the term $b_4 v^4$ in the original equation (2), while similar reasoning allows us to set $a_0 = a_1 = 0$. Under the aforementioned circumstances the original equation reads³

$$w^2 + a_2 v^2 w = u(b_0 u^3 + b_1 u^2 v + b_2 u v^2 + b_3 v^3) \quad (4)$$

To recover the Weierstraß form with global section associated to P , one has to find sections $H^0(kP)$. Since from group structure this is $H^0(kM - kQ)$ one has to look for $H^0(kM)$ vanishing k times at Q .

Starting with $k = 1$, we have already assumed that the section u vanishes at P, Q and thus one can set $u := z$. For $k = 2$ one section is u^2 while the other must be a linear combination of all possible degree-2 sections. Let

$$w = \gamma u^2 + \beta u v + \alpha v^2$$

Substituting in Eq. (2) while organising in powers of u , we get

$$(\beta^2 + \gamma(2\alpha + a_2) - b_2)u^2 + (\beta(2\alpha + a_2) - b_3)u + \alpha(\alpha + a_2)$$

The vanishing of the coefficients of zeroth and first order powers in u above, gives the solutions

$$\alpha = -a_2, \quad \beta = -\frac{b_3}{a_2}$$

and

$$\alpha = 0, \quad \beta = \frac{b_3}{a_2}$$

Therefore, (setting $\gamma = 0$ since section u^2 has already been included) we can have two possible forms of the section x given by

$$x = b_3 u v + a_2 w + a_2^2 v^2$$

$$x = b_3 u v - a_2 w \quad (5)$$

To find y we examine $H^0(3M)$. In general we expect another combination of the form

$$w = \mu u^2 + \lambda u v + \kappa v^2$$

We substitute as before, and demand vanishing of the coefficients up to second order in u :

$$\kappa(a_2 + \kappa) = 0, \quad \lambda(a_2 + 2\kappa) - b_3 = 0$$

$$\mu(a_2 + 2\kappa) - b_2 + \lambda^2 = 0$$

³ Notice that the singularity is resolved by blowing up $w \rightarrow sw$ and $u \rightarrow su$ so that

$$sw^2 + a_2 v^2 w = u(b_0 u^3 s^3 + b_1 u^2 s^2 v + b_2 u s v^2 + b_3 v^3).$$

Again, we obtain two distinct solutions which imply two forms of y :

$$y = a_2^3 v^3 + a_2^2 v w + a_2 b_2 u^2 v - \frac{b_3^2 u^2 v}{a_2} + a_2 b_3 u v^2$$

$$y = a_2^2 v w - a_2 b_2 u^2 v + \frac{b_3^2 u^2 v}{a_2} - a_2 b_3 u v^2 \quad (6)$$

To recover the Weierstraß form of the original equation, we must invert the equations of $x(u, v, w)$, $y(u, v, w)$ and substitute them into the original equation. One can observe that both sets of x, y solutions lead to the same Weierstraß form. For the first solution

$$v = \frac{a_2 y}{a_2^2 (b_2 u^2 + x) - b_3^2 u^2}$$

$$w = -\frac{a_2^3 y^2}{(b_3^2 u^2 - a_2^2 (b_2 u^2 + x))^2} + \frac{b_3 u y}{b_3^2 u^2 - a_2^2 (b_2 u^2 + x)} + \frac{x}{a_2}$$

$$u = z \quad (7)$$

while, inverting the second solution for x, y we obtain

$$v = \frac{a_2 y}{b_3^2 u^2 - a_2^2 (b_2 u^2 + x)}$$

$$w = \frac{b_3 u y}{b_3^2 u^2 - a_2^2 (b_2 u^2 + x)} - \frac{x}{a_2}$$

$$u = z \quad (8)$$

These lead to the Weierstraß equation in Tate's form

$$y^2 + 2 \frac{b_3}{a_2} x y z \pm b_1 a_2 y z^3$$

$$= x^3 \pm \left(b_2 - \frac{b_3^2}{a_2^2} \right) x^2 z^2 - b_0 a_2^2 x z^4 - b_0 a_2^2 \left(b_2 - \frac{b_3^2}{a_2^2} \right) z^6 \quad (9)$$

with the upper signs corresponding to the first case and the lower ones to the second solution.

