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INTEGRATION TECHNIQUES FOR ISOPARAMETRIC AND HIGHER ORDER BASES ON FINITE ELEMENTS WITH A CURVED SIDE

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Abstract—Efficient integration techniques are developed for a class of integrals over finite elements bounded by two straight sides and a parabolic arc. The techniques can be used to speed up the evaluation of the element matrices for both high order transformation bases and for isoparametric bases.

1. INTRODUCTION

There has been considerable interest in curved-sided finite elements in recent years. Among the best known of the techniques for dealing with such elements is undoubtedly the isoparametric transformation method [1]. However, while functions linear in the coordinates of the problem domain are accurately interpolated by the isoparametric method, functions which are quadratic in the problem coordinates are only approximately interpolated whenever one or more sides are curved. The fact that this is true even when the isoparametric basis functions are of high degree in their local coordinates has motivated a search for higher order bases which will accurately represent second or higher degree polynomials in the problem coordinates. A number of techniques [2–5] proposed for producing high order bases have been developed from the first order direct (as opposed to transformation) methods considered in Refs. [6–8]. There are two wide classes of high order bases for curved elements. These are the rational bases [2] and the bases produced by what has been called the High Order Transformation (HOT) method [3]. The rational basis is direct in that the individual basis functions are defined in the coordinates of the problem, i.e. no local transformation to some standard shape is made. In contrast the HOT method employs a local transformation which may or may not be the same as that used by the isoparametric method. Both classes of high order methods use the same geometric considerations in their construction, possess almost identical properties, and are general enough to produce high order bases for a very wide class of elements. For the element bounded by two straight sides and a conic arc a second order rational basis consists of functions which are ratios of a cubic to a linear polynomial, while the functions in a HOT basis can be chosen to be polynomials in the local coordinates. But in both cases the coefficients of the monomial terms are functions of the nodal positions. This is in contrast to the isoparametric technique where the basis functions in the local coordinates are simple polynomials unaffected by nodal positions. There can be no doubt, therefore, that the high order basis functions are more complicated to describe than their isoparametric counterparts.

One of the major components of any finite element calculation is likely to be the evaluation of a large number of integrals over individual element domains. Each of the integrals may be written as a linear combination of integrals whose integrands are products of the element's basis functions and/or derivatives of basis functions. The fact that the coefficients in the high order bases depend explicitly on the nodal coordinates greatly complicates the evaluation of the integrals and until recently has deterred the implementation of high order methods. Initial results on the performance of an HOT basis in the solution of test problems reported by

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McLeod and Murphy[9] gave sufficient encouragement to examine more closely the question of the integral evaluation when using this type of basis. The integrals can be evaluated using numerical quadrature, but then accuracy is achieved only with considerable computational expense.

For the purpose of comparing the computational speeds attainable with HOT and isoparametric bases, a three-sided element with one curved side and two straight sides has been chosen. The quadratic isoparametric basis for such an element has six nodes and approximates the element boundaries by parabolic arcs. Since in most cases such arcs provide good approximations to curved element boundaries, we adopt this element shape to use with both the isoparametric and HOT bases. The second order HOT basis for this element has eight nodes.

The specific objectives of this study are to develop and compare integration techniques for a six-node isoparametric basis and for an eight-node second-order basis where the elements are bounded by two straight sides and a parabolic arc. In both cases computation times are considerably reduced by making use of linear relations among the integrals and limiting the use of numerical quadrature. A count of floating point operations indicates that the evaluation of an element stiffness matrix for a generalized Laplace equation involves less computation for the eight-node basis than for the corresponding six-node isoparametric basis if the isoparametric computation is performed in a conventional manner. However, with the improved method of evaluation for the isoparametric basis presented herein, we can conclude only that the computation times per stiffness coefficient (i.e. per element stiffness matrix component) are approximately equal.

It is expected that the techniques presented herein will be useful in the evaluation of a wide variety of integrals and for a wide variety of isoparametric and high order bases.

2. THE ELEMENT

The element under consideration is shown in Fig. 1. The user specifies the geometry of the element by providing the coordinates of the three vertex nodes (the nodes numbered 1, 2 and 3) and, in a suitable way, specifying the locus of the parabolic arc. One of the many ways in which the parabolic arc can be specified[10, 11] is by also providing the position of node 4 somewhere on the curved side. The positions of the remaining nodes are then automatically determined, nodes 5 and 6 as midpoints on the straight sides and nodes 7 and 8 by the prescription which follows. The accuracy with which the resulting parabolic arc approximates the original curve depends on the position chosen for node 4. To deal with this problem one may use the prescription given in Ref. [10] by which two points on the original curve are selected. From these points the position of node 4 (now not necessarily a point on the original curve) is determined.

