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Lelek's problem is not a metric problem

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A B S T R A C T

We show that Lelek's problem on the chainability of continua with span zero is not a metric problem: from a non-metric counterexample one can construct a metric one.

0. Introduction

The notion of span of a metric continuum was introduced by Lelek in [9], where he showed that chainable continua have span zero, and in [10] he asked whether continua with span zero are chainable. This has become one of the classic problems of Continuum Theory, see [11] for a recent survey.

The purpose of this paper is not to solve Lelek's problem; our goal is more modest: we show that a non-metrizable counterexample to the problem may be converted into a metrizable one. This makes the tools of infinitary combinatorics available to those searching for a counterexample.

Our proof makes use of methods from Model Theory, most notably the Löwenheim–Skolem theorem. Given a non-metric continuum one can use this theorem to obtain a metric quotient that shares many properties with the original space. Indeed, we shall prove that the quotient will be chainable iff the original space is and likewise for having span zero. The proof of one of the four implications is much more involved than that of the others as it relies on Shelah's Ultrapower Isomorphism theorem from [12]. This suggests an obvious question that we shall discuss at the end of this paper.

Section 1 contains some preliminaries. We repeat the definitions of chainability and the various forms of span. We also describe the results from Model Theory that will be used in the proofs. In Section 2 we prove our main results and in Section 3 we discuss some questions related to the proofs.

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1. Preliminaries

1.1. Chainability and span

Let X be a continuum, i.e., a connected compact Hausdorff space. We say X is *chainable* if every finite open cover has a refinement that is a chain, which means that it can be enumerated as $\langle V_i\colon i<n\rangle$ such that $V_i \cap V_j \neq \emptyset$ iff $|i-j| \leqslant 1$.

We shall deal with four kinds of span: span, semispan, surjective span, and surjective semispan. Each is defined, for a *metric* continuum (X,d), as the supremum of all $\epsilon \geqslant 0$ for which there is a subcontinuum Z of $X \times X$ with the property that $d(x,y) \geqslant \epsilon$ for all $(x,y) \in Z$ and

- $\pi_1[Z] = \pi_2[Z]$, in the case of span;
- $\pi_1[Z] \subseteq \pi_2[Z]$, in the case of semispan;
- $\pi_1[Z] = \pi_2[Z] = X$, in the case of surjective span; or
- $\pi_2[Z] = X$, in the case of surjective semispan.

Thus any one of the spans is equal to zero if *every* subcontinuum of $X \times X$ with the corresponding property from the list must intersect the diagonal Δ_X of X. This then yields four definitions of having span zero for general continua.

There are relations between these four kinds of span zero, corresponding to the inclusion relations between the defining collections of subcontinua of $X \times X$; see [5] for a diagram and also for a proof that chainability implies that all spans are zero.

The diagram in [5] also mentions (surjective) symmetric span, but, as reported in [3], the dyadic solenoid, which is not chainable, has symmetric span zero, so that symmetric span zero does not characterize chainability. The reader will be able to check that having (surjective) symmetric span zero is also covered by our reflection results.

1.2. Wallman representation

In the construction of the metric quotient we employ the Wallman representation of distributive lattices.

We start with a compact Hausdorff space X and consider its lattice of closed sets 2^X. Any sublattice, L, of 2^X gives rise to a continuous image of X: the space wL of ultrafilters on L. If $a \in L$ then $\bar{a}$ denotes $\{u \in wL\colon a \in u\}$; the family $\{\bar{a}\colon a \in L\}$ is used as a base for the closed sets in wL. In general this yields a T_1-space; the space wL is Hausdorff iff L is *normal*, which means that disjoint elements of L can be separated by disjoint open sets that are complements of members of L.

In general, a lattice embedding $h : L \to K$ yields a continuous onto map $wh : wK \to wL$, where $wh(u)$ is the unique ultrafilter on L that contains $\{a\colon h(a) \in u\}$ (this family is a prime filter), so that in our case we obtain a continuous onto map $q_L : X \to wL$.

