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The relation of the *d*-orthogonal polynomials to the Appell polynomials

Khalfa Douak

Université Pierre et Marie Curie, Laboratoire d'Analyse Numérique, Tour 55–65, 4 place Jussieu, 75252 Paris Cedex 05, France

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Abstract

We are dealing with the concept of d-dimensional orthogonal (*abbreviated* d-orthogonal) polynomials, that is to say polynomials verifying one standard recurrence relation of order $d + 1$. Among the d-orthogonal polynomials one singles out the natural generalizations of certain classical orthogonal polynomials. In particular, we are concerned, in the present paper, with the solution of the following problem (P): Find all polynomial sequences which are at the same time Appell polynomials and d-orthogonal. The resulting polynomials are a natural extension of the Hermite polynomials.

A sequence of these polynomials is obtained. All the elements of its $(d + 1)$-order recurrence are explicitly determined. A generating function, a $(d + 1)$-order differential equation satisfied by each polynomial and a characterization of this sequence through a vectorial functional equation are also given. Among such polynomials one singles out the d-symmetrical ones (Definition 1.7) which are the d-orthogonal polynomials analogous to the Hermite classical ones. When $d = 1$ (ordinary orthogonality), we meet again the classical orthogonal polynomials of Hermite.

Keywords: Appell polynomials; Hermite polynomials; Orthogonal polynomials; Generating functions; Differential equations; Recurrence relations

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0. Introduction

The orthogonal polynomials in general and the classical orthogonal polynomials in particular have been the object of extensive works. They are connected with numerous problems of applied mathematics, theoretical physics, chemistry, approximation theory and several other mathematical branches. In particular, their applications are being widely used in theories as Padé approximants, continued fractions, spectral study of Schrödinger discrete operators, polynomial solutions of second-order differential equations and others.

The notions of d-dimensional orthogonality for polynomials [14], vectorial orthogonality as defined and studied in [19] or simultaneous orthogonality in [7] are obviously generalizations of

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the notion of ordinary orthogonality for polynomials. Such polynomials are characterized by the fact that they satisfy an order $d+1$ recurrence relationship, that is a relation between $d+2$ consecutive polynomials [19]. All these new notions of d-orthogonality for polynomials and, equivalently, $1/d$-orthogonality [5] appear as particular cases of the general notion of biorthogonality studied in [6]. Recently they have been the subject of numerous investigations and applications. In particular, they are connected with the study of vector Padé approximants [19, 20], simultaneous Padé approximants [7] and other problems such as vectorial continued fractions, polynomials solutions of the higher-order differential equations, spectral study of multidiagonal nonsymmetric operators [4].

Otherwise, according to a point of view based only upon the d-orthogonality conditions, especially the $(d+1)$-order recurrence relation, some studies which look for generalizations of the orthogonal polynomials properties in general and the classical orthogonal polynomial ones in particular, are made in [5, 8–11, 15]. In this paper we present the results in this direction.

It is well known that the Hermite polynomials (up to a linear change of variable) form the only sequence of polynomials that are simultaneously orthogonal and Appell polynomials. This characterization of the Hermite polynomials was first given by Angelesco [2], and later by other authors (see, e.g., [18] and for additional references [1]).

Let $\{H_n\}_{n>0}$ be the sequence of Hermite polynomials and $\{\hat{H}_n\}_{n\geqslant 0}$ the corresponding *monic* polynomials, i.e., $\hat{H}_n(x) = 2^{-n}H_n(x)$, $n \geqslant 0$.

The previous property is translated by the fact that the sequence $\{\hat{H}_n\}_{n\geqslant 0}$ verifies:

(a) the three-term recurrence relation

$$\hat{H}_{n+2}(x) = x\hat{H}_{n+1}(x) - \tfrac{1}{2}(n+1)\hat{H}_n(x), \quad n \geqslant 0;$$

$$\hat{H}_0(x) = 1, \qquad \hat{H}_1(x) = x$$

and

(b) the Appell character

$$\hat{H}'_{n+1}(x) = (n+1)\hat{H}_n(x), \quad n \geqslant 0.$$

Our main objective in this paper is the investigation of sequences $\{P_n\}_{n\geqslant 0}$ of monic polynomials which are simultaneously Appell polynomials and d-orthogonal.

Section 1 is devoted to some preliminaries and notations necessary for the sequel. In particular, the exposition of the definition and characterizations of the d-orthogonal polynomials. In Section 2, we state and solve the problem (P). The proof is based only on the $(d+1)$-order recurrence where an explicit expression for each of its elements is obtained. In Section 3, we give some characterizations of these polynomials. First, by using the $(d+1)$-order recurrence and the Appell property, we obtain a generating function and a $(d+1)$-order differential equation of the polynomials $P_n(x)$, $n \geqslant 0$. Next, by introducing the d-dimensional functional with respect to which the sequence $\{P_n\}_{n\geqslant 0}$ is d-orthogonal, we give a characterization of this novel sequence through a vectorial functional equation. Lastly, we express the derivative of the product of two consecutive polynomials in the form of a differential relation which generalizes McCarthy's characterization of a Hermite classical polynomials.

1. Preliminaries and notations

Let $\mathscr{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathscr{P}'$ be its dual. We denote by $\langle u, f \rangle$ the effect of $u \in \mathscr{P}'$ on $f \in \mathscr{P}$. In particular, we denote by $(u)_n = \langle u, x^n \rangle$, $n \geqslant 0$, the moments of the functional u. Since a linear functional is uniquely determined by its action on a basis, u is uniquely determined by the sequence of constants $(u)_n$.

By a polynomial set (PS), we mean a sequence of monic polynomials $\{P_n\}_{n \geqslant 0}$ in which $\deg P_n(x) = n$ for all n, say, $P_n(x) = x^n + \dots, n \geqslant 0$.

Let $\{P_n\}_{n \geqslant 0}$ be a polynomial set; there exists a sequence of linear functionals $\{u_n\}_{n \geqslant 0}$, such that

$$\langle u_m, P_n \rangle = \delta_{m,n}, \quad m, n \geqslant 0. \tag{1.1}$$

The sequence $\{u_n\}_{n \geqslant 0}$ is called the *dual sequence* of $\{P_n\}_{n \geqslant 0}$; it is unique [14].