Three coordinate systems are used for discussing the element. The x,y- (alias x^1,x^2- or $\underline{x}$-) coordinates are the coordinates of the problem domain. The l,m- (alias l^1,l^2- or $\underline{l}$-) coordinates are local coordinates related to the $\underline{x}$-coordinates by the linear transformation

$$x^\mu = S^{\mu\nu} l^\nu + x_1{}^\mu \quad (\mu,\nu = 1,2) \tag{1}$$

where $(x_1{}^1,x_1{}^2)$ are the coordinates of node 1, the vertex which joins the two straight sides, and the constants $S^{\mu\nu}$ are determined by the requirement that the second and third nodes have $\underline{l}$-coordinates equal to (1,0) and (0,1), respectively. The index summation convention is used on

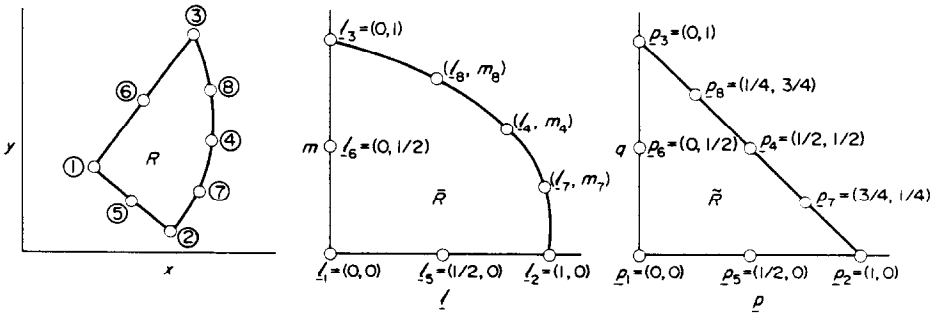

Fig. 1. Three coordinate systems used in the description of the six-node isoparametric and eight-node HOT bases. Nodes 7 and 8 are to be ignored for the six-node isoparametric basis.

the Greek indices throughout this paper since their range is always from 1 to 2. The p,q- (alias p^1,p^2- or $\underline{p}$-) coordinates are a second set of local coordinates and are related to the $\underline{l}$-coordinates by the nonlinear transformation

$$l = p(1+\alpha q); \qquad m = q(1+\beta p) \tag{2}$$

where α and β are determined by the requirement that the $\underline{p}$-coordinates of node 4 are (1/2, 1/2). Consequently

$$\alpha = 2(2l_4 - 1); \qquad \beta = 2(2m_4 - 1) \tag{3}$$

where $(l_4, m_4) = (l_4^1, l_4^2)$ are the $\underline{l}$-coordinates of node 4. The nonlinear transformation which relates the $\underline{x}$- and $\underline{p}$-spaces (as given by eqns 1–3) is the usual isoparametric transformation. Since the transformation from $\underline{x}$- to $\underline{l}$-spaces is linear, we can, without loss of generality, work in the $\underline{l}$- and $\underline{p}$-spaces and apply the linear transformation of eqn (1) as the last step in the process. This greatly simplifies the calculations and basis definitions. The $\underline{p}$-coordinates of nodes 7 and 8 are somewhat arbitrarily chosen as (3/4, 1/4) and (1/4, 3/4), respectively. This choice, together with eqns (2) and (3) determines their $\underline{l}$-coordinates, and, finally, with eqn (1) their $\underline{x}$-coordinates. Thus nodes 7, 4 and 8 all lie on a parabolic arc approximating the original curve. The Jacobian of the linear transformation from the $\underline{l}$- to $\underline{x}$-coordinates is given by

$$S = \frac{\partial(x, y)}{\partial(l, m)} = S^{11}S^{22} - S^{12}S^{21}, \tag{4}$$

and the Jacobian from the $\underline{p}$- to $\underline{l}$-coordinates is given by

$$\frac{\partial(l, m)}{\partial(p, q)} = 1 + \alpha q + \beta p. \tag{5}$$

3. BASIS FUNCTIONS

The quadratic isoparametric basis functions are a well-known set of six second degree polynomials $W^i(\underline{p})$ which satisfy

$$W^i(\underline{p}_j) = \delta^i_j \quad (i, j = 1, \ldots, 6) \tag{6}$$

and hence

$$\sum_{i=1}^{6} W^i(\underline{p}) = 1. \tag{7}$$

The coefficients in the expressions for the $W^i(\underline{p})$ are independent of the nodal coordinates. The basic isoparametric relation written in $\underline{l}$-space has the form

$$\sum_{i=1}^{6} l_i^{\mu} W^i(\underline{p}) = l^{\mu} \tag{8}$$

which is just another form of eqn (2).