Defining the functions

$$f = b_1 b_3 - a_2^2 b_0 - \frac{b_3^2}{3}$$

$$g = b_0 b_3^2 + \frac{1}{12} a_2^2 (3 b_1^2 - 8 b_0 b_2) + \frac{2}{27} b_3^3 - \frac{1}{3} b_1 b_3 b_2 \quad (10)$$

we may also write down the compact Weierstraß form of the latter, which is just the form given in (1).

3. Constraints on gauge group structure of the effective model

After this short review we proceed with the investigation of the obtained Weierstraß form. The main point we wish to stress is that in the specific form given above, the coefficients satisfy certain relations and therefore are strongly constrained. In this work we restrict our analysis to Weierstraß equation given by the original Tate's algorithm [16,17].⁴ Since the specific type of the non-abelian singularity depends on the form of these coefficients, these aforementioned relations are expected to impose restrictions on the gauge group of the effective theory. However, before abandoning the simple Tate algorithm, it is worth considering whether there are viable GUT symmetries left over to accommodate the Standard Model gauge group. To see this, we should compare (9) with the general Tate form given by

Table 1

Tate's algorithm for the most common non-abelian groups [16,17]. Table shows the gauge group, the order of vanishing of the coefficients $\alpha_k \sim a_{k,n} \xi^n$, the discriminant Δ and the corresponding singularity type.

Group	α_1	α_2	α_3	α_4	α_6	Δ	Type
$SU(2n)$	0	1	n	n	$2n$	$2n$	I_{2n}^3
$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$	I_{2n+1}^3
$SO(10)$	1	1	2	3	5	7	I_1^{*5}
E_6	1	2	2	3	5	8	IV^{*5}
E_7	1	2	3	3	5	9	III^{*5}
E_8	1	2	3	4	5	10	II^5

$$y^2 + \alpha_1 x y z + \alpha_3 y z^3 = x^3 + \alpha_2 x^2 z^2 + \alpha_4 x z^4 + \alpha_6 z^6 \quad (11)$$

Comparing the two equations, we can extract the relations

$$\alpha_1 = \pm 2 \frac{b_3}{a_2}$$

$$\alpha_2 = b_2 - \frac{b_3^2}{a_2^2}$$

$$\alpha_3 = \pm b_1 a_2$$

$$\alpha_4 = -b_0 a_2^2$$

$$\alpha_6 = -\left(b_2 - \frac{b_3^2}{a_2^2} \right) b_0 a_2^2 \quad (12)$$

Inspecting these equations, we can easily observe that the following relation holds among the coefficients

$$\alpha_6 = \alpha_2 \alpha_4 \quad (13)$$

Notice now that each of the coefficients can be represented locally by an expansion in the 'normal' coordinate ξ

$$\alpha_n(\xi) = \alpha_{n,0} + \alpha_{n,1} \xi + \dots$$

As is well known, the type of the geometric singularity associated to the non-abelian gauge group is determined by the vanishing order of the coefficients $\alpha_n(\xi)$ with respect to ξ . For the most common non-abelian symmetries these data are summarised in Table 1.

We can examine now whether a relation of the form (13) can be fulfilled.

• From the first row of the table we can read off the relations of the coefficients for the $SU(2n)$ case. Indeed, the vanishing order of a_2 is one, thus we may write $a_2 = a_{2,1} \xi$, meaning that $a_{2,1}$ has a constant part plus possible ξ -dependent terms. Similarly, in the same notation we write $a_4 = a_{4,n} \xi^n$ and $a_6 = a_{6,2n} \xi^{2n}$. Hence,

$$\alpha_2 \alpha_4 \propto \alpha_{2,1} \alpha_{4,n} \xi^{n+1}, \quad \alpha_6 \propto \alpha_{6,2n} \xi^{2n}$$

therefore the equation $a_2 a_4 = a_6$ now reads

$$\alpha_{2,1} \alpha_{4,n} \xi^{n+1} = \alpha_{6,2n} \xi^{2n} \Rightarrow n = 1$$

i.e., it is satisfied for $n = 1$, corresponding to the $SU(2)$ group.

