Discrete Applied Mathematics 6 (1983) 319-322
North-Holland

NOTE

A CHARACTERIZATION OF GRAPHS OF COMPETITION NUMBER *m*

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Received 27 April 1982

In this note we give a characterization of graphs with competition number less than or equal to m. We also give an alternate proof of a theorem characterizing competition graphs.

In 1968 Cohen [1] introduced competition graphs associated with food web models of an ecosystem. Let D be a digraph. The competition graph of D is the undirected graph G obtained as follows: G has the same vertices as D and $\{x, y\}$ is an edge in G if and only if for some vertex z in D there are arcs (x, z) and (y, z) in D. If we let A be the adjacency matrix of D, then we see that G is simply the row graph of A (RG(A)) studied by Greenberg, Lundgren, and Maybee [6,7]. That is, the rows of A correspond to the points of G, and two rows are adjacent in G if and only if they have nonzero entry in the same column of A.

Ecologists usually assume that the digraph D is acyclic. This case has been studied by Cohen [1,2,3], Dutton and Brigham [5], Opsut [9], and Roberts [10,11]. Recently Dutton and Brigham [5] and Roberts and Steif [12] have introduced the study of competition graphs of digraphs D which may not be acyclic. In this note G will be called a competition graph if it is the competition graph of an acyclic digraph D.

If G is a competition graph, then it must have an isolated vertex since D has a vertex with no outgoing arcs. Letting I_k be the graph of k isolated vertices, Roberts [11] shows that $G \cup I_e$ is a competition graph where e is the number of edges in G. The competition number $k(G)$ is the least number k so that $G \cup I_k$ is a competition graph. In this note we characterize graphs G satisfying $k(G) \leq m$. We also present an alternate proof to a theorem of Dutton and Brigham [5] which characterizes competition graphs.

Before proceeding, we need the following lemma which follows from Theorem 10.1 of [8].

Lemma. *If A is the adjacency matrix of a digraph D, then D is acyclic if and only if it is possible to order the points of D so that A is strictly lower triangular.*

Given a graph G with n points, we would like to determine if G is a competition graph. By the lemma this is equivalent to determining if $G = \mathrm{RG}(A)$, where A is a strictly lower triangular $n \times n$ matrix. In Theorem 1 of [7] we gave a method of finding a matrix A such that $\mathrm{RG}(A) = G$ using a clique cover of G. However, that method produced a regular matrix A (i.e., a matrix with a nonzero in each row and column). Hence, we need a modification of that method here, since rows corresponding to points of outdegree zero and columns corresponding to points of indegree zero will have only zero entries.

Observe that if A is a strictly lower triangular matrix, then columns of A either determine a clique in $\mathrm{RG}(A)$ or consist entirely of zeros. Each edge in $\mathrm{RG}(A)$ is in at least one of these cliques as is each point corresponding to a row with at least one nonzero. However, points corresponding to zero rows are not in any of these cliques. Furthermore, if point v_i is in clique C_j corresponding to column j, then $i > j$ since A is strictly lower triangular.

We will call a collection $\mathscr{S}$ of sets of points from G an *edge cover* of G if each set in $\mathscr{S}$ is either a clique in G or the empty set, and every edge in G is in at least one of the cliques in $\mathscr{S}$. We now give an alternate proof of the following theorem of Dutton and Brigham [5].

Theorem 1. *A graph G with n points is a competition graph if and only if the points of G can be labeled so that G has an edge cover $\mathscr{S} = \{C_1, \ldots, C_n\}$ such that if $v_i \in C_j$, then $i > j$.*

Proof. If G is a competition graph, then $G = \mathrm{RG}(A)$ for a strictly lower triangular matrix, and the above discussion shows that the columns of A determine an edge cover $\mathscr{S}$ satisfying the necessary conditions.

Suppose G has an edge cover $\mathscr{S} = \{C_1, \ldots, C_n\}$ such that if $v_i \in C_j$, then $i > j$. Then we construct an $n \times n$ matrix A using the procedure of Theorem 1 of [7] with the following modification. If C_j is the empty set, we let column j consist entirely of zeros. Otherwise, column j has a zero in row i if $v_i \notin C_j$ and a one in row i if $v_i \in C_j$. Then, from the conditions on $\mathscr{S}$, A is strictly lower triangular, and by Theorem 1 of [7], $\mathrm{RG}(A) = G$. Hence, G is a competition graph.