Preliminary to the definition of the second order HOT basis, a set of six second degree polynomials in $\underline{l}$-space is required. These polynomials, $T^i(\underline{l})$, are given by

$$T^i(\underline{l}_j) = \delta^i_j \quad (i, j = 1, \ldots, 6) \tag{9}$$

where (l_j^1, l_j^2) are the $\underline{l}$-coordinates of the jth node. This implies that

$$\begin{aligned}
T^1 &= 1 - 3(l+m) + 2(l^2+m^2) - \frac{(2m_4^2 - 3m_4 + 2l_4^2 - 3l_4 + 1)}{l_4 m_4} lm \\
T^2 &= -l\left(1 - 2l - \frac{1-2l_4}{m_4} m\right); \qquad T^3 = -m\left(1 - 2m - \frac{1-2m_4}{l_4} l\right); \\
T^4 &= \frac{lm}{l_4 m_4}; \qquad T^5 = 4l\left(1 - l - \frac{(1-l_4)}{m_4} m\right); \qquad T^6 = 4m\left(1 - m - \frac{(1-m_4)}{l_4} l\right).
\end{aligned} \tag{10}$$

The set of T^i is a generalization of the set of isoparametric basis functions W^i referred to in eqns (6), (7) and (8) in that the W^i may be obtained from the T^i by replacing (l, m, l_4, m_4) in eqn (10) by $(p, q, 1/2, 1/2)$. Because the six nodes $1, 2, \ldots, 6$ do not lie on a conic, the $T^i(\underline{l})$ are linearly independent and hence span the set of polynomials of degree two or less in the $\underline{l}$-coordinates. This is crucial to the basis definition because a linear transformation relates the $\underline{x}$- and $\underline{l}$-systems and the $T^i(\underline{l})$ must also span polynomials of degree two or less in the $\underline{x}$-coordinates. The definition of this set of simple functions $T^i(\underline{l})$ in terms of l, m also facilitates the calculation of the derivatives of basis functions. Since the $T^i(\underline{l})$ span polynomials of degree two, we have the relations

$$\sum_{i=1}^{6} T^i(\underline{l}) = 1; \qquad \sum_{i=1}^{6} l_i^{\mu} T^i(\underline{l}) = l^{\mu}; \qquad \sum_{i=1}^{6} l_i^{\mu} l_i^{\nu} T^i(\underline{l}) = l^{\mu} l^{\nu}. \tag{11}$$

We note, however, that the set T^i is unsuitable as a basis because as such it would be nonconforming.

Let $P^7(\underline{l})$ be the linear form which takes the value zero at nodes 4 and 8 and value unity at node 7. Similarly, let $P^8(\underline{l})$ be the linear form which takes the value zero at nodes 4 and 7 and value one at node 8. Because of the relation between $\underline{l}$ and $\underline{p}$ given by eqn (2), the quantities P^7 and P^8 (as well as the T^i) may be regarded as polynomials in $\underline{p}$. We will therefore economize in our notation by writing the basis functions as functions of p and q only, even though they are indeed functions of the dependent set of variables l, m, p and q. The second order HOT basis is then given by

$$W^7(\underline{p}) = (16/3)pqP^7(\underline{l}) \tag{12}$$

$$W^8(\underline{p}) = (16/3)pqP^8(\underline{l}) \tag{13}$$

$$W^i(\underline{p}) = T^i(\underline{l}) - T_7^i W^7(\underline{p}) - T_8^i W^8(\underline{p}) \qquad (i = 1, \ldots, 6) \tag{14}$$

where T_j^i is defined by

$$T_j^i = T^i(\underline{l}_j) \qquad (i = 1, \ldots, 6; j = 7, 8). \tag{15}$$

By this construction[3] we have a conforming second order basis which satisfies

$$W^i(\underline{p}_j) = \delta_j^i \qquad (i, j = 1, \ldots, 8) \tag{16}$$

$$\sum_{i=1}^{8} W^i(\underline{p}) = 1; \qquad \sum_{i=1}^{8} l_i^{\mu} W^i(\underline{p}) = l^{\mu}; \qquad \sum_{i=1}^{8} l_i^{\mu} l_i^{\nu} W^i(\underline{p}) = l^{\mu} l^{\nu}, \tag{17}$$

i.e. the basis is a normalized set of functions which spans polynomials of degree two or less in l, m—and hence also in x, y. Equations (8), (11) and (17) may also be written in $\underline{x}$-space, but use of the $\underline{l}$-space forms given here results in a more efficient computer implementation.