Lemma 1.1 (Maroni [14]). *For each $u \in \mathscr{P}'$ and $p \geqslant 1$ integer, the following two propositions are equivalent*:

(a) $\langle u, P_{p-1} \rangle \neq 0$; $\langle u, P_n \rangle = 0$, $n \geqslant p$;

(b) $\exists \lambda_\nu \in \mathbb{C}$, $0 \leqslant \nu \leqslant p-1$, $\lambda_{p-1} \neq 0$, *such that*

$$u = \sum_{\nu=0}^{p-1} \lambda_\nu u_\nu . \tag{1.2}$$

We now consider a polynomial set $\{P_n\}_{n \geqslant 0}$ with its derivative sequence $\{Q_n\}_{n \geqslant 0}$ defined by $Q_n(x) = [1/(n+1)] P'_{n+1}(x)$, $n \geqslant 0$. We denote by $\{v_n\}_{n \geqslant 0}$ the dual sequence of $\{Q_n\}_{n \geqslant 0}$.

Proposition 1.2 (Maroni [14]). *We have*

$$Dv_n = -(n+1) u_{n+1}, \quad n \geqslant 0, \tag{1.3}$$

where Du, the derivative of a linear functional u, is defined by $\langle Du, f \rangle := -\langle u, f' \rangle$, $\forall u \in \mathscr{P}'$ and $\forall f \in \mathscr{P}$.

1.1. The regular orthogonality

Let us recall the definition of regular orthogonality. The linear functional u will be called *regular* if we can associate with it a sequence of polynomials $\{P_n\}_{n \geqslant 0}$ such that

$$\langle u, P_n P_m \rangle = r_n \delta_{n,m}, \quad n, m \geqslant 0, \quad r_n \neq 0, \ n \geqslant 0.$$

Then for all n the degree of $P_n(x)$ is exactly n and we can always suppose each $P_n(x)$ to be monic. Therefore, the PS $\{P_n\}_{n \geqslant 0}$ is unique. In this case, it is called the orthogonal polynomial set (OPS) relative to u. Necessarily, $u = \lambda u_0$, $\lambda \neq 0$.

It is an old result that such OPSs are characterized by the fact that they verify a three-term recurrence relation, namely:

$$P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \quad n \geqslant 0 \ \text{(monic form)}; \tag{1.4}$$

$$P_0(x) = 1, \qquad P_1(x) = x - \beta_0; \tag{1.5}$$

with

$$\gamma_{n+1} \neq 0, \quad n \geqslant 0.$$

Definition 1.3. The OPS $\{P_n\}_{n \geqslant 0}$ will be called *classical* if $\{Q_n\}_{n \geqslant 0}$ is also an OPS [13].

This gives (1.4), (1.5) and

$$Q_{n+2}(x) = (x - \alpha_{n+1}) Q_{n+1}(x) - \delta_{n+1} Q_n(x), \quad n \geqslant 0; \tag{1.6}$$

$$Q_0(x) = 1, \qquad Q_1(x) = x - \alpha_0; \tag{1.7}$$

with

$$\delta_{n+1} \neq 0, \quad n \geqslant 0.$$

Theorem 1.4 (Geronimus [12]). *The sequence $\{P_n\}_{n \geqslant 0}$, orthogonal with respect to u, is classical if and only if there exist two polynomials ϕ and ψ with* $\deg \phi = t \leqslant 2$, $\deg \psi = 1$, *such that*

$$\psi u + D(\phi u) = 0 \quad (\phi \textit{ monic}),$$

writing $\psi(x) = a_1 x + a_0$, we must have $a_1 \notin \mathbb{N}^*$ if $t = 2$.

1.2. The d-orthogonality

Let us consider d linear functionals $\Gamma^1, \ldots, \Gamma^d$ ($d \geqslant 1$).

Definition 1.5. A polynomial set $\{P_n\}_{n \geqslant 0}$ is called a d-dimensional orthogonal polynomial set or simply a d-orthogonal polynomial set (d-*OPS*) with respect to $\Gamma = (\Gamma^1, \ldots, \Gamma^d)^{\mathrm{T}}$ if it fulfils [19]:

$$\langle \Gamma^\alpha, P_m P_n \rangle = 0, \quad n \geqslant md + \alpha, \ m \geqslant 0, \tag{1.8}$$

$$\langle \Gamma^\alpha, P_m P_{md+\alpha-1} \rangle \neq 0, \quad m \geqslant 0, \tag{1.9}$$

for each integer α with $1 \leqslant \alpha \leqslant d$.

Remark 1.6. The inequalities (1.9) are the regular conditions equivalent to the ones given in [14, p. 110] or [20, p. 142]. They are a natural generalization of those given in the regular orthogonality case. In this case, the d-dimensional functional Γ is called *regular*. It is not unique.

Indeed, from Lemma 1.1, we have

$$\Gamma^\alpha = \sum_{\nu=0}^{\alpha-1} \lambda_\nu^\alpha u_\nu, \quad \lambda_{\alpha-1}^\alpha \neq 0, \quad 1 \leqslant \alpha \leqslant d \ \Leftrightarrow \ u_\nu = \sum_{\alpha=1}^{\nu+1} \tau_\alpha^\nu \Gamma^\alpha, \quad \tau_{\nu+1}^\nu \neq 0, \quad 0 \leqslant \nu \leqslant d-1.$$

Therefore, from now on, we shall work uniquely with the dual functionals $u_0, \ldots, u_{d-1}$, that is to say $\mathscr{U} = (u_0, \ldots, u_{d-1})^{\mathrm{T}}$: it is the canonical regular d-dimensional functional with respect to which the PS $\{P_n\}_{n \geqslant 0}$ is d-orthogonal.

Note that if $d = 1$, we meet again the ordinary orthogonality, say, $\{P_n\}_{n \geqslant 0}$ is an OPS.

The remarkable characterization of the d-OPSs is the one given by Van Iseghem in her thesis [19]: A polynomial set $\{P_n\}_{n \geqslant 0}$ is d-OPS if and only if it satisfies the $(d+1)$-order recurrence relation:

$$P_{m+d+1}(x) = (x - \beta_{m+d})\,P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \quad m \geqslant 0, \tag{1.10}$$

with the initial conditions

$$P_0(x) = 1, \qquad P_1(x) = x - \beta_0$$

and if $d \geqslant 2$:

$$P_n(x) = (x - \beta_{n-1})\,P_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} P_{n-2-\nu}(x), \quad 2 \leqslant n \leqslant d, \tag{1.11}$$

and the regularity conditions

$$\gamma_{m+1}^{0} \neq 0, \quad m \geqslant 0. \tag{1.12}$$

Now, when the derivative PS $\{Q_n\}_{n\geqslant 0}$ is also d-OPS with respect to $\mathscr{V} = (v_0, \ldots, v_{d-1})^{\mathrm{T}}$, the polynomial set $\{P_n\}_{n\geqslant 0}$ is called a "*classical*" d-OPS [8, 9].