It should be clear that X is the Wallman space of 2^X. However, one space may correspond to many lattices. Indeed, if $\mathcal{C}$ is a base for the closed sets of X that is closed under finite unions and intersections then $X = w\mathcal{C}$.

The article [1] gives a good introduction to Wallman representations.

1.3. Elementarity

To construct the metric quotient mentioned in the Introduction we need a special sublattice of 2^X, an *elementary* sublattice.

In general a substructure A of some structure B (a group, a field, a lattice) is said to be an *elementary* substructure if every sentence in the language for the structure, with parameters from A, that is true in B is also true in A. A sentence is a formula without free variables and such a formula is true in a structure if it holds with all its quantifiers bound by that structure.

As a quick example consider the subfield $\mathbb{Q}$ of $\mathbb{R}$: it is *not* an elementary subfield because of the following sentence:

$$(\exists x)\big(x^2 = 2\big)$$

The parameter 2 belongs to $\mathbb{Q}$; the sentence holds in $\mathbb{R}$ but does not hold in $\mathbb{Q}$. This example illustrates the source of the power of elementarity: because all existential statements true in the larger structure must be true in the substructure this substructure is very rich. In fact, an elementary subfield of $\mathbb{R}$ must contain all real algebraic numbers and it is a non-trivial result that these numbers do in fact form an elementary subfield of $\mathbb{R}$.

By a straightforward closing-off argument one shows that every subset of a structure can be expanded to an elementary substructure — this is the Löwenheim–Skolem theorem [6, Corollary 3.1.4]. In full it states that a subset, C, of a structure B can be expanded to an elementary substructure A whose cardinality is at most $\aleph_0 \cdot |C| \cdot |\mathcal{L}|$, where $\mathcal{L}$ is the language used to describe the structures. In the case of lattices the language is countable: one needs $\wedge$, $\vee$ and $=$ as well as logical symbols and (countably many) variables. Thus every lattice has a countable elementary sublattice.

As we discuss in Section 3 the expressive power of the language of lattices is not strong enough for our purposes; therefore we consider structures for the language of Set Theory. Any reasonably large set will do but usually one takes a 'suitably large' regular cardinal number θ and considers the set $H(\theta)$ of sets that are hereditarily of cardinality less than θ, which means that they and their elements and their elements' elements and … all have cardinality less than θ. The advantage of these sets is that they satisfy all the axioms of Set Theory, except possibly the power set axiom.

What will be particularly useful to us is that if M is an elementary substructure of $H(\theta)$ then ω is both an element and a subset of M; this is because ω and each finite ordinal are uniquely defined in $H(\theta)$ by a formula with just one free variable; therefore they automatically belong to M. As a consequence of this every finite subset of M is an element of M and this will give us the extra flexibility that we need.

We refer to [8, Chapters IV and V] for information on the sets $H(\theta)$ and elementarity in the context of Set Theory. Note that the language of Set Theory has even fewer non-logical symbols than that of lattice theory: $\in$ and $=$. The lattice operations, $\cap$ and $\cup$, are derived from these.

1.4. Ultrapowers and ultracopowers

We shall be using ultrapowers of lattices so we need to fix some notation. Let L be a lattice; given an ultrafilter u on a cardinal number κ we define the ultrapower $\prod_u L$ of L by u to be the quotient of L^κ by the equivalence relation $\sim_u$ defined by $\langle x_\alpha\colon \alpha<\kappa\rangle \sim_u \langle y_\alpha\colon \alpha<\kappa\rangle$ iff $\{\alpha\colon x_\alpha = y_\alpha\}\in u$. We turn $\prod_u L$ into a lattice by defining the operations pointwise. There is an obvious embedding $\triangle : L \to \prod_u L$, the diagonal embedding, defined by sending an element a to the (class of the) sequence $\langle a\colon \alpha<\kappa\rangle$.

Dual to this is the notion of ultracopower of a compact Hausdorff space X by an ultrafilter u. One can define it in two equivalent ways. The first is as the Wallman representation of the ultrapower $\prod_u 2^X$ of the lattice 2^X by u.