Though both the isoparametric and HOT basis functions are polynomials in p and q, their first derivatives with respect to l and m (or with respect to x and y) are not polynomials in p and q. When expressed in $\underline{p}$-coordinates, they have the form

$$\text{derivative} = \frac{\text{polynomial in } p \text{ and } q}{1 + \alpha q + \beta p}. \tag{18}$$

The two basic differences between a high order method and the corresponding isoparametric method are (i) the coefficients required in the defining functions for a high order method depend on the particular element geometry; and (ii) the isoparametric basis is only first order whereas a high order basis spans polynomials of greater degree. We have highlighted the fact that (i) makes the high order basis more complicated. However (ii) reduces the amount of computation required to evaluate certain of the integrals. This has been noted in [12] though we shall outline the idea here. If we rewrite eqn (14) without specifying the coordinate dependence we get

$$W^i = T^i - T_7^i W^7 - T_8^i W^8 \quad (i = 1, \ldots, 6). \tag{19}$$

This equation defines six of the eight basis functions in terms of W^7, W^8 and a much simpler set of polynomial functions, the T^i. This equation applies to any high order basis, whether rational or high order transformation. In fact, a rational basis can be constructed by simply defining a different W^7 and W^8 but using the same set of T^i and the same eqn (19) to define a rational set of functions $W^1, W^2, \ldots, W^6$. We have chosen a particular high order transformation method, but eqn (19) is in fact quite general.

4. BASIC INTEGRALS

The components of the element stiffness matrix as well as other desired integrals (e.g. components of a load vector or mass matrix), may be written as a linear combination of basic integrals of the form

$$I^{i1,i2,\ldots,is,j1,j2,\ldots,jt}_{\mu 1,\mu 2,\ldots,\mu s} = \iint_R \frac{\partial W^{i1}}{\partial x^{\mu 1}} \frac{\partial W^{i2}}{\partial x^{\mu 2}} \cdots \frac{\partial W^{is}}{\partial x^{\mu s}} W^{j1} W^{j2} \cdots W^{jt} \, dx \, dy \tag{20}$$

where R is the element domain, the μ's take the values 1 and 2, and the i's and j's take the values 1–8. Typical basic integrals which may be needed are

$$A^{ij} = I^{ij} = \iint_R W^i W^j \, dx \, dy \tag{21}$$

$$B_\mu^{ij} = I_\mu^{j,i} = \iint_R W^i \frac{\partial W^j}{\partial x^\mu} dx \, dy \tag{22}$$

$$C_{\mu\nu}^{ij} = I_{\mu\nu}^{ij,} = \int_R \int \frac{\partial W^i}{\partial x^\mu} \frac{\partial W^j}{\partial x^\nu} dx \, dy. \tag{23}$$

When transformed to p-space the integrands of the A- and B-integrals of eqns (21) and (22) are polynomials and thus may be evaluated in closed form. However, the C-integrals of eqn (23) have the form

$$\int_0^1 \int_0^{1-q} \frac{\text{polynomial in } p \text{ and } q}{1 + \alpha q + \beta p} \, dp \, dq \tag{24}$$

and thus are much more difficult to evaluate. For the eight-node second order basis the polynomial in the integrand is of sixth degree, while for the six-node isoparametric element it is of fourth degree.

The evaluation of the stiffness coefficients and other desired integrals can thus be divided into two separate problems: (a) the determination of the coefficients which multiply the basic integrals given in eqns (20) (e.g. the A-, B- and C-integrals of eqns 21, 22 and 23), and (b) the actual evaluation of these integrals. Since (a) is relatively simple and since the C-integrals appear to be more difficult to evaluate than the A- and B-integrals, we focus attention on the evaluation of the C-integrals. However, we observe that the solutions to Laplace's equation require the evaluation of the C-integrals only and thus provide a practical test of the techniques presented here. Efficient computation of the A- and B-integrals for an element with three curved sides and a six-node isoparametric basis is discussed in Refs. [13, 14]. The evaluation of integrals is somewhat simpler for the element shape considered in the present study. We remark that for the isoparametric case, at least, it does not seem to be useful for the computation of A- and B-integrals to make use of the $\underline{l}$-coordinate system, but instead to transform directly from the $\underline{x}$- to the $\underline{p}$-coordinate system.

5. EVALUATION OF THE C-INTEGRALS FOR THE SECOND ORDER BASIS

Because $C_{\mu\nu}^{ij} = C_{\nu\mu}^{ji}$ with $i, j = 1, \ldots, 8$, there are 136 distinct C-integrals to be evaluated per element. The evaluation of the C-integrals is simplified by the following observations. First,

from eqn (17) it follows that

$$\sum_{i=1}^{8} \frac{\partial W^i}{\partial x^\mu} = 0 \tag{25}$$

and consequently

$$\sum_{i=1}^{8} C^{ij}_{\mu\nu} = 0 \qquad (j = 1, \ldots, 8). \tag{26}$$

This simple relation can be used to evaluate 31 of the C-integrals in terms of the remaining 105.

