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Brauer Characters of q'-Degree in p-Solvable Groups

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INTRODUCTION

Let G be a finite group and let p be a prime. We let $\mathrm{Irr}(G)$ and $\mathrm{IBr}_p(G)$ denote the ordinary irreducible characters and the irreducible Brauer characters of G. Then $p \nmid \chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if G has normal abelian Sylow-p-subgroup. For p-solvable G, this is a well-known result of Ito. Michler [9] recently proved the general case using the classification of simple groups. Also G has a normal Sylow-p-subgroup if and only if $p \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}(G)$. This is due to Okuyama [11] for $p=2$ and Michler [10] for p odd.

Suppose that G is p-solvable, q is a prime different from p, and $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}_p(G)$. We show that $O^{q'}(G)$ is indeed solvable and that the q-length of G is at most two. In particular, since q-factors of G are necessarily abelian, it follows that the Sylow-q-subgroups of G are metabelian. For groups of odd order, these bounds were first obtained by Manz [8]. We also show that the p-length of $O^{q'}(G)$ is bounded by at most

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two if $(p, q) \neq (7, 3)$ and by three in all cases. As a corollary, it follows that if $\beta(1)$ is a power of p for all $\beta \in \mathrm{IBr}_p(G)$, then the p-length of G is at most 3. We prove our results for solvable G and then extend them to p-solvable G via the classification of finite simple groups.

Section 1 concerns a situation where a solvable group G acts faithfully and irreducibly on a vector space V such that each vector $v \in V$ is centralized by a Sylow-q-subgroup of G. Assuming $O^{q'}(G) = G$, a reasonably complete description of this situation is given. This uses and generalizes some results in [2], where it was assumed that $q^2 \nmid |G|$. In Section 2, we use the results in Section 1 to prove the results mentioned in the last paragraph.

1. Module Actions

Gluck [1] determines all solvable permutation groups (G, Ω) in which every subset Δ of Ω has a non-trivial stabilizer in G. The following lemma is a corollary of this. We let D_{2n} denote the dihedral group of order $2n$ and let J be the group of all semi-linear mappings on $GF(8)$. Then J is solvable of order 168 and acts 2-transitively on eight points.

1.1. Lemma. *Let G be a solvable primitive permutation group on Ω. Assume that $q \mid |G|$ and for each $\Delta \subseteq \Omega$, $\mathrm{stab}_G(\Delta) = \{x \in G \mid \Delta^x = \Delta\}$ contains a Sylow-q-subgroup of G. Then*

(a) $|\Omega| = 3$, $q = 2$, *and* $G \cong D_6$;

(b) $|\Omega| = 5$, $q = 2$, *and* $G \cong D_{10}$;

(c) $|\Omega| = 8$, $q = 3$, *and* $G \cong J$.

Proof. Let M be a minimal normal subgroup of G and let I be a point stabilizer in G. A standard argument shows that $MI = G$, $M \cap I = 1$, $M = C_G(M)$, and M acts regularly on Ω. Thus $|M| = |\Omega|$ and $q \mid |I|$. Since each subset of Ω has a non-trivial stabilizer, it follows from Theorem 1 of [1] that the conclusion of the lemma is satisfied or

(i) $|\Omega| = 4$ and $|I| = 3, 6$;

(ii) $|\Omega| = 5$ and I is cyclic of order 4;

(iii) $|\Omega| = 7$ and I is cyclic of order 6;

(iv) $|\Omega| = 9$ and $|I| = 8, 16, 24$, or 48.