Then, both $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ satisfy an order $d+1$ recurrence relation. Consequently, the recurrence (1.10) with (1.11) as well as the following recurrence relation verified by $\{Q_n\}_{n\geqslant 0}$ hold simultaneously:

$$Q_{m+d+1}(x) = (x - \alpha_{m+d})\,Q_{m+d}(x) - \sum_{\nu=0}^{d-1} \delta_{m+d-\nu}^{d-1-\nu} Q_{m+d-1-\nu}(x), \quad m \geqslant 0, \tag{1.13}$$

with the initial conditions

$$Q_0(x) = 1, \qquad Q_1(x) = x - \alpha_0$$

and if $d \geqslant 2$:

$$Q_n(x) = (x - \alpha_{n-1})\,Q_{n-1}(x) - \sum_{\nu=0}^{n-2} \delta_{n-1-\nu}^{d-1-\nu} Q_{n-2-\nu}(x), \quad 2 \leqslant n \leqslant d, \tag{1.14}$$

$$\delta_{m+1}^{0} \neq 0, \quad m \geqslant 0.$$

1.3. d-symmetric polynomials

Let k be an integer with $0 \leqslant k \leqslant d$. By ξ_k, $k = 0, 1, \ldots, d$, we denote the $d+1$ roots of unity, namely: $\xi_k = \exp(2ik\pi/(d+1))$, $\xi_k^{d+1} = 1$.

Definition 1.7. The sequence $\{P_n\}_{n\geqslant 0}$ is called d-symmetric if it fulfils:

$$P_n(\xi_k x) = \xi_k^n P_n(x), \quad n \geqslant 0, \tag{1.15}$$

for each $k = 0, 1, \ldots, d$.

When $d = 1$, then $\xi_0 = 1$ and $\xi_1 = -1$, this means that the sequence $\{P_n\}_{n\geqslant 0}$ is symmetric, that is $P_n(-x) = (-1)^n P_n(x)$, $n \geqslant 0$.

Definition 1.8. The d-dimensional functional $\mathscr{U} = (u_0, \ldots, u_{d-1})^{\mathrm{T}}$ is called d-symmetric if it fulfils:

$$(u_\nu)_{(d+1)n+\mu} = 0, \quad \mu = 0, 1, \ldots, d, \ \mu \neq \nu, \ n \geqslant 0, \tag{1.16}$$

for each $0 \leqslant \nu \leqslant d-1$.

When $d = 1$, then $\mathscr{U} = u_0$ ($\mathscr{U}$ is reducible to a linear functional), we find again the symmetrical functional: $(u_0)_{2n+1} = 0$, $n \geqslant 0$ (all the moments of odd order vanish).

It is of interest to interpret the d-symmetrical property (1.15) in terms of the moment conditions (1.16). That is given in

Theorem 1.9 (Douak and Maroni [9]). *Let $\{P_n\}_{n\geqslant 0}$ be a d-OPS with respect to $\mathscr{U} = (u_0, \ldots, u_{d-1})^{\mathrm{T}}$. Then the following propositions are equivalent:*
(a) *$\mathscr{U}$ is d-symmetric.*
(b) *$\{P_n\}_{n\geqslant 0}$ is d-symmetric.*
(c) *$\{P_n\}_{n\geqslant 0}$ satisfies the recurrence relation:*

$$\begin{aligned} &P_n(x) = x^n, && 0 \leqslant n \leqslant d, \\ &P_{n+d+1}(x) = xP_{n+d}(x) - \gamma_{n+1}^0 P_n(x), && n \geqslant 0 \end{aligned} \tag{1.17}$$

that is to say $\beta_n = 0$, $n \geqslant 0$; $\gamma_{n+1}^{\nu} = 0$, $n \geqslant 0$, for each $\nu = 1, \ldots, d-1$.

Corollary 1.10. *Moreover, if $\{P_n\}_{n\geqslant 0}$ is a "classical" d-OPS, then $\{Q_n\}_{n\geqslant 0}$ is also d-symmetric.*

2. Statement of the problem

Now we pose the problem (P): *Find all d-OPSs $\{P_n\}_{n\geqslant 0}$ which are also Appell PS.*
Thus, the polynomial sets $\{P_n\}_{n\geqslant 0}$ necessarily must enjoy the two properties:
(a) the $(d+1)$-order recurrence relation (1.10) with (1.11), and
(b) the Appell character:

$$Q_n(x) = P_n(x), \quad n \geqslant 0. \tag{2.1}$$

The following remark is in order here: The d-OPS $\{P_n\}_{n\geqslant 0}$ is necessarily a "classical" one.

Indeed, from the equality (2.1) its derivative PS $\{Q_n\}_{n\geqslant 0}$ again forms a d-OPS. Then, the two $(d+1)$-order recurrence relations (1.10) with (1.11) and (1.13) with (1.14) hold simultaneously.

Before solving the problem (P), it is very important to recall the following well-known characterizations of Appell polynomials [3]:

(i) A polynomial set $\{P_n\}_{n\geqslant 0}$ is an Appell PS if and only if the polynomials $P_n(x)$, $n \geqslant 0$, are defined by the generating function

$$A(t)\,\mathrm{e}^{xt} = \sum_{n\geqslant 0} P_n(x)\frac{t^n}{n!}, \qquad A(t) = \sum_{n\geqslant 0} a_n t^n, \quad a_0 = 1, \tag{2.2}$$

which we denote by $G(x, t)$, that is $G(x, t) = A(t)\,\mathrm{e}^{xt}$.

(ii) $\{P_n\}_{n\geqslant 0}$ is an Appell PS if and only if there exists a sequence of numbers $\{a_n\}_{n\geqslant 0}$, such that

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k} x^k, \quad n = 0, 1, \ldots, \text{ with } a_0 = 1. \tag{2.3}$$

Examples of Appell polynomials are: The sequence of powers $\{x^n\}_{n\geqslant 0}$, the Hermite polynomials $\{\hat{H}_n\}_{n\geqslant 0}$ and the Bernoulli polynomials $\{B_n\}_{n\geqslant 0}$. Their corresponding generating functions are given, respectively, by e^{xt}, $e^{xt-t^2/4}$ and $[t/(e^t-1)]\,e^{xt}$.

Now, we are ready to sketch the solution.