The second is via the Čech–Stone compactification. Consider the product $\kappa \times X$, where κ carries the discrete topology, and the two projections $\pi_X : \kappa \times X \to X$ and $\pi_\kappa : \kappa\times X \to \kappa$. These have extensions, $\beta\pi_X : \beta(\kappa\times X) \to X$ and $\beta\pi_\kappa : \beta(\kappa\times X) \to \beta\kappa$ respectively. The preimage $\beta\pi_\kappa^{\leftarrow}(u)$ is homeomorphic to the Wallman representation of $\prod_u 2^X$. This follows from the facts that

(1) $\beta(\kappa\times X)$ is the Wallman representation of $2^{\kappa\times X}$, which in turn is isomorphic to $(2^X)^\kappa$; and
(2) if F and G are closed subsets of $\kappa\times X$ then the intersections $\mathrm{cl}_\beta F \cap \beta\pi_\kappa^{\leftarrow}(u)$ and $\mathrm{cl}_\beta G \cap \beta\pi_\kappa^{\leftarrow}(u)$ are equal iff the set of αs for which $F\cap(\{\alpha\}\times X) = G\cap(\{\alpha\}\times X)$ belongs to u.

The topological viewpoint enables us to see easily that one may use any base, $\mathcal{C}$, for the closed sets that is closed under finite unions and finite intersections to construct the ultracopower. Indeed, if F and G are closed and disjoint in $\kappa\times X$ then a compactness argument applied to $\{\alpha\}\times X$ for each α will yield sequences $\langle B_\alpha\colon \alpha<\kappa\rangle$ and $\langle C_\alpha\colon \alpha<\kappa\rangle$ in $\mathcal{C}$ such that $B_\alpha\cap C_\alpha=\emptyset$ for all α, and $F\subseteq \bigcup_\alpha\{\alpha\}\times B_\alpha$ and $G\subseteq\bigcup_\alpha\{\alpha\}\times C_\alpha$.

This then can be used to show that the dual to the inclusion map $\mathcal{C}^\kappa \to (2^X)^\kappa$ is injective, so that $\beta(\kappa\times X) = w(\mathcal{C}^\kappa)$, and, similarly, that the dual to the inclusion map $\prod_u \mathcal{C} \to \prod_u 2^X$ is injective, which gives us that $\beta\pi_\kappa^{\leftarrow}(u)$ is the Wallman representation of $\prod_u \mathcal{C}$.

We denote the ultracopower of X by u as $\coprod_u X$. Also, if $\langle F_\alpha\colon \alpha<\kappa\rangle$ is a sequence of closed subsets of X then we let F_u be the intersection of $\mathrm{cl}_\beta(\bigcup_\alpha\{\alpha\}\times F_\alpha)$ with $\coprod_u X$; in case $F_\alpha = F$ for all α the set F_u corresponds to the image of F under the diagonal embedding into $\prod_u 2^X$.

The restriction of $\beta\pi_X$ to $\coprod_u X$ is induced by the diagonal embedding $\triangle$, we shall denote it by $\triangledown$.

2. Reflections

We fix a continuum X, a suitably large cardinal θ and a countable elementary substructure M of $H(\theta)$ with $X\in M$; as θ was taken large enough the entities $X\times X$, 2^X and $2^{X\times X}$ belong to M as well, by elementarity. We let $L = M\cap 2^X$ and $K = M\cap 2^{X\times X}$. The family $\mathcal{B}_L = \{wL\setminus F\colon F\in L\}$ is a base for the open sets of L.

As M is countable, so are L and K. Therefore wL and wK are compact *metrizable* spaces. We shall have proved our main result once we establish that wL is chainable iff X is and that wL has span zero iff X does.

2.1. Chainability

We first show that X is chainable if and only if wL is. The forward implication is easiest to establish.