Secondly, because of the linear transformation given in eqn (1), the C-integrals of eqn (23) may be expressed as linear combinations of integrals which are functions of l_4 but do not otherwise depend on the nodal positions, i.e.

$$C^{ij}_{\mu\nu} = S(S^{-1})_{\mu\kappa}(S^{-1})_{\nu\lambda}F^{ij}_{\kappa\lambda} \qquad (i,j = 1, \ldots, 8) \tag{27}$$

where

$$F^{ij}_{\mu\nu} = \iint_{\bar{R}} \frac{\partial W^i}{\partial l^\mu}\frac{\partial W^j}{\partial l^\nu}\, \mathrm{d}l\, \mathrm{d}m \qquad (i,j = 1, \ldots, 8). \tag{28}$$

The $F^{ij}_{\mu\nu}$, though considerably simpler than the $C^{ij}_{\mu\nu}$, still have the form (24).

Some useful relations among the $F^{ij}_{\mu\nu}$ can be formed by application of the linear operators $L^j_{\mu\nu}$ defined by

$$L^j_{\mu\nu}(f) = \iint_{\bar{R}} \frac{\partial f}{\partial l^\mu}\frac{\partial W^j}{\partial l^\nu}\, \mathrm{d}l\, \mathrm{d}m \qquad (j = 1, \ldots, 8) \tag{29}$$

to eqn (19). This results in

$$F^{ij}_{\mu\nu} = G^{ij}_{\mu\nu} - T_7{}^i F^{7j}_{\mu\nu} - T_8{}^i F^{8j}_{\mu\nu} \qquad (i = 1, \ldots, 6;\ j = 1, \ldots, 8) \tag{30}$$

where

$$G^{ij}_{\mu\nu} = \iint_{\bar{R}} \frac{\partial T^i}{\partial l^\mu}\frac{\partial W^j}{\partial l^\nu}\, \mathrm{d}l\, \mathrm{d}m \qquad (i = 1, \ldots, 6;\ j = 1, \ldots, 8). \tag{31}$$

Since $F^{ij}_{\mu\nu} = F^{ji}_{\nu\mu}$, once the 10 distinct $F^{mn}_{\mu\nu}$ $(m, n = 7, 8)$ and the set of integrals $G^{ij}_{\mu\nu}$ are known, the remaining $F^{ij}_{\mu\nu}$ can readily be deduced by repeated application of eqn (30).

The $G^{ij}_{\mu\nu}$ are simpler than the $F^{ij}_{\mu\nu}$ in that they can be expressed as integrals over polynomials in p-space and thus may be evaluated in closed form. However, there are a number of relations which can be used to simplify the evaluation of the $G^{ij}_{\mu\nu}$.

For example, it follows from eqn (11) that

$$\sum_{i=1}^{6} \frac{\partial T^i}{\partial l^\mu} = 0; \qquad \sum_{i=1}^{6} l_i^\mu \frac{\partial T^i}{\partial l^\nu} = \delta_\nu{}^\mu; \qquad \sum_{i=1}^{6} l_i^\mu l_i^\nu \frac{\partial T^i}{\partial l^\kappa} = \delta_\kappa{}^\mu l^\nu + \delta_\kappa{}^\nu l^\mu \tag{32}$$

and consequently

$$\sum_{i=1}^{6} G^{ij}_{\mu\nu} = 0 \qquad (j = 1, \ldots, 8) \tag{33}$$

$$\sum_{i=1}^{6} l_i^\mu G^{ij}_{\nu\kappa} = \delta_\nu{}^\mu F_\kappa{}^j \qquad (j = 1, \ldots, 8) \tag{34}$$

$$\sum_{i=1}^{6} l_i^\mu l_i^\nu G^{ij}_{\kappa\lambda} = \delta_\kappa{}^\mu H^j_{\nu\lambda} + \delta_\kappa{}^\nu H^j_{\mu\lambda} \qquad (j = 1, \ldots, 8) \tag{35}$$

where

$$F_\kappa{}^j = \iint_{\bar{R}} \frac{\partial W^j}{\partial l^\kappa}\, \mathrm{d}l\, \mathrm{d}m \ (j = 1, \ldots, 8) \tag{36}$$

and

$$H^j_{\mu\nu} = \int\int_{\bar{R}} l^\mu \frac{\partial W^j}{\partial l^\nu}\, dl\, dm \qquad (j = 1, \ldots, 8). \tag{37}$$

Similar relations follow from considering derivatives of eqn (17). For example

$$\sum_{j=1}^{8} G^{ij}_{\mu\nu} = 0; \qquad \sum_{j=1}^{8} l^\mu G^{ij}_{\nu\kappa} = \delta_\kappa{}^\mu \int\int_{\bar{R}} \frac{\partial T^i}{\partial l^\nu}\, dl\, dm \qquad (j = 1, \ldots, 8). \tag{38}$$