Let $\{P_n\}_{n\geqslant 0}$ be d-orthogonal and satisfies the equality (2.1). First, by differentiating (1.10) and shifting $m\to m+1$, we get

$$P_{m+d+1}(x) = (m+d+2)\,Q_{m+d+1}(x) - (m+d+1)(x-\beta_{m+d+1})\,Q_{m+d}(x) + \sum_{\nu=0}^{d-1}(m+d-\nu)\,\gamma_{m+d+1-\nu}^{d-1-\nu}\,Q_{m+d-1-\nu}(x), \quad m\geqslant 0. \tag{2.4}$$

Lastly, writing $P_{m+d+1}(x)+(m+d+1)\,Q_{m+d+1}(x)$ and making use of (1.13), we obtain

$$P_{m+d+1}(x) = Q_{m+d+1}(x) + (m+d+1)(\beta_{m+d+1}-\alpha_{m+d})\,Q_{m+d}(x) + \sum_{\nu=0}^{d-1}\left[(m+d-\nu)\,\gamma_{m+d+1-\nu}^{d-1-\nu} - (m+d+1)\,\delta_{m+d-\nu}^{d-1-\nu}\right]Q_{m+d-1-\nu}(x), \quad m\geqslant 0. \tag{2.5}$$

Likewise, differentiating the initial conditions (1.11) and making use of (1.14), we get

$$P_0(x) = Q_0(x),$$

$$P_1(x) = Q_1(x) + (\beta_1-\alpha_0)\,Q_0(x)$$

and if $d\geqslant 2,\ 2\leqslant n\leqslant d$: (2.6)

$$P_n(x) = Q_n(x) + n(\beta_n-\alpha_{n-1})\,Q_{n-1}(x) + \sum_{\mu=0}^{n-2}\left[(n-1-\mu)\,\gamma_{n-\mu}^{d-1-\mu} - n\,\delta_{n-1-\mu}^{d-1-\mu}\right]Q_{n-2-\mu}(x).$$

Otherwise, since $Q_n(x)=P_n(x)$, $n\geqslant 0$, we have necessarily

$$\alpha_m = \beta_m, \quad m\geqslant 0;$$

$$\delta_{m+1}^{\nu} = \gamma_{m+1}^{\nu},\ m\geqslant 0, \text{ for each } \nu \text{ with } 0\leqslant\nu\leqslant d-1,$$

so that, the two relations (2.5) and (2.6) take, respectively, the forms:

$$P_{m+d+1}(x) = P_{m+d+1}(x) + (m+d+1)(\beta_{m+d+1}-\beta_{m+d})\,P_{m+d}(x) + \sum_{\nu=0}^{d-1}\left[(m+d-\nu)\,\gamma_{m+d+1-\nu}^{d-1-\nu} - (m+d+1)\,\gamma_{m+d-\nu}^{d-1-\nu}\right]P_{m+d-1-\nu}(x), \quad m\geqslant 0 \tag{2.7}$$

and for any integer n with $0\leqslant n\leqslant d$, we have

$$P_n(x) = P_n(x) + n(\beta_n-\beta_{n-1})\,P_{n-1}(x) + \sum_{\mu=0}^{n-2}\left[(n-1-\mu)\,\gamma_{n-\mu}^{d-1-\mu} - n\,\gamma_{n-1-\mu}^{d-1-\mu}\right]P_{n-2-\mu}(x), \tag{2.8}$$

with $P_{-n}(x)=0$ and $\sum^{-n}=0$ for $n>1$.

Since $\{P_n\}_{n\geqslant 0}$ is a basis of $\mathscr{P}$, then from (2.7) and (2.8) it is easy to obtain

$$\beta_{n+1} = \beta_n, \quad n \geqslant 0,$$
$$(m+d-\nu)\gamma_{m+d+1-\nu}^{d-1-\nu} = (m+d+1)\gamma_{m+d-\nu}^{d-1-\nu}, \quad m \geqslant 0,\ \nu = 0, 1, \ldots, d-1,$$
$$(n-1-\mu)\gamma_{n-\mu}^{d-1-\mu} = n\gamma_{n-1-\mu}^{d-1-\mu}, \quad 0 \leqslant \mu \leqslant n-2,\ 2 \leqslant n \leqslant d,$$

it follows that

$$\beta_n = \beta_0, \quad n \geqslant 0;$$
$$\gamma_{m+1}^{\nu} = \gamma_1^{\nu}\binom{m+d-\nu}{d-\nu}, \quad m \geqslant 0;\ \nu = 0, 1, \ldots, d-1;$$
$$\gamma_1^0 \neq 0.$$

It is a matter of simple calculus. We can always choose $\beta_0 = 0$ and $\gamma_1^0 = 1/(d+1)$ for d fixed, then

$$P_{m+d+1}(x) = xP_{m+d}(x) - \sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\binom{m+d}{\nu+1}P_{m+d-1-\nu}(x)$$
$$- (d+1)^{-1}\binom{m+d}{d}P_m(x), \quad m \geqslant 0 \tag{2.9}$$

and the initial conditions

$$P_0(x) = 1, \qquad P_1(x) = x$$

and if $d \geqslant 2$: (2.10)

$$P_n(x) = xP_{n-1}(x) - \sum_{\nu=0}^{n-2}\gamma_1^{d-1-\nu}\binom{n-1}{\nu+1}P_{n-2-\nu}(x), \quad 2 \leqslant n \leqslant d,$$

where $\gamma_1^1, \gamma_1^2, \ldots, \gamma_1^{d-1}$ are $d-1$ arbitrary constants.

The two relations (2.9) and (2.10) can be written in the form

$$P_{n+1}(x) = xP_n(x) - \sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\binom{n}{\nu+1}P_{n-1-\nu}(x), \quad n \geqslant 0, \tag{2.11}$$

with $P_0(x) = 1$, $P_{-n}(x) = 0$ for $n \geqslant 1$ and $\binom{n}{\nu} = 0$ if $\nu > n$.

Remarks 2.1. (a) By differentiating k times the polynomial $P_n(x)$ and taking (2.1) into account, we get

$$P_{n-k}(x) = \frac{(n-k)!}{n!}P_n^{(k)}(x), \quad 0 \leqslant k \leqslant n,\ n \geqslant 0, \tag{2.12}$$

where $P_n^{(k)}(x) = D^kP_n(x) = (n+1)^{-1}P_{n+1}^{(k+1)}(x)$.

(b) In $\mathscr{P}'$, the equality (2.1) becomes

$$v_n = u_n \quad \text{for } n \geqslant 0, \tag{2.13}$$

where $\{u_n\}_{n\geqslant 0}$ (resp. $\{v_n\}_{n\geqslant 0}$) is the dual sequence associated to $\{P_n\}_{n\geqslant 0}$ (resp. $\{Q_n\}_{n\geqslant 0}$). Hence,

$$\mathscr{V} = \mathscr{U}, \tag{2.14}$$

where $\mathscr{U}$ (resp. $\mathscr{V}$) is the canonical regular d-dimensional functional with respect to which the polynomial set $\{P_n\}_{n\geqslant 0}$ (resp. $\{Q_n\}_{n\geqslant 0}$) is d-orthogonal.

Otherwise, by making use of (2.13), we obtain from (1.3) the relation:

$$Du_n = -(n+1)u_{n+1}, \quad n \geqslant 0.$$

By recurrence, it follows that

$$u_n = \frac{(-1)^n}{n!} D^n u_0, \quad n \geqslant 0, \tag{2.15}$$

which is another characteristic of the Appell polynomial sets.