Proposition 2.1. *([13, Section 7.2]) If X is chainable then so is wL.*

Proof. Let $\mathcal{U}$ be a finite open cover of wL. By compactness we can find a finite subfamily $\mathcal{B}$ of $\mathcal{B}_L$ that refines $\mathcal{U}$. Because every finite subset of M belongs to M we have $\mathcal{B}\in M$. Now the formula that expresses '$\mathcal{C}$ is a chain refinement of $\mathcal{B}$' — with $\mathcal{C}$ as its only free variable — is satisfied by a member of $H(\theta)$ and hence by an element of M. The latter consists of members of $\mathcal{B}_L$ and is a finite chain refinement of $\mathcal{B}$, and hence of $\mathcal{U}$. □

The converse implication is slightly harder to establish; in the proof we employ the notion of a precise refinement. A *precise* refinement of a cover $\mathcal{U}$ is a refinement, $\{V_U\colon U \in \mathcal{U}\}$, indexed by $\mathcal{U}$ such that $V_U \subseteq U$ for all U.

Proposition 2.2. *([13, Section 7.3]) If X is not chainable then neither is wL.*

Proof. There is an open cover of X that does not have an open chain refinement. This statement can be expressed by a formula, with parameters in M, that is quite complicated: expressing that a cover does not have a chain refinement involves a quantification over all finite sequences of elements of 2^X.

By elementarity this formula holds in M, so we can take an open cover, $\mathcal{U}$, of X that belongs to M and that satisfies the formula *with all quantifiers restricted to M*, which means that $\mathcal{U}$ has no chain refinements that consist of members of $\mathcal{B}_L$.

As $\mathcal{U}$ is a subset of $\mathcal{B}_L$ it also forms an open cover of wL. We must show that $\mathcal{U}$ does not have any open chain refinement at all. Let $\mathcal{V}$ be any finite open refinement of $\mathcal{U}$. By normality we can find a closed cover $\{F_V\colon V \in \mathcal{V}\}$ of wL such that $F_V \subseteq V$ for all V. By compactness we can find finite subfamilies $\mathcal{B}_V$ of $\mathcal{B}_L$ such that $F_V \subseteq \bigcup \mathcal{B}_V \subseteq V$ for all V. Then $\mathcal{W} = \{\bigcup \mathcal{B}_V\colon V \in \mathcal{V}\}$ is a refinement of $\mathcal{U}$ that consists of members of $\mathcal{B}_L$, hence it is not a chain refinement. As $\mathcal{W}$ is a precise refinement of $\mathcal{V}$ the latter is not a chain refinement of $\mathcal{U}$ either. □

2.2. Products

To establish that (non-)zero span is reflected we need to explore the relationship between $wL \times wL$ and wK.

It is clear, by elementarity, that K contains the families $\{A \times X\colon A \in L\}$ and $\{X \times A\colon A \in L\}$. We use L' to denote the sublattice of K generated by these families. We trust that the reader will recognize the formula implicit in the following proof.

Lemma 2.3. *If F and G are elements of K with empty intersection then there are F' and G' in L' such that $F \subseteq F'$, $G \subseteq G'$ and $F' \cap G' = \emptyset$.*

Proof. By compactness there are finite families $\mathcal{U}$ and $\mathcal{V}$ of basic open sets such that $F \subseteq \bigcup \mathcal{U}$, $G \subseteq \bigcup \mathcal{V}$ and $\operatorname{cl} \bigcup \mathcal{U} \cap \operatorname{cl} \bigcup \mathcal{V} = \emptyset$. By elementarity, and because $F, G \in M$ there are in M two sequences $\langle \langle A_i, B_i\rangle, i < n\rangle$ and $\langle \langle C_j, D_j\rangle, j < m\rangle$ of pairs of closed sets such that $F \subseteq \bigcup_{i<n}(A_i \times B_i)$, $G \subseteq \bigcup_{j<m}(C_j \times D_j)$ and $\bigcup_{i<n}(A_i \times B_i) \cap \bigcup_{j<m}(C_j \times D_j) = \emptyset$. The two unions belong to L' and are the sets F' and G' that we seek. □

This lemma implies that $wK = wL'$ in the sense that $u \mapsto u \cap L'$ is a homeomorphism between the two spaces. Furthermore it should be clear that L' serves as a lattice base for the closed sets of $wL \times wL$, so that $wL' = wL \times wL$.