In case (iv), we must have $q = 2$. Let $Q \in \mathrm{Syl}_2(I)$. Since $C_M(Q) = 1$, Q has exactly one fixed point in Ω (see [4, Hilfssatz II.2.2]). Since Q must stabilize a set of size j for all $j \leqslant 9$, the orbit sizes of Q are $\{1, 2, 2, 2, 2\}$ or

$\{1, 2, 2, 4\}$. Let s denote the number of subsets of Ω of size 4 that are fixed by Q and note that $s \leqslant 6$. Since $|\mathrm{Syl}_2(G)| \cdot s \geqslant \binom{9}{4}$ and since $|\mathrm{Syl}_2(G)| \mid 27$, it follows that $|\mathrm{Syl}_2(G)| = 27$, and $s \geqslant 5$. Thus the orbit sizes of Q are $\{1, 2, 2, 2, 2\}$ and since Q acts faithfully on Ω, Q is elementary abelian. Since $|Q| \geqslant 8$ and $GL(2, 3)$ does not have an elementary abelian group of order 8, we have a contradiction. This eliminates case (iv). Cases (i)–(iii) are eliminated by counting orbit sizes of Sylow-subgroups, except possibly when $|\Omega| = 7$ and $q = 2$. But in this case, counting shows that some $\Delta \subseteq \Omega$ with $|\Delta| = 3$ is not stabilized by a Sylow-2-subgroup. ∎

1.2. LEMMA. *Assume that a solvable group G acts faithfully and irreducibly on a finite vector space V, that $q \mid |G|$ and $q \nmid |G: C_G(x)|$ for all $x \in V$. Assume that $C \leqslant G$ is maximal with respect to $C \trianglelefteq G$ and V_C is not homogeneous. Let $V_1, ..., V_n$ be the homogeneous components of V_C. Then*

(a) *G/C permutes the V_i faithfully and primitively;*

(b) *$n = 3$, 5 or 8 and (resp.) $q = 2$, 2, or 3;*

(c) *G/C is isomorphic to D_6, D_{10}, or J (resp.); and*

(d) *$C/C_C(V_i)$ acts transitively on $V_i^{\#}$, the non-identity elements of V_i for each i.*

Proof. When $q^2 \nmid |G|$, this is [2, Lemma 3.2]. Lemma 1.1 above makes this additional hypothesis unnecessary. ∎

If W is a faithful vector space for a solvable group H that transitively permutes the non-identity elements of W, then the semi-direct product WH is a solvable doubly transitive permutation group. Such permutation groups have been determined by Huppert (see [6, Theorem XII, 7.3]) and thus the following corollary holds.

1.3. COROLLARY. *In Lemma 1.2, C is metabelian or $|V_i| = 3^2$, 3^4, 5^2, 7^2, 11^2, or 23^2 for all i.*

Using wreath products with D_6, D_{10}, or J; one can construct examples where the hypotheses of Lemma 1.2 are satisfied (see [12, Example 4.5]).

1.4. LEMMA. *Assume that G is solvable and acts faithfully on a finite vector space V such that $2 \nmid |G: C_G(x)|$ for all $x \in V$. If $|G|$ is even, then* $\mathrm{char}(V) = 2$.

Proof. View V multiplicatively and form the semi-direct product $H = VG$. For $v \in V$ and an involution $t \in H$, let $y = v^{-1}v^t \in V$ and note that $y^t = y^{-1}$. The hypotheses imply that $|N_H(\langle y \rangle)/C_H(y)|$ is odd and thus $y^{-1} = y$. Hence $\mathrm{char}(V) = 2$. ∎

We will need to give upper bounds for the orders of solvable linear groups.

1.5. LEMMA. *Let V be a finite, faithful, and completely reducible module for a solvable group G. Then*

(a) $|G| \leqslant |V|^{9/4}/2$;

(b) *If $|G|$ is odd, then* $|G| \leqslant |V|^2/2$; *and*

(c) *If $|G|\,|V|$ is odd and G is abelian, then* $|G| \leqslant |V|/2$.