Example. For the powers $\{x^n\}_{n\geqslant 0}$, $u_0 = \delta$ (Dirac delta function) and $u_n = [(-1)^n/n!]\, D^n\delta$, $n \geqslant 0$.

In order to produce a d-OPS analogous to the Hermite PS, we have to choose the arbitrary constants $\gamma_1^\nu = 0$ for any $\nu = 1, \dots, d-1$ (choice is always possible, because orthogonality is kept through a shift), then $\{P_n\}_{n\geqslant 0}$ is reducible to a d-symmetric PS.

In this case, $\{P_n\}_{n\geqslant 0}$ is a "classical" d-OPS analogous to the Hermite OPS $\{\hat{H}_n\}_{n\geqslant 0}$. It will be called the Hermite-type d-OPS. One reason for giving this name is that these polynomials have some properties that are analogous to those of the Hermite polynomials (that is the object of the next section).

Therefore, from now on, we denote this d-OPS of type Hermite by $\{\hat{H}_n(\cdot\,; d)\}_{n\geqslant 0}$ and for the corresponding nonmonic PS we adopt the notation $\{H_n(\cdot\,; d)\}_{n\geqslant 0}$, such that $\hat{H}_n(x; d) = k_n H_n(x; d)$, $n \geqslant 0$ where k_n is a normalization constant which will be determined below.

Thus the recurrence (1.17) becomes

$$\begin{aligned} &\hat{H}_{m+d+1}(x; d) = x\hat{H}_{m+d}(x; d) - (d+1)^{-1}\binom{m+d}{d}\hat{H}_m(x; d), \quad m \geqslant 0; \\ &\hat{H}_n(x; d) = x^n, \quad n = 0, 1, \dots, d. \end{aligned} \tag{2.16}$$

Otherwise, since $\{\hat{H}_n(\cdot\,; d)\}_{n\geqslant 0}$ is also d-symmetric, we can write [9]:

$$\hat{H}_{(d+1)n+\mu}(x; d) = x^\mu R_n^\mu(x^{d+1}), \quad n \geqslant 0,\ 0 \leqslant \mu \leqslant d, \tag{2.17}$$

where $\{R_n^\mu\}_{n\geqslant 0}$, $0 \leqslant \mu \leqslant d$ are $d+1$ polynomial sets which are not d-symmetric. Such PSs are called the $d+1$ components of the d-OPS $\{\hat{H}_n(\cdot\,; d)\}_{n\geqslant 0}$. They are defined by the system

$$\rho_{n+1}(x) = -\,\Omega_n(x)\,\rho_n(x), \quad n \geqslant 0, \tag{2.18}$$

with

$$\rho_n(x) = \begin{pmatrix} R_n^0(x) \\ \vdots \\ R_n^d(x) \end{pmatrix}$$

and

$$\Omega_n(x) = \begin{pmatrix} \omega_{n+1} & 0 & 0 & \cdots & 0 & -x \\ \omega_{n+1} & \omega_{n+2} & 0 & \cdots & 0 & -x \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \omega_{n+1} & \omega_{n+2} & \omega_{n+3} & \cdots & 0 & -x \\ \omega_{n+1} & \omega_{n+2} & \omega_{n+3} & \cdots & \omega_{n+d} & -x \\ \omega_{n+1} & \omega_{n+2} & \omega_{n+3} & \cdots & \omega_{n+d} & -x+\omega_{n+d+1} \end{pmatrix},$$

where

$$\omega_{n+\tau} = \gamma^0_{(d+1)n+\tau} = (d+1)^{-1}\binom{(d+1)n+d+\tau}{d}, \quad n \geqslant 0,\ 1 \leqslant \tau \leqslant d+1.$$

According to [9, Théorème 5.2], each component $\{R^\mu_n\}_{n\geqslant 0}$ is d-OPS. Moreover, the first component $\{R^0_n\}_{n\geqslant 0}$ is a "classical" one (see [9] for further details).

When $d = 1$ (ordinary orthogonality case), we find again the Hermite PS $\{\hat{H}_n\}_{n\geqslant 0}$ with its two quadratic components $\{L^{(-1/2)}_n\}_{n\geqslant 0}$ and $\{L^{(1/2)}_n\}_{n\geqslant 0}$, where $\{L^{(\alpha)}_n\}_{n\geqslant 0}$, $\alpha > -1$, are the Laguerre PS.

3. Some characterizations of the sequence $\{P_n\}_{n\geqslant 0}$

At last, it is interesting to show that some properties satisfied by the d-OPS $\{P_n\}_{n\geqslant 0}$ (solution of the problem (P)) and, in particular, those satisfied by the Hermite-type d-OPS $\{\hat{H}_n(.;d)\}_{n\geqslant 0}$ are a natural extension of the Hermite classical OPS. This is given by the following results.

3.1. Generating function

Theorem 3.1. *Let $\{P_n\}_{n\geqslant 0}$ be a PS with the generating function $G(x,t) = \sum_{n\geqslant 0} P_n(x)t^n/n!$. Then $\{P_n\}_{n\geqslant 0}$ is a solution of* (P) *if and only if $G(x,t)$ has the form $G(x,t) = A(t)\mathrm{e}^{xt}$ with*

$$A(t) = \exp\left(-\sum_{\nu=0}^{d-1} \gamma_1^{d-1-\nu}\frac{t^{\nu+2}}{(\nu+2)!}\right). \tag{3.1}$$

Proof. The necessity. Let $\{P_n\}_{n\geqslant 0}$ be a solution of (P), i.e., it satisfies the $(d+1)$-order recurrence relation (1.10) with (1.11) and the equality (2.1).