• For the $SU(2n+1)$ groups we have

$$\alpha_2 \alpha_4 \propto \alpha_{2,1} \alpha_{4,n+1} \xi^{n+1}, \quad \alpha_6 \propto \alpha_{6,2n+1} \xi^{2n+1}$$

therefore the equation yields

$$\alpha_{2,1} \alpha_{4,n+1} \xi^{n+1} = \alpha_{6,2n+1} \xi^{2n+1} \Rightarrow n = 1$$

which is satisfied for $n = 1$ implying an $SU(3)$ group.

The above analysis shows that, in the context of Tate's form for the $P_{(1,1,2)}$ case and the simple mapping to Weierstraß model

⁴ A generalisation of these results can be found in [32].

Table 2

The vanishing order of the coefficients $b_k \sim b_{k,n}\xi^n$, of Eq. (4) for the \mathcal{E}_6 and \mathcal{E}_7 models.

Group	a_2	b_0	b_1	b_2	b_3
\mathcal{E}_6	1	1	1	2	2
	0	3	1	2	1
\mathcal{E}_7	1	1	2	2	2
	0	3	3	2	1

$P_{(1,2,3)}$ described in Section 2, the only groups compatible with the constraints of one additional rational section are $SU(3)$ and $SU(2)$. Extending our investigation to $SO(n)$ singularities, we infer that, if we restrict to the lower bounds on the vanishing orders of the coefficients $\alpha_n(\xi)$ in Tate's algorithm, the most common GUT groups such as $SU(5)$ and $SO(10)$ are not accommodated. To resolve this issue a more detailed treatment is required and a non-minimal version of the coefficients should be sought to meet these conditions. In fact, such GUT models can appear within the so called 'top' constructions of toric geometry, which have been studied in [21–33]. Recently, the implementation of the latter technique was shown to give rise to explicit constructions of various codimension one singularities. However, we stress in this note that the familiar local Tate's forms are not completely excluded. Indeed, repeating the analysis for the exceptional groups, we will find out immediately, that the required criteria are fulfilled by two of them.

- For \mathcal{E}_6 we have

$$\alpha_2\alpha_4 \propto \alpha_{2,2}\alpha_{4,3}\xi^5, \quad \alpha_6 \propto \alpha_{6,5}\xi^5$$

i.e., the ξ powers match and therefore we only need to impose the equality constraint

$$\alpha_{2,2}\alpha_{4,3} = \alpha_{6,5}$$

Once this condition is satisfied, we also need to check the remaining coefficients constrained by Eqs. (12). To investigate these relations, we express all coefficients in terms of a_2 . Assuming that the latter is given in terms of an unspecified power of the coordinate, $a_2 \propto \xi^n$, we find that a consistent solution exists in accordance with

$$\begin{aligned} b_0 &= -\alpha_{43}\xi^{3-2n}, & b_1 &= \alpha_{32}\xi^{2-n} \\ b_2 &= (a_{22} + a_{11}^2/2)\xi^2, & b_3 &= (a_{11}/2)\xi^{n+1} \end{aligned} \quad (14)$$

Requiring the b_0 coefficient to be a positive power in ξ we see that this leaves two possibilities for the integer n , namely $n = 0, 1$.

Substituting (14) into Eqs. (12) we find

$$\begin{aligned} \alpha_1 &= \alpha_{11}\xi, & \alpha_2 &= \alpha_2\xi^2, & \alpha_3 &= \alpha_{32}\xi^2 \\ \alpha_4 &= \alpha_{43}\xi^3, & \alpha_6 &= \alpha_{65}\xi^5 \end{aligned}$$

As can be checked in Table 2 this is just the requirement to obtain an \mathcal{E}_6 singularity. We compute the discriminant to find

$$\Delta = -27\alpha_{32}^4\xi^8 + \mathcal{O}(\xi^9)$$

which, as expected has vanishing order 8.

- Repeating the analysis of the \mathcal{E}_7 case, we end up with the conditions on b_i 's listed in the corresponding rows of Table 2. Here, compared to the previous case, we require also the vanishing of the coefficient α_{32} so that $\alpha_3 = \alpha_{3,3}\xi^3$. It is also straightforward to see that $\Delta \propto \xi^9$ in accordance with Table 1. Finally, notice that for the \mathcal{E}_8 case, the condition $a_2a_4 = a_6$ cannot be fulfilled.