Proof. Parts (a) and (b) follow from [13, Theorem 3.1]. To prove (c), an induction argument can be used to reduce to the case where V is irreducible. But then G is cyclic and $|G| \mid |V| - 1$. ∎

1.6. THEOREM. *Let G be a solvable group that acts faithfully and irreducibly on a finite vector space V. Assume $O^{q'}(G) = G$, $q \mid |G|$ and $q^2 \nmid |G|$. Suppose that $q \nmid |G\colon C_G(x)|$ for all $x \in V$. If V_N is homogeneous for all characteristic subgroups N of G and if all normal abelian subgroups of G are cyclic, then*

(i) $O^q(G)$ *is cyclic*; *or*

(ii) $G \cong SL(2, 3)$, $|V| = 9$, *and* $q = 3$.

Proof. This is proved in [2, Theorem 2.3] under the stronger hypothesis that V_N is homogeneous for all $N \trianglelefteq G$. But there the condition on homogeneity is needed only to conclude that normal abelian subgroups are cyclic and that V_M is homogeneous for characteristic subgroups M of $F(G)$. In particular, the bounds in Step 7(iii, iv) there are dependent only on the fact that V_L is homogenous for all Sylow-subgroups L of $F(G)$. Consequently the same proof is valid for this Theorem. ∎

We wish to weaken the hypotheses of Theorem 1.6, not only to allow q^2 to divide $|G|$, but also to assume that V_M is homogeneous for only characteristic M and allow us to use an inductive argument. We must consider another group.

1.7. PROPOSITION. *Assume that $F = F(G)$ is extra-special of order 27 and exponent 3, that $|G : F| = 2$, $|Z(G)| = 3$, and $O^{2'}(G) = G$. Let V be the unique faithful, irreducible G-module over $GF(2)$. Then $|V| = 4^3$ and*

(a) $2 \mid |C_G(x)|$ *for all* $x \in V$;

(b) *There exists $y \in V$ such that* $|C_G(y)| = 2$.

Proof. We many choose $C \trianglelefteq G$ with C elementary abelian of order 3^2. Since V_F is irreducible, it follows that $V_C = V_1 \oplus V_2 \oplus V_3$ with the V_i

homogeneous components of V_C, $|V_i|=4$, and G/C transitively and faithfully permuting the V_i and also the $C_C(V_i)$. For $i \neq j$, $C_C(V_i) \cap C_C(V_j)=1$. Let $\Delta=\{(v_1, v_2, v_3) \mid \text{exactly two } v_i \text{ are non-zero}\}$. Then $C_C(x)=1$ for all $x \in \Delta$. A Sylow-2-subgroup Q interchanges two of the V_i and stabilizes the third. Counting yields that each element of Δ is centralized by exactly one Sylow-2-subgroup. This establishes (b). Furthermore, if $|C_V(Q)|=4^2$, then every element of V is centralized by a Sylow-2-subgroup of G. But this is the case, because if $H=\mathrm{Stab}(V_1)$, then $|H/C|=2$, $O_2(H/C_H(V_1))=1$, $C=Z \times C_C(V_1)$, and $Z \leqslant Z(H)$. ∎

The next theorem is the main result of this section.

1.8. THEOREM. *Let G be a solvable group acting completely reducibly and faithfully on a finite vector space V such that $q \nmid |G: C_G(v)|$ for all $v \in V$. Assume that $O^{q'}(G)=G$, $q \mid |G|$, and V_N is homogeneous for all characteristic subgroups N of G. Then V is an irreducible G-module and one of the following occurs:*

(a) *$O^q(G)$ is cyclic;*

(b) *$G \cong SL(2,3)$, $q=3$, and $|V|=9$; or*

(c) *$O^q(G)$ is extra-special of order 3^3 and exponent 3, $q=2=|G: O^q(G)|$, $|Z(G)|=3$, and $|V|=4^3$.*

Proof. We argue by induction on $|G|\,|V|$. If V is not irreducible, we may write $V=X \oplus Y$ for faithful G-modules X and Y with X irreducible. Applying the inductive hypothesis, we may assume G and X satisfy the conclusion of the theorem. If $O^q(G)$ is cyclic or $G=SL(2,3)$, then $C_G(w) \in \mathrm{Syl}_q(G)$ for all $0 \neq w \in X$. By Proposition 1.7, we may choose $x \in X$ with $C_G(x) \in \mathrm{Syl}_q(G)$. Now choose $y \in Y$ such that $C_G(x)$ does not centralize y. Now the vector (x, y) is not centralized by a Sylow-q-subgroup of G. Hence V is an irreducible G-module.