We find that $wK = wL \times wL$ by means of a natural homeomorphism f: the diagonal of the two maps p_1 and p_2 from wK to wL: $p_1(u) = \{A \in L\colon A \times X \in u\}$ and $p_2(u) = \{A \in L\colon X \times A \in u\}$.

This implies that the product map $q_L \times q_L : X \times X \to wL \times wL$ can be factored as $f \circ q_K$; here $q_L : X \to wL$ and $q_K : X \times X \to wK$ are the maps dual to the inclusions $L \subseteq 2^X$ and $K \subseteq 2^{X\times X}$ respectively. It also follows that p_1 and p_2 correspond to the projections from $wL \times wL$ to wL.

Where possible we will suppress mention of the map f and simply identify wK with $wL \times wL$; we also use q_K in stead of $q_L \times q_L$.

2.3. Reflecting non-zero span

Using the above result on products we prove the first reflection result on span.

Proposition 2.4. *([13, Section 7.4]) If the span (of any kind) of X is non-zero then the span (of the same kind) of wL is non-zero too.*

Proof. Because having non-zero span is an existential statement we immediately apply elementarity to conclude that there is $Z \in M$ that is a subcontinuum of $X \times X$, that is disjoint from the diagonal Δ_X of X and has the corresponding property from the list in Section 1.1.

Since Z and Δ_X belong to K their images under q_K are disjoint as well, so that $q_K[Z]$ is a continuum in $wL \times wL$ that is disjoint from Δ_{wL}.

Using the properties of the maps q_L and q_K derived above it follows that $q_K[Z]$ satisfies the same property as Z. For example, if $\pi_1[Z] \subseteq \pi_2[Z]$ then $\pi_1[q_K[Z]] = q_L[\pi_1[Z]] \subseteq q_L[\pi_2[Z]] = \pi_2[q_K[Z]]$.

Thus wL inherits any kind of non-zero span that X may have. □

2.4. Reflecting span zero

We now turn to showing that having span zero (of any kind) is reflected down from X to wL. We do this by proving the contrapositive, i.e., that having non-zero span reflects upward from wL to X.

To this end we assume that Z is a subcontinuum of $wL \times wL$ that does not meet the diagonal Δ_{wL} of wL and satisfies the property associated to the type of span under consideration. The obvious thing to do would be to find a continuum Z' in $X \times X$ with the same property as Z and such that $Z = q_K[Z']$, for then Z' is a witness to X having non-zero span of the same kind as wL.

The only way to obtain this Z' seems to be via Shelah's Ultrapower theorem from [12], which says that if two structures, A and B, for the same language are elementarily equivalent then there are a cardinal κ and an ultrafilter u on κ such that the ultrapowers of A and B by u are isomorphic.

It was noted by Gurevič in [4] that if A is an elementary substructure of B then the isomorphism $h : A_u \to B_u$ can be chosen in such a way that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\;e\;} & B \\ \downarrow{\scriptstyle\triangle} & & \downarrow{\scriptstyle\triangle} \\ A_u & \xrightarrow{\;h\;} & B_u \end{array}$$

here $\triangle$ is the diagonal embedding of a structure into its ultrapower and e is the elementary embedding of A into B. Inspection of the proof in [12] will reveal that one can start its recursive construction with the identity map on the diagonal in A^κ.

In [2, Lemma 2.8] Bankston used this observation to show that if $e : A \to B$ is an elementary embedding of lattices then every continuum in wA is the image, under the map dual to e, of a continuum in wB. We shall use the proof of this result with a few extra twists to find the desired continuum Z' in $X \times X$.

We expand the language of lattices by adding three unary function symbols: p_1, p_2 and i. In the case of the lattice $2^{X\times X}$ we interpret these as follows:

- $p_1(F) = \pi_1[F] \times X$;
- $p_2(F) = X \times \pi_2[F]$; and
- $i(F) = \{\langle x, y\rangle\colon \langle y, x\rangle \in F\}$.