4. $\mathcal{E}_6 \times U(1)$

From the previous analysis, we have seen that in the presence of an additional rational section which is associated to an extra $U(1)$ symmetry, as long as the minimal requirements on α_n of Table 1 are implemented, the available non-abelian groups compatible with the restrictions are $SU(3)$, $SU(2)$ and the \mathcal{E}_6 and \mathcal{E}_7 . From these, only the exceptional groups are adequate to include the complete gauge symmetry of the SM.

The \mathcal{E}_6 model has been extensively analysed in the literature. In the present context the corresponding effective model is based on the extended gauge group

$$G_{GUT} = \mathcal{E}_6 \times U(1)$$

In the resulting effective theory all available matter is included in 78 and 27 representations. We can reduce the gauge symmetry down to the Standard Model using appropriate $U(1)$ fluxes. We can reach the properties of the representations by successive decompositions of the \mathcal{E}_6 representations. The decomposition $\mathcal{E}_6 \rightarrow SO(10) \times U(1)_y$ gives

$$78 \rightarrow 45_0 + 16_{-3} + \overline{16}_3 + 1_0$$

$$27 \rightarrow 16_1 + \overline{10}_{-2} + 1_4$$

Under $SO(10) \rightarrow SU(5) \times U(1)_x$ the non-trivial representations obtain the following quantum numbers

$$45_0 \rightarrow 24_{(0,0)} + 10_{(4,0)} + \overline{10}_{(-4,0)} + 1_{(0,0)} \quad (15)$$

$$16_{-3} \rightarrow 10_{(-1,-3)} + \overline{5}_{(3,-3)} + 1_{(-5,-3)} \quad (16)$$

$$\overline{10}_{-2} \rightarrow 5_{(2,-2)} + \overline{5}_{(-2,-2)} \quad (17)$$

and analogously for the other representations, while the $SU(5)$ singlet emerging from 27 is $1_{(0,4)}$.

Observe that 10, 5's of $SU(5)$ emerge from 27 as well as 78 so it is possible to accommodate families in both. In the simplest scenario the third family fermions and the Higgs fields reside in $27_q, 27_{q'}$. To write down superpotential terms of the effective model, we need the charges q, q' under the Mordell–Weil $U(1)$. This computation is rather involved and goes beyond the scope of this short note. However, in analogy with $SU(5)$ models, we might expect a solution where the allowed charges are multiples of $1/3$ so that a tree level coupling of the form could be allowed

$$27_{\frac{1}{3}} 27_{\frac{1}{3}} 27_{-\frac{2}{3}} \rightarrow 10_M 10_M 5_{h_u} + 10_M \overline{5}_M \overline{5}_{h_d} \rightarrow m_t, m_b \quad (18)$$

As indicated, this is suitable to derive the top and bottom quark entries, while higher order terms involving powers of the 78-representation can give higher order contributions to the fermion masses of the lighter generations

$$(78 + 78^2) 27_{\frac{1}{3}} 27_{\frac{1}{3}} 27_{-\frac{2}{3}} \rightarrow m_{u_{ij}}, m_{d_{ij}} \quad (19)$$

A detailed analysis of the \mathcal{E}_6 F-theory models is beyond the scope of this note and can be found in [35].

5. Conclusions

In this note we investigated constraints on GUTs in F-theory compactifications with an extra rational section which corresponds to an additional abelian factor in the gauge group of the final effective theory model. Elliptic fibrations with two sections can be represented by a quartic polynomial of definite form written in terms of three homogeneous coordinates in the ambient space $P_{(1,1,2)}$. Converting the quadratic equation to a local Tate from we

find that the Tate coefficients are subject to constraints which restrict the number of non-abelian gauge groups that can be realised in the local Tate form. Models emerging in this context which can accommodate the Standard Model gauge symmetry are based on $\mathcal{E}_6 \times U(1)$ and $\mathcal{E}_7 \times U(1)$. We discuss briefly the salient features of the $\mathcal{E}_6 \times U(1)$ case.

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