So that its generating function $G(x,t)$ has the form $G(x,t) = A(t)\mathrm{e}^{xt}$. We shall show that $A(t)$ has the form (3.1). Indeed, by differentiating $G(x,t) = \sum_{n\geqslant 0} P_n(x)t^n/n!$ with respect to t, and using the notation $\partial_t = \partial/\partial t$, we have

$$\partial_t G(x,t) = \sum_{n\geqslant 1} P_n(x)\frac{t^{n-1}}{(n-1)!} = \sum_{n\geqslant 0} P_{n+1}(x)\frac{t^n}{n!}. \tag{3.2}$$

Now, in virtue of (2.11), the equality (3.2) can be written

$$\begin{aligned}
\partial_t G(x,t) &= \sum_{n\geqslant 0}\left[xP_n(x) - \sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\binom{n}{\nu+1}P_{n-1-\nu}(x)\right]\frac{t^n}{n!}\\
&= xG(x,t) - \sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\frac{t^{\nu+1}}{(\nu+1)!}\sum_{n\geqslant 0}P_{n-1-\nu}(x)\frac{t^{n-1-\nu}}{(n-1-\nu)!}\\
&= xG(x,t) - \left(\sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\frac{t^{\nu+1}}{(\nu+1)!}\right)G(x,t).
\end{aligned}$$

It follows that $G(x,t)$ satisfies the partial differential equation:

$$\partial_t G(x,t) = \left(x - \sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\frac{t^{\nu+1}}{(\nu+1)!}\right)G(x,t). \tag{3.3}$$

Otherwise, using the Appell character, i.e., the generating function has the form $G(x,t) = A(t)\,\mathrm{e}^{xt}$ and differentiating it always with respect to t, we get

$$\partial_t G(x,t) = A'(t)\,\mathrm{e}^{xt} + xG(x,t). \tag{3.4}$$

From (3.3) and (3.4), it is easy to see that $A(t)$ satisfies the differential equation:

$$\begin{aligned}
&A'(t) + \left(\sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\frac{t^{\nu+1}}{(\nu+1)!}\right)A(t) = 0;\\
&A(0) = 1
\end{aligned} \tag{3.5}$$

of which the solution has the form (3.1): $A(t) = \exp(-\sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}t^{\nu+2}/(\nu+2)!)$. Thus, the necessity of the condition stated is established.

Conversely, suppose that $G(x,t) = A(t)\,\mathrm{e}^{xt}$ is the generating function of the polynomials $P_n(x)$, $n \geqslant 0$, where $A(t)$ has the form (3.1), we shall show that the PS $\{P_n\}_{n\geqslant 0}$ is a solution of the problem (P). Since $G(x,t) = A(t)\,\mathrm{e}^{xt}$ with (3.1), it is clear that $\{P_n\}_{n\geqslant 0}$ is an Appell PS and $G(x,t)$ satisfies the partial differential equation (3.3).

Therefore, it suffices to show that $\{P_n\}_{n\geqslant 0}$ is d-OPS, i.e., that the $(d+1)$-order recurrence relation (2.11) holds. Indeed, replacing $G(x,t)$ by $\sum_{n\geqslant 0}P_n(x)\,t^n/n!$ into (3.3), we obtain

$$\sum_{n\geqslant 0}P_{n+1}(x)\frac{t^n}{n!} = x\sum_{n\geqslant 0}P_n(x)\frac{t^n}{n!} - \left(\sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\frac{t^{\nu+1}}{(\nu+1)!}\right)\sum_{n\geqslant 0}P_n(x)\frac{t^n}{n!}$$

it follows that

$$\sum_{n\geqslant 0}P_{n+1}(x)\frac{t^n}{n!} = \sum_{n\geqslant 0}xP_n(x)\frac{t^n}{n!} - \sum_{n\geqslant 0}\left(\sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\frac{t^{\nu+1}}{(\nu+1)!}\right)P_n(x)\frac{t^n}{n!}$$

or

$$\sum_{n\geqslant 0}P_{n+1}(x)\frac{t^n}{n!} = \sum_{n\geqslant 0}\left[xP_n(x) - \sum_{\nu=0}^{d-1}\gamma_1^{d-1-\nu}\binom{n}{\nu+1}P_{n-1-\nu}(x)\right]\frac{t^n}{n!}.$$

Comparing the coefficients of t^n, the recurrence (2.11) follows immediately.

Hence, the converse statement of Theorem 3.1 is thus established. □

Remarks 3.2. (a) When $\gamma_1^\nu = 0$ for $\nu = 1, 2, \ldots, d-1$, we obtain the generating function of the Hermite-type d-OPS $\{\hat{H}_n(.; d)\}_{n \geqslant 0}$:

$$G(x,t) = A(t)\mathrm{e}^{xt} \text{ with } A(t) = \exp\left(-\frac{t^{d+1}}{d!(d+1)^2}\right).$$

In particular, for $d = 1$, we meet again $G(x,t) = \mathrm{e}^{xt - t^2/4}$ which is the generating function of the Hermite PS $\{\hat{H}_n\}_{n \geqslant 0}$.

(b) Setting $x = 0$ in $G(x,t) = A(t)\mathrm{e}^{xt}$, expanding $\exp(-t^{d+1}/d!(d+1)^2)$ in a power series, and comparing coefficients of powers of t in both sides of the resulting equation, we find that

$$\hat{H}_{(d+1)n}(0; d) = (-1)^n \frac{((d+1)n)!}{(d+1)^{2n}(d!)^n n!},$$

$$\hat{H}_{(d+1)n+\mu}(0; d) = 0, \quad 1 \leqslant \mu \leqslant d$$

and it is easy to see that 0 is a zero of $\hat{H}_{(d+1)n+\mu}(x; d)$ of multiplicity μ.

(c) Otherwise, in order to produce a d-orthogonal polynomials of type Hermite $\{H_n(.; d)\}_{n \geqslant 0}$ analogous to the nonmonic polynomials of Hermite, we have to choose the normalization coefficient $k_n = \lambda^{-n}$, $n \geqslant 0$ with $\lambda = (d!(d+1)^2)^{1/(d+1)}$.

Indeed, replacing t by λt into

$$\exp\left(xt - \frac{t^{d+1}}{d!(d+1)^2}\right) = \sum_{n \geqslant 0} \hat{H}_n(x; d)\frac{t^n}{n!},$$

we obtain that

$$\exp(\lambda xt - t^{d+1}) = \sum_{n \geqslant 0} \lambda^n \hat{H}_n(x; d)\frac{t^n}{n!} = \sum_{n \geqslant 0} H_n(x; d)\frac{t^n}{n!},$$

which is the generating function of the Hermite-type d-OPS $\{H_n(\cdot; d)\}_{n \geqslant 0}$.

When $d = 1$, we meet again the generating function of the Hermite polynomials $\{H_n\}_{n \geqslant 0}$

$$\exp(2xt - t^2) = \sum_{n \geqslant 0} H_n(x)\frac{t^n}{n!}.$$

3.2. $(d+1)$-order differential equation

Let us adopt the following notation for a finite differential operator on $\mathscr{P}$:

$$L[y(x)] = \alpha_{d+1} y^{(d+1)}(x) + \alpha_d y^{(d)}(x) + \cdots + \alpha_2 y''(x) + \alpha_1 xy'(x),$$

where the α_ν's, $1 \leqslant \nu \leqslant d+1$ are constants.