Hence we can deduce the set of $G^{ij}_{\mu\nu}$ from even simpler integrals. We see from eqn (26) that not all of the integrals $G^{ij}_{\mu\nu}$ are needed and that there is some freedom in selecting the ones to be evaluated. Furthermore, there are many ways of evaluating those that are used. Through the use of the MACSYMA symbolic manipulation system [15, 16], we are able to develop closed-form expressions for all of the $G^{ij}_{\mu\nu}$, compare the complexity of these expressions, and thereby estimate the computational efficiency of many of the alternatives suggested by the above equations, as well as some *ad hoc* relations. As a result, eqn (26) is used to express the $C^{ij}_{\mu\nu}$ with i and/or j equal to one in terms of other $C^{ij}_{\mu\nu}$ and consequently the integrals $G^{1j}_{\mu\nu}$ are not needed. Then for $\mu = \nu = 1$ the following relations can be used.

$$\begin{gathered} G^{6j}_{11} = -4m_4(1-m_4)G^{4j}_{11}; \qquad G^{3j}_{11} = m_4(1-2m_4)G^{4j}_{11}; \qquad G^{5j}_{11} = 2(F_1{}^j - G^{2j}_{11} - l_4 G^{4j}_{11}); \\ G^{2j}_{11} = 4[F_1{}^j - H^j_{11} - l_4(1-l_4)G^{4j}_{11}]; \qquad G^{4j}_{11} = \frac{1}{l_4 m_4} H^j_{21}. \end{gathered} \tag{39}$$

Relations similar to eqn (39) apply for the other values of μ and ν. The integrals $F_\kappa{}^j$, H^j_{12} and H^j_{21} of eqns (36) and (37) are simple polynomials in p-space and are almost trivial to evaluate. The integrals H^j_{11} and H^j_{22} are somewhat more complicated but are not difficult.

6. EVALUATION OF THE LAST TEN INTEGRALS

Only 10 integrals remain to give us trouble. These are the $F^{ij}_{\mu\nu}$ with $i, j = 7, 8$ and $\mu, \nu = 1, 2$. Each of them has the form shown in eqn (24). They can be calculated through the use of recurrence techniques [3] by which a set of integrals of the form

$$\int_0^1 \int_0^{1-q} \frac{p^m q^n}{1+\alpha q+\beta p}\, dp\, dq \qquad (m, n = 0, 1, \ldots, 4) \tag{40}$$

are evaluated as an intermediate step. The integrals of eqn (40) can efficiently be evaluated with high accuracy without the use of numerical quadrature, but evaluation of the appropriate linear combinations of these integrals to form the C-integrals appears to be a slow process. Consequently, for this study we resort to numerical quadrature.

The most appropriate formulae to use for quadrature over a triangular region appear to be the symmetric Gaussian-type quadrature formulae given in Ref. [17]. Formulae of the symmetric type are known for only moderately high degree polynomials but should give sufficient accuracy for finite element work. As shown in Table 1, fewer quadrature points are required with the symmetric formulae to accurately integrate a polynomial of given total degree than

Table 1. Comparison of the number of quadrature points required for the accurate integration over a triangular region of a polynomial of given total degree

Total degree of polynomial	Number of quadrature points using symmetric quadrature	Number of quadrature points using product Gauss quadrature
3	4	4
5	7	9
7	13	16
9	19	25
11	28	36

with the product Gaussian quadrature formulae described in Ref. [18]. However, if for test purposes higher accuracy is desired, the product Gauss quadrature points and weights are known [18, 19] for polynomials of total degree up to 59.

In the limit that the parabolic arc becomes a straight line, both α and β go to zero, and accurate values for the integrals are obtained by the use of 13 symmetric quadrature points. This limit should be avoided, however, with the particular basis discussed here [20]. For only 10 integrals it is not expensive to use 19 or 28 quadrature points, and these numbers seem to give sufficient accuracy.

7. OPERATION COUNTS FOR THE SECOND ORDER BASIS

To demonstrate that efficient computer implementations can be constructed using a HOT basis, we present here a set of operation counts taken from a program which evaluates the C-integrals of eqn (23) for the eight-node element of Fig. 1. The operation counts presented here are for floating point adds, multiplies and divides. No attempt has been made to count integer operations because the integer operations are largely hidden from the programmer's view and their numbers depend on which FORTRAN compiler is used.

Some operations are performed for the first element but are not repeated for subsequent elements. Their number is not large and is ignored here because it is presumed that any practical finite element computation will involve several elements, and thus the set up operations are insignificant.

The operation counts in only one phase of the computation are dependent on the accuracy desired. The 10 distinct integrals $F_{\mu\nu}^{mn}$ $(m, m = 7, 8)$ are evaluated by numerical quadrature. If N is the number of quadrature points (see Table 1), the quadrature subroutine takes $9+24N$ adds, $16+23N$ multiplies, and $1+N$ divisions. In the sequel we will assume that the 19-point symmetric quadrature formula is used.