These interpretations belong to M so that K is also an elementary substructure of $2^{X\times X}$ with respect to the extended language.

We apply Gurevič's remark to K and $2^{X\times X}$ to obtain a cardinal κ and an ultrafilter u on κ such that there is an isomorphism, with respect to the extended language, $h : \prod_u K \to \prod_u 2^{X\times X}$ for which $\triangle \circ e = h \circ \triangle$. The dual, wh, of h is a homeomorphism between $\coprod_u (X \times X)$ and $\coprod_u wK$ for which the dual equality $q_K \circ \nabla = \nabla \circ wh$ holds. By the remark at the end of Section 1.4 we know that $\coprod_u wK$ is the Wallman representation of both $\prod_u K$ and $\prod_u 2^{wK}$.

We consider the closed subset Z_u of $\coprod_u wK$. We know that $Z = \nabla[Z_u]$ and that Z_u is a continuum, so $Z^+ = (wh)^{-1}[Z_u]$ is a continuum as well. We let $Z' = \nabla[Z^+]$. Then Z' is a subcontinuum of $X \times X$ and

$$q_K\big[Z'\big] = q_K\big[\nabla\big[Z^+\big]\big] = \nabla\big[wh\big[(wh)^{-1}[Z_u]\big]\big] = \nabla[Z_u] = Z$$

Thus far we have followed Bankston's argument; we now turn to showing that Z' has the same property as Z. Because $q_K[Z'] = Z$ we know that Z' is disjoint from Δ_X. As to the mapping properties: we shall prove that $\pi_1[Z] \subseteq \pi_2[Z]$ implies $\pi_1[Z'] \subseteq \pi_2[Z']$, leaving any obvious modifications for the other cases to the reader.

Let $K_Z = \{F \in K\colon Z \subseteq \bar{F}\}$. Since K is a base for the closed sets of wK we know that $Z = \bigcap\{\bar{F}\colon F \in K_Z\}$. Next we observe that for $F \in K_Z$ there is $G \in K_Z$ such that $G \subseteq F$ and $\pi_1[G] \subseteq \pi_2[F]$. Indeed, let $G = F \cap \pi_1^{\leftarrow}[\pi_2[F]]$, then $G \in K_Z$ because $\pi_1[Z] \subseteq \pi_2[Z]$, and $\pi_1[G] \subseteq \pi_1[F] \cap \pi_2[F]$. When we reformulate this in terms of our extended language we find that for every $F \in K_Z$ there is $G \in K_Z$ such that $G \subseteq F$ and $i(p_1(G)) \subseteq p_2(F)$.

Even though Z is not (necessarily) a member of K this carries over to $\coprod_u wK$, *because* $\prod_u K$ is a base for the closed sets of $\coprod_u wK$ and because for every element $\langle F_\alpha\colon \alpha < \kappa\rangle$ of K^κ such that $Z \subseteq F_\alpha$ for all α we can find $\langle G_\alpha\colon \alpha < \kappa\rangle$ such that $Z \subseteq G_\alpha \subseteq F_\alpha$ and $i(p_1(G_\alpha)) \subseteq p_2(F_\alpha)$ for all α.

Thus we find that $Z_u = \bigcap\{\bar{F}\colon F \in \prod_u K_Z\}$ and for every $F \in \prod_u K_Z$ there is $G \in \prod_u K_Z$ such that $G \subseteq F$ and $i(p_1(G)) \subseteq p_2(F)$.

Now apply the homeomorphism $(wh)^{-1}$ (and the isomorphism h) to see that the same holds for Z^+ and the family $h[\prod_u K_Z]$, the latter is equal to $\{G \in \prod_u 2^{X\times X}\colon Z^+ \subseteq \bar{G}\}$.