Theorem 3.3. *Let $\{P_n\}_{n \geqslant 0}$ be a PS. A necessary and sufficient condition that $\{P_n\}_{n \geqslant 0}$ is a solution of the problem* (P) *is that each polynomial $P_n(x)$, $n = 0, 1, \ldots,$ satisfies the following $(d+1)$-order differential equation of type*:

$$\frac{\gamma_1^0}{d!} P_n^{(d+1)}(x) + \frac{\gamma_1^1}{(d-1)!} P_n^{(d)}(x) + \cdots + \frac{\gamma_1^{d-1}}{1!} P_n''(x) - xP_n'(x) + nP_n(x) = 0, \quad n \geqslant 0, \tag{3.6}$$

i.e., using the previous differential operator notation:

$$L[P_n(x)] = nP_n(x), \quad n \geqslant 0$$

with

$$\alpha_1 = 1$$

and

$$\alpha_\nu = -\frac{\gamma_1^{d+1-\nu}}{(\nu-1)!}$$

for each $2 \leqslant \nu \leqslant d+1$.

Proof. The necessity of (3.6). Let $\{P_n\}_{n\geqslant 0}$ be a solution of (P), so that it satisfies the $(d+1)$-order recurrence relation (2.11) and the Appell character (2.1).

In virtue of (2.12), replacing $P_{n-k}(x)$, $n \geqslant 0$, $k = 0, 1, \ldots, d$, by the successive derivatives $P_{n+1}^{(k+1)}(x)$ into the recurrence (2.11), the $(d+1)$-order differential equation (3.6) follows immediately.

Conversely, suppose that the sequence $\{P_n\}_{n\geqslant 0}$ satisfies the differential equation (3.6). It is verified that $Q_0(x) = P_0(x) = 1$.

Assuming that $Q_m(x) = P_m(x)$ for $m = 1, 2, \ldots, n-1$, we shall prove it for $m = n$. Differentiating (3.6), we get

$$\frac{\gamma_1^0}{d!} P_n^{(d+2)}(x) + \frac{\gamma_1^1}{(d-1)!} P_n^{(d+1)}(x) + \cdots + \frac{\gamma_1^{d-1}}{1!} P_n^{(3)}(x) - xP_n''(x) + (n-1)P_n'(x) = 0, \quad n \geqslant 0. \tag{3.7}$$

Changing $n \to n+1$ and $P_{n+1}'(x)$ by $(n+1)Q_n(x)$, we find that the sequence $\{Q_n\}_{n\geqslant 0}$ also satisfies a $(d+1)$-order differential equation of type (3.6):

$$\frac{\gamma_1^0}{d!} Q_n^{(d+1)}(x) + \frac{\gamma_1^1}{(d-1)!} Q_n^{(d)}(x) + \cdots + \frac{\gamma_1^{d-1}}{1!} Q_n''(x) - xQ_n'(x) + nQ_n(x) = 0, \quad n \geqslant 0, \tag{3.8}$$

that is

$$L[Q_n(x)] = nQ_n(x), \quad n \geqslant 0.$$

Hence, since the differential operator L is nonsingular, we have $Q_n(x) = c_n P_n(x)$.

But, the polynomials $Q_n(x)$ and $P_n(x)$ are monics, comparing coefficients of x^n, we find that $c_n = 1$, therefore $Q_n(x) = P_n(x)$ for all $n \geqslant 0$, i.e., $\{P_n\}_{n\geqslant 0}$ is an Appell PS.

Now, making use of (2.12) and $P_n^{(k)}(x)$ by the corresponding expression of $P_{n-k}(x)$, $0 \leqslant k \leqslant n$, in the differential equation (3.6), it is easy to obtain the $(d+1)$-order recurrence relation (2.11). Hence, $\{P_n\}_{n\geqslant 0}$ is at the same time d-OPS and Appell PS, then it is a solution of the problem (P). This proves the converse. □

Remarks 3.4. (a) The differential equation (3.6) is a particular case of the one given in [17].

(b) For the d-symmetrical case, i.e., the Hermite-type d-OPS $\{\hat{H}_n(\cdot;d)\}_{n\geqslant 0}$, the differential equation (3.6) becomes

$$\hat{H}_n^{(d+1)}(x;d) - (d+1)!\,x\hat{H}_n'(x;d) + (d+1)!\,n\hat{H}_n(x;d) = 0, \quad n \geqslant 0. \tag{3.9}$$

In particular, for $d = 1$ we get again the second-order differential equation satisfied by the Hermite polynomials $\{\hat{H}_n\}_{n\geqslant 0}$:

$$\hat{H}_n''(x) - 2x\hat{H}_n'(x) + 2n\hat{H}_n(x) = 0, \quad n \geqslant 0.$$

3.3. A vectorial functional equation and moments

We now consider the d-dimensional functional $\mathscr{U} = (u_0, \ldots, u_{d-1})^{\mathrm{T}}$ with respect to which the PS $\{P_n\}_{n\geqslant 0}$ is d-orthogonal.

We begin by recalling the following characterization of the "classical" d-OPS [10].

Theorem 3.5. *A d-OPS $\{P_n\}_{n\geqslant 0}$ is "classical" if and only if there exist two $d\times d$ polynomial matrices $\Psi = (\psi_\mu^\nu)_{\nu,\mu=0}^{d-1}$, $\Phi = (\phi_\mu^\nu)_{\nu,\mu=0}^{d-1}$ with $\deg\psi_\mu^\nu \leqslant 1$, $\deg\phi_\mu^\nu \leqslant 2$, such that*

$$\Psi\mathscr{U} + D(\Phi\mathscr{U}) = 0 \tag{3.10}$$

with conditions about regularity (see [10, Théorème 3.1]).

Here the action of the d-dimensional functional $\mathscr{U} = (u_0, \ldots, u_{d-1})^{\mathrm{T}}$ on a polynomial f as well as the left-multiplication of a functional by a polynomial are defined, respectively, by

$$\mathscr{U}(f) := \begin{pmatrix} \langle u_0, f\rangle \\ \langle u_1, f\rangle \\ \vdots \\ \langle u_{d-1}, f\rangle \end{pmatrix} \quad \text{and} \quad \langle hu, f\rangle := \langle u, hf\rangle, \quad \forall\, u\in\mathscr{P}';\ \forall\, h, f\in\mathscr{P}.$$

According to the above theorem (which we adopt to our situation), we obtain easily the following result.

Theorem 3.6. *The PS $\{P_n\}_{n\geqslant 0}$ is solution of the problem* (P) *if and only if the d-dimensional functional $\mathscr{U}$ satisfies the vectorial functional equation*:

$$\Psi\mathscr{U} + D(\Phi\mathscr{U}) = 0, \tag{3.11}$$

where Ψ and Φ are the two $d\times d$ matrices

$$\Psi(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d-1 \\ (d/\gamma_1^0)x & \zeta_1 & \zeta_2 & \cdots & \zeta_{d-1} \end{pmatrix}, \qquad \Phi(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

with $\zeta_\mu = -\,d\gamma_1^{d-\mu}/\gamma_1^0$, $\mu = 1, \ldots, d$ are arbitrary constants.