We let $K=O^q(G)$. For $S \trianglelefteq G$ and $Q \in \mathrm{Syl}_q(G)$, we have that $Q \cap S \in \mathrm{Syl}_q(S)$. Hence $q \nmid |S: C_S(v)|$ for all $v \in V$. Since G acts faithfully on V, $O_q(G)=1$ and $F(G) \leqslant K$. We let $F=F(G)$ and $Z=Z(G)$. The hypotheses imply that every characteristic abelian subgroup of G is cyclic. Hence Z is cyclic and central in $G' \geqslant K$. We may assume that $F>Z$, since otherwise $K \leqslant G' \leqslant C_G(F) \leqslant F=Z$ and conclusion (a) of the theorem holds. We proceed in a number of steps.

Step 1. Let $P_1/Z, \ldots, P_k/Z$ be the distinct Sylow-subgroups of F/Z and let $C_i=C_G(P_i/Z)$. Then

(a) *P_i/Z is elementary abelian of order $p_i^{2n_i}$ for a prime p_i and integer n_i;*

(b) *Let* $e^2 = |F : Z|$, *let* W *be an irreducible* Z*-submodule of* V *and let* X *be an irreducible* F*-submodule of* V. *Then* $e \mid \dim(X)/\dim(W)$. *In particular,* $\dim(V) = te \dim(W)$ *for an integer* t; *and*

(c) *If* $C_i \leqslant H \leqslant G$ *and* $Z \leqslant Z(H)$, *then* H/C_i *is isomorphic to a subgroup of the symplectic group* $\mathrm{Sp}(2n_i, p_i)$.

Proof. Let $P \in \mathrm{Syl}_p(F)$. Every characteristic abelian subgroup of P is cyclic and thus central in $K \leqslant G'$ and central in F. This step now follows from [14, Proposition 1.6, Lemma 1.7]. We note an omission in the conclusion of [14, Lemma 1.7]. It should read $\dim(V) = me \dim(W)$. We also note that P is a central product of an extra-special p-group and $Z(P)$, and provided $P \neq 1$, $p \mid |Z|$.

Step 2. *Assume* $Z \leqslant M \trianglelefteq G$ *and* $q \mid |M|$. *Then*

(a) $|\mathrm{Syl}_q(M)| \geqslant |W|^{te/3}$, *and*

(b) $|\mathrm{Syl}_q(M)| \geqslant |W|^{te/2}$ *if* $q \geqslant 3$ *or if* $2^2 \nmid |M|$ *and* $Z \nleqslant Z(M)$.

Proof. Let $Q \in \mathrm{Syl}_q(M)$. Since $q \nmid |M : C_M(v)|$ for all $v \in V$, we have that $|\mathrm{Syl}_q(G)|\ |C_V(Q)| \geqslant |V| = |W|^{te}$. Let $Q_0 \leqslant Q$ with $|Q_0| = q$. Since $F \geqslant C_G(F)$ and $Q_0 \nleqslant F$, we may choose a Sylow-subgroup E/Z of F/Z such that $Q_0 \nleqslant C_G(E)$. Since $Z = Z(E)$, we apply Lemma 1.7 of [12] to conclude $|C_V(Q_0)| \leqslant |V|^{2/3}$ and furthermore $|C_V(Q_0)| \leqslant |V|^{1/2}$ unless $q = 2$ and Q_0 centralizes Z. Since $C_V(Q) \leqslant C_V(Q_0)$, this step follows.

Step 3. $C_G(F/Z) = F/Z$.