Finally, let z be a point outside $\pi_2[Z']$; we show it is not in $\pi_1[Z']$ either. To begin, Z' and $X \times \{z\}$ are disjoint. By compactness we can find open sets U and V with disjoint closures such that $z \in U$ and $Z' \subseteq X \times V$. Let $P = X \times (X \setminus U)$ and $Q = X \times (X \setminus V)$. Now $Q_u \subseteq \nabla^{\leftarrow}[Q]$, so that $Q_u \cap Z^+ = \emptyset$; but $P_u \cup Q_u = \coprod_u (X \times X)$, hence $Z^+ \subseteq P_u$. Hence there is $\langle R_\alpha\colon \alpha < \kappa\rangle$ in $\prod_u 2^{X\times X}$ such that $Z^+ \subseteq R_u \subseteq P_u$ and $\pi_1[R_\alpha] \subseteq \pi_2[P]$ for all α. It follows that $\pi_1[Z'] \subseteq \operatorname{cl} \bigcup_\alpha \pi_1[R_\alpha] \subseteq \pi_2[P]$, so that $z \notin \pi_1[Z']$.

3. Remarks and questions

3.1. Elementarity, I

The reader will undoubtedly have reflected on the amount of machinery that we brought to bear on the seemingly simple properties of chainability and having span zero. One would expect that taking an elementary sublattice of 2^X would be enough. In the case of chainability this is not the case. The proofs of Propositions 2.1 and 2.2 show that chainability is what one would call a base-independent property: a continuum is chainable iff some/every lattice-base satisfies the chainability condition. On the other hand, as shown in [5] no ultracopower $\coprod_u[0,1]$ of the unit interval by an ultrafilter on ω is chainable. Now $2^{[0,1]}$ is an elementary substructure of its corresponding ultrapower; hence $[0,1]$ and $\coprod_u[0,1]$ have elementarily equivalent bases: they satisfy the same first-order lattice-theoretic sentences. Because one space is chainable and the other is not we conclude that chainability is not expressible by a first-order sentence in the language of lattices.

This changes when we use the language of set theory; chainability is first-order when expressed in this language: for every finite set $\mathcal{U}$ that is an open cover there are a finite ordinal n and an indexed family $\langle V_i \colon i < n\rangle$ of open sets such that $\ldots$. We needed the expressive power of set theory to be able to take finite subsets of our lattice of unspecified cardinality.

The proofs on span relied on the equality $wK = wL \times wL$, which again needed the availability of all possible finite subsets of the substructure.

3.2. Elementarity, II

The proof on reflection of span zero used Shelah's Ultrapower Isomorphism theorem to associate to a continuum in wK a continuum in $X \times X$. This raises an obvious question.

Question 3.1. Can one obtain the continuum Z' and prove its properties by more elementary (pun intended) means?

The reflection of surjective (semi)span zero can be established by elementary means, though without actually exhibiting a continuum Z' as in the question above.

To see this for the case of surjective semispan let Z be a subcontinuum of $wL \times wL$ that is disjoint from the diagonal and find $Y \in K$ that contains Z and is also disjoint from the diagonal. Back in $X \times X$ the closed set Y has the property that none of its components maps onto X under the map π_2. Let C be such a component and take $x \in X \setminus \pi_2[C]$; as $C \cap (X \times \{x\}) = \emptyset$ there must be a relatively clopen subset D of Y that contains C and that is also disjoint from $X \times \{x\}$. This yields a finite partition of Y into closed sets, none of which maps onto X under π_2. By elementarity there is such a partition in M; since Z must be a subset of one of the pieces of this partition we find that $\pi_2[Z] \neq wL$.

If the case of surjective span each piece, D, of the partition will satisfy '$\pi_1[D] \neq X$ or $\pi_2[D] \neq X$', resulting in '$\pi_1[Z] \neq wL$ or $\pi_2[Z] \neq wL$'.

Another question is related to the result in [5] that no ultracopower of $[0,1]$ by an ultrafilter on ω has span zero.

Question 3.2. Is having span zero a base-independent property?

If it is base-independent then the formulation cannot be first-order.

Note added 16-08-2011

Since this paper was accepted for publication the third-named author has constructed a metric counterexample to Lelek's problem, see [7].

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