Proof. It is a particular case of the result given in [10]. □

Remark 3.7. For the d-OPS of type Hermite $\{\hat{H}_n(\cdot;d)\}_{n\geqslant 0}$, we have $\gamma_1^0=(d+1)^{-1}$ and $\gamma_1^\nu=0$ for $\nu=1,\dots,d-1$, then $\zeta_\mu=0$, $\mu=1,\dots,d-1$. *In the sequel we consider only this case.*

For any polynomial f, the vectorial equation (3.11) leads to

$$(\Psi\mathscr{U})(f)+D(\Phi\mathscr{U})(f)=0,$$

that is to say

$$\begin{pmatrix} \langle u_1,f\rangle \\ 2\langle u_2,f\rangle \\ \vdots \\ (d-1)\langle u_{d-1},f\rangle \\ d(d+1)\ \langle xu_0,f\rangle \end{pmatrix} - \begin{pmatrix} \langle u_0,f'\rangle \\ \langle u_1,f'\rangle \\ \vdots \\ \langle u_{d-2},f'\rangle) \\ \langle u_{d-1},f'\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

In particular, for $f(x)=x^{m+1}$, $m\geqslant 0$, we have the recurrence of the moments of the linear functionals $u_0,u_1,\dots,u_{d-1}$, which we write in the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \dfrac{1}{m+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \dfrac{d-1}{m+1} & 0 \\ 0 & 0 & \cdots & 0 & \dfrac{d(d+1)}{m+1} \end{pmatrix} \begin{pmatrix} (u_0)_{m+1} \\ (u_1)_{m+1} \\ \vdots \\ (u_{d-1})_{m+1} \\ (u_0)_{m+2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} (u_0)_m \\ (u_1)_m \\ \vdots \\ (u_{d-1})_m \\ (u_0)_{m+1} \end{pmatrix}.$$

Then

$$\left.\begin{aligned} (u_1)_{m+1} &= (m+1)(u_0)_m \\ (u_2)_{m+1} &= \frac{1}{2}(m+1)(u_1)_m \\ &\vdots \\ (u_{d-1})_{m+1} &= \frac{1}{d-1}(m+1)(u_{d-2})_m \\ (u_0)_{m+2} &= \frac{1}{d(d+1)}(m+1)(u_{d-1})_m \end{aligned}\right\} \quad m\geqslant 0. \tag{3.12}$$

It follows that

$$(u_0)_{m+d+1}=(d+1)^{-1}\binom{m+d}{d}(u_0)_m,\quad m\geqslant 0. \tag{3.13}$$

Corollary 3.8. *The moments of the linear functionals $u_0, \ldots, u_{d-1}$ are given by*
– *for the linear functional u_0:*

$$(u_0)_0 = 1,$$

$$(u_0)_{(d+1)m} = (d+1)^{-1} \prod_{n=0}^{m-1} \binom{(d+1)n+d}{d}, \quad m \geqslant 1, \tag{3.14}$$

$$(u_0)_{(d+1)m+\mu} = 0, \quad \mu = 1, \ldots, d, \ m \geqslant 0;$$

– *and for each u_α with $\alpha = 1, \ldots, d-1$:*

$$\begin{aligned}(u_\alpha)_m = \langle u_\alpha, x^m \rangle &= \frac{(-1)^\alpha}{\alpha!} \langle D^\alpha u_0, x^m \rangle \\ &= \binom{m}{\alpha} \langle u_0, x^{m-\alpha} \rangle \\ &= \binom{m}{\alpha} (u_0)_{m-\alpha}, \quad \alpha = 1, 2, \ldots, d-1, \ m \geqslant 0,\end{aligned}$$

which we can write

$$(u_\alpha)_{(d+1)m+\mu} = \delta_{\alpha,\mu} \binom{(d+1)m+\mu}{\alpha} (u_0)_{(d+1)m}, \quad \mu = 0, 1, \ldots, d; \ m \geqslant 0, \tag{3.15}$$

where we have taken $(u_0)_{-n} = 0$, $n \geqslant 1$.

Proof. From (3.13) we have immediately (3.14) and from (3.12) we obtain (3.15). □

3.4. A differential relation

Finally, in order to complete the analogy between the Hermite-type d-OPS and the classical OPS of Hermite, we conclude this section with the following differential relation which generalizes the one given by McCarthy [16] in the ordinary orthogonality case for the Hermite polynomials $\{\hat{H}_n\}_{n \geqslant 0}$.

The derivative of the two consecutive polynomials' product $\hat{H}_n(x;d)\hat{H}_{n+1}(x;d)$ is given by

$$\begin{aligned}(\hat{H}_{n+1}(x;d)\hat{H}_n(x;d))' = (n+1)\hat{H}_n^2(x;d) &+ \frac{n!\,(d+1)!}{(n+d-1)!} x\hat{H}_{n+1}(x;d)\hat{H}_{n+d-1}(x;d) \\ &- \frac{n!(d+1)!}{(n+d-1)!} \hat{H}_{n+1}(x;d)\hat{H}_{n+d}(x;d), \quad n \geqslant 0.\end{aligned} \tag{3.16}$$

Indeed, differentiating the product $\hat{H}_{n+1}(x;d)\hat{H}_n(x;d)$, we get

$$(\hat{H}_{n+1}(x;d)\hat{H}_n(x;d))' = \hat{H}'_{n+1}(x;d)\hat{H}_n(x;d) + \hat{H}_{n+1}(x;d)\hat{H}'_n(x;d). \tag{3.17}$$

Hence, since $\{\hat{H}_n(\cdot;d)\}_{n \geqslant 0}$ is an Appell sequence, we have

$$\hat{H}'_{n+1}(x;d) = (n+1)\hat{H}_n(x;d) \quad \text{and} \quad \hat{H}'_n(x;d) = n\hat{H}_{n-1}(x;d).$$

Now, from the recurrence relation (2.16), it is possible to express $\hat{H}_{n-1}(x;d)$ in terms of $\hat{H}_{n+d-1}(x;d)$ and $\hat{H}_{n+d}(x;d)$, substituting into (3.17), the relation (3.16) follows immediately.

When $d=1$, the previous identity is reduced to the quadratic diferential relation satisfied by the Hermite polynomials $\{\hat{H}_n\}_{n\geqslant 0}$ [16]:

$$(\hat{H}_{n+1}(x)\hat{H}_n(x))' = (n+1)\hat{H}_n^2(x) + 2x\hat{H}_{n+1}(x)\hat{H}_n(x) - 2\hat{H}_{n+1}^2(x), \quad n \geqslant 0.$$

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