The major steps in the program and their respective operation counts are given in Table 2. If all the integrals $F_{\mu\nu}^{ij}$ are evaluated by numerical quadrature, the total operation counts are approximately 4085 adds, 4541 multiplies, and 23 divides, or about three times as many multiplies as shown in Table 2. The operation counts are reduced to 3366 adds, 4251 multiplies and 23 divides if eqn (25) is taken into account. For a rational basis they would be larger if eqn (19) were not taken into consideration. In any case the approach given in Table 2 is clearly the more efficient one.

Table 2. Floating point operation counts for the major steps in the program for computing C-integrals for the eight-node HOT basis. Column one shows the number of quantities evaluated in the step. The number of quadrature points is denoted by N

Quantities evaluated	Adds	Multiplies	Divides	Comments
$8\alpha, \beta, l_4^\mu, l_7^\mu, l_8^\mu$	15	12	1	
$10F_{\mu\nu}^{mn}$ $(m, n = 7, 8)$	$9+24N$	$19+33N$	$1+N$	eqns (13), (14), (28)
$32F_\mu^j, H_{\mu\nu}^j$ $(j = 1, \ldots, 8)$	116	131	6	eqns (36), (37)
$160G_{\mu\nu}^{ij}$ $(i, j = 2, \ldots, 6; j = 1, \ldots, 8)$	250	160	1	eqns (31), (33)–(35), (38), (39)
$105F_{\mu\nu}^{ij}$ $(i, j = 2, \ldots, 8)$	228	217	1	eqn (30)
$105C_{\mu\nu}^{ij}$ $(i, j = 2, \ldots, 8)$	308	350	0	eqn (27)
$31C_{\mu\nu}^{ij}$ $(i, j = 1, \ldots, 8)$	186	0	0	eqn (26)
Totals for 19-point quadrature	1568	1516	29	

In Table 2 more than a third of the computational effort was required to obtain the initial subset of ten integrals. It is possible that further investigation could result in more attractive ways of obtaining these integrals, in which case the total operation count would be reduced even further.

8. EVALUATION OF C-INTEGRALS FOR THE 6-NODE ISOPARAMETRIC BASIS

Many of the techniques presented herein for the evaluation of C-integrals for the HOT basis are also applicable for an isoparametric basis. We again work with the element and coordinate systems of Fig. 1. The procedure is as follows: We evaluate by numerical quadrature the 21

distinct integrals

$$F^{ij}_{\mu\nu} = \int\int_{\bar{R}} \frac{\partial W^i}{\partial l^\mu} \frac{\partial W^j}{\partial l^\nu} \, dl \, dm \qquad (i, j = 2, 3, 4) \tag{41}$$

where $W^i(p, q)$ are now the shape functions referred to in eqns (6)–(8). These particular integrals were chosen for numerical quadrature because their integrands appear to be simpler to evaluate than the other integrands. Next we evaluate from their (almost trivial) exact analytic expressions the 10 integrals $F_\mu{}^i$ defined by

$$F_\mu{}^i = \int\int_{\bar{R}} \frac{\partial W^i}{\partial l} \, dl \, dm = \epsilon_{\mu\nu}\epsilon_{\kappa\lambda} \int\int_{\bar{R}} \frac{\partial W^i}{\partial p^\kappa} \frac{\partial l^\nu}{\partial p^\lambda} \, dp \, dq \qquad (i = 2, \ldots, 6) \tag{42}$$

where $\epsilon_{\mu\nu}$ is the permutation symbol. Then, through the use of the relations

$$\sum_{i=1}^{6} l_j^\mu F^{ij}_{\nu\kappa} = \delta_\nu{}^\mu F_\kappa{}^j \qquad (j = 1, \ldots, 6) \tag{43}$$

we evaluate 34 more of the $F^{ij}_{\mu\nu}$ so that all $F^{ij}_{\mu\nu}$ with $(i, j = 2, \ldots, 6)$ are known. Through the use of eqn (26) the $C^{ij}_{\mu\nu}$ with $i, j = 2, \ldots, 6$ are evaluated, and finally the integrals $C^{ij}_{\mu\nu}$ with i and/or j equal to one are evaluated through the use of eqn (26). The respective operation counts are shown in Table 3.