A graph is called a rigid circuit graph if it does not contain Z_n, $n \geq 4$, as a generated subgraph. A vertex b of a graph is called simplicial if $N(b)$, the set of points adjacent to b, is a clique. By a result of Dirac [4], every rigid circuit graph has a simplicial vertex. We can now apply Theorem 1 to get the following result of Roberts [10].

Theorem 2. *If G is a rigid circuit graph with an isolated vertex, then G is a competition graph.*

Proof. Let a_1 be the isolated vertex, $G_1 = G$, and $C_n = \{\emptyset\}$. Let $G_2 = G_1 - a_1$, then this is a rigid circuit graph, so it has a simplicial vertex a_2. If $N(a_2) \neq \emptyset$, let $C_1 = N(a_2)$ in G_2 and $G_3 = G_2 - a_2$. If $N(a_2) = \emptyset$, let $C_{n-1} = \{\emptyset\}$. Continue this process. Note that if $a_i \in C_j$, then $a_i \in N(a_k)$ for $k > j$, so $i \geq k > j$.

We claim that $\mathscr{S} = \{C_1, \ldots, C_n\}$ is an edge cover of G. Suppose $\{a_j, a_k\}$ is an edge in G with $j < k$. Then $a_k \in N(a_j)$, so $\{a_j, a_k\}$ belongs to the clique determined by $N(a_j)$. Hence, by Theorem 1, G is a competition graph.

If G does not have an isolated vertex, then G cannot be a competition graph. One method of determining if G is a competition graph is to compute $k(G)$, the competition number of G. We now characterize those graphs G satisfying $k(G) \leq 1$.

Theorem 3. *If G is a graph with n points, then $k(G) \leq 1$ if and only if G has an edge cover $\mathscr{S} = \{C_1, \ldots, C_n\}$ and a sequence of points $a_1, \ldots, a_n$ such that if $a_i \in C_j$, then $i \geq j$.*

Proof. Suppose $k(G) \leq 1$. If $k(G) = 0$, the result follows from Theorem 1; so suppose $k(G) = 1$ and let $G_1 = G \cup \{b\}$. Then by Theorem 1, G_1 has an edge cover $\mathscr{S}_1 = \{C_1, \ldots, C_{n+1}\}$ and a sequence of points $b_1, \ldots, b_{n+1}$ such that if $b_i \in C_j$, then $i > j$. Also, $C_{n+1} = \{\emptyset\}$ and b_1 is isolated in G_1. Hence, $G \cong G_1 - \{b_1\}$, $\mathscr{S} = \{C_1, \ldots, C_n\}$ is an edge cover of $G_1 - \{b\}$, and if we let $a_i = b_{i+1}$, $i = 1, \ldots, n$, the set $\mathscr{S}$ and sequence of points $a_1, \ldots, a_n$ satisfies the conditions of the theorem.

Suppose G has an edge cover $\mathscr{S} = \{C_1, \ldots, C_n\}$ and a sequence of points $a_1, \ldots, a_n$ such that if $a_i \in C_j$, then $i \geq j$. Then if we let $G_1 = G \cup \{b_1\}$, $b_i = a_{i-1}$ for $i = 2, \ldots, n+1$, and $C_{n+1} = \{\emptyset\}$, then we have satisfied the conditions of Theorem 1; so G_1 is a competition graph and $k(G) \geq 1$.

The following theorem, which gives an upper bound on $k(G)$, has a similar proof.

Theorem 4. *If G is a graph with n points, and $m \leq n$, then $k(G) \leq m$ if and only if G has an edge cover $\mathscr{S} = \{C_1, \ldots, C_n\}$ and a sequence of points $a_1, \ldots, a_n$ such that if $a_i \in C_j$, then $i \geq j - m + 1$.*

These theorems can be applied to obtain results of Roberts [10] on triangle free graphs, and should also be useful in determining the competition numbers of other classes of graphs. It is easy to find classes of graphs that are competition graphs or have $k(G) \leq m$ by using RG(A) where A is strictly lower triangular or has a triangular form determined by Theorem 4.

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