Table 3. Floating point operation counts for the major steps in the program for computing C-integrals for the 6-node isoparametric basis. Column one shows the number of quantities evaluated in the step. The number of quadrature points is denoted by N

Quantities evaluated	Adds	Multiplies	Divides	Comments
$4\alpha, \beta, l_4{}^\mu$	15	8	1	
$21F^{mn}_{\mu\nu}$ $(m, n = 2, 3, 4)$	$25N$	$33N$	N	eqn (41)
$10F_\mu{}^j$ $(j = 1, \ldots, 8)$	1	1	0	eqn (42)
$34F^{ij}_{\mu\nu}$ $(i, j = 2, \ldots, 6)$	83	34	0	eqn (43)
$55C^{ij}_{\mu\nu}$ $(i, j = 2, \ldots, 6)$	165	194	0	eqn (27)
$23C^{ij}_{\mu\nu}$ $(i, j = 1, \ldots, 6)$	92	0	0	eqn (26)
Totals for 19-point quadrature	831	864	20	

If all of the $F^{ij}_{\mu\nu}$ are evaluated by the 19-point quadrature formula followed by evaluation of the C-integrals through the use of eqn (27), then the addition and multiplication counts are more than twice as high as given in Table 3. We come up with 2032 adds, 2174 multiplies, and 20 divides. The operation counts are reduced to 1469 adds, 1627 multiplies and 20 divides by taking eqn (23) into account but are considerably higher if the C-integrals are evaluated directly by numerical quadrature without benefit of the transformation to $\underline{l}$-coordinates.

9. OTHER INTEGRALS AND OTHER BASES

The integration techniques just presented may also be used to evaluate integrals such as $I^{ij,k}_{\mu\nu}$, $I^{ij,kl}_{\mu\nu}$, $D^{ijk}_{\mu\nu\kappa} = I^{ijk,}_{\mu\nu\kappa}$ and $E^{ijkl}_{\mu\nu\kappa\lambda} = I^{ijkl,}_{\mu\nu\kappa\lambda}$ (see eqn 20). Since the numerators of the integrands which occur in such integrals are higher degree polynomials than what we have been considering, more quadrature points are needed for comparable accuracy. Thus it becomes all the more important to reduce the number of integrals evaluated by numerical quadrature. For example, for the 816 distinct D-integrals for the second order basis, numerical quadrature is needed for only 50 integrals—20 integrals of the form

$$F^{kmn}_{\mu\nu\kappa} = \int\int_{\bar{R}} \frac{\partial W^k}{\partial l^\mu} \frac{\partial W^m}{\partial l^\nu} \frac{\partial W^n}{\partial l^\kappa} \, dl \, dm \qquad (k, m, n = 7, 8); \tag{44}$$

20 integrals of the form

$$H^{mn}_{\mu\nu\kappa} = \iint_{\bar{R}} l^{\mu} \frac{\partial W^m}{\partial l^{\nu}} \frac{\partial W^n}{\partial l^{\kappa}} \, dl \, dm \qquad (m, n = 7, 8); \tag{45}$$

and the 10 integrals $F^{mn}_{\mu\nu}$ with $m, n = 7, 8$ referred to in Table 2. The numerators of the D-integrals are ninth degree polynomials in p and q.

The integration techniques may also be used for HOT bases of order three and higher and for isoparametric bases of dimension higher than six. For example, a third order HOT basis for the element shape of Fig. 1 can be constructed with dimension 12. Such a basis satisfies not only six conditions of the type given in eqn (17) but also the four conditions

$$\sum_{i=1}^{12} l_i^{\mu} l_i^{\nu} l_i^{\kappa} W^i(\underline{p}) = l^{\mu} l^{\nu} l^{\kappa}. \tag{46}$$

It follows that only ten $F^{ij}_{\mu\nu}$ need be evaluated by numerical quadrature in order to calculate the 300 distinct C-integrals.

The isoparametric cubic basis has dimension 10 but still is a first order basis in the $\underline{x}$-coordinates. Consequently 105 of the $F^{ij}_{\mu\nu}$ must be evaluated by quadrature to calculate the 210 distinct C-integrals. Even though a relatively high percentage of the integrals require numerical quadrature the integration techniques may still be practical for this basis.

Finally, the integration techniques have application to other finite element shapes than the one considered herein.

10. CONCLUDING REMARKS

The calculation of integrals is a major part of any finite element computation involving curved-sided elements. The technique of using the known order of the basis to deduce many of the integrals once a certain subset of them have been found is shown to be useful and efficient. The generality of the technique should be stressed since the idea is independent of the method used to obtain the original subset. The amount of benefit depends, among other things, on the difference between the number of conditions on the basis and the dimension of the basis. Thus, although there is a clear advantage in using the method for an isoparametric basis which is only first order, the advantage is much greater when used with a second or higher order basis. That is, since a isoparametric basis is only linear, we have only three conditions, regardless of dimension, which we can use to our advantage. However, with high order bases as the order goes up we have more conditions which we can use to deduce more and more of the integrals. Thus, for the six-node isoparametric basis about two thirds of the computational effort is expended on numerical quadrature as contrasted with little more than one third for the eight-node second order basis. The authors feel that the use of this simple idea alone answers the criticism that a high order method is impractical because of the computational expense associated with the evaluation of integrals containing derivatives of basis functions.

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