Proof. Assume not and let D/F be a chief factor of G with $D \leqslant \bigcap C_i$ (see Step 1). Then $D \nleqslant C_G(Z) \geqslant K$, since otherwise $[D, D, D, D] = 1$ and $D \leqslant F$. Since Z is cyclic, it follows that D/F is cyclic of order q. A Sylow-q-subgroup M/Z of D/Z is normal in D/Z and G/Z. We apply Step 2 to conclude that

$$|W|^{e/2} \leqslant |\mathrm{Syl}_q(M)| \leqslant |Z|.$$

Since W is a faithful module for the cyclic group Z, $e = 1$. This is a contradiction since $F > Z$.

Step 4. *We may assume that* $K > F$.

Proof. Suppose $K = F$. We first assume all normal abelian subgroups of G are cyclic. Let $0 \neq v \in V$. We may choose $Q \in \mathrm{Syl}_q(G)$ with $Q \leqslant C_G(v)$. Then $Q \leqslant N_G(C_K(v))$. Since F/Z is abelian and $C_Z(v) = 1$, we have that $Z \times C_K(v) \trianglelefteq FQ = KQ = G$. Thus $C_K(v) = 1$. Since K is nilpotent but not cyclic and since K acts fixed-point-freely on V, it follows that $K = S \times T$, where S is a quaternion 2-group and T is cyclic of odd order (see [4, Hauptsatz V.8.7]). Since $T \leqslant Z$, it follows that $|F:Z| = 4$. Because G/F acts faithfully on F/Z, $q = 3$ and $3^2 \nmid |G|$. We now apply Theorem 1.6 to show that the conclusion of this theorem is satisfied. Thus we now assume that G has a non-cyclic abelian normal subgroup.

Choose $C \trianglelefteq G$ maximal such that V_C is not homogeneous. Since G/K is a q-group and K is nilpotent, it follows from Lemma 1.2 that $G/C \cong D_{2r}$ for $r = 3$ or 5, that $q = 2$, and $V_C = V_1 \oplus \cdots \oplus V_r$ for C-modules V_i that are transitively permuted by G/C. Since Z centralizes C, it follows that $Z < C < K$, $r \mid e$, and $r \mid |Z|$. Choose $K \leqslant H \trianglelefteq G$ with $|H:K| = 2$. Since $|H/K| = 2$ and K is not abelian, H/K does not induce a fixed-point-free automorphism on K and thus $|\mathrm{Syl}_2(H)| \leqslant |K|/3$. Since $|Z| < |W|$, it follows from Step 2 that

$$e^2 \geqslant |W|^{e/3} \qquad \text{if} \quad Z \leqslant Z(H)$$

or

$$e^2 \leqslant 3\,|W|^{e/2-1} \qquad \text{if} \quad Z \nleqslant Z(H).$$

If $|W| \geqslant 16$, both inequalities yield that $e < 3$, a contradiction as $r \mid e$. By Lemma 1.4, $\mathrm{char}(W) = 2$. Since $r \mid |Z|$ and $|Z| \mid |W| - 1$, it follows that $r = 3 = |Z|$, $|W| = 4$, and K is an extra-special 3-group. Let $|V_1| = 2^d$. By Lemma 1.2(d), $2^d - 1$ is a power of 3. Thus $|V_1| = 4$ and $|V| = 4^3 = |W|^3$. Hence $e = 3$.

Since $|V_i| = 4$, it follows that $C/C_C(V_i)$ is cyclic of order 3 for all i or isomorphic to D_6 for all i. In the latter case, $2 \mid |C|$ and a Sylow-2-subgroup of C centralizes one non-zero vector in each V_i and thus centralizes exactly one vector of $\{(v_1, v_2, v_3) \mid \text{all } v_i \neq 0\}$. Then, since $C \trianglelefteq G$, $|\mathrm{Syl}_2(C)| \geqslant 27$, a contradiction since $|\mathrm{Syl}_2(C)| \leqslant |F \cap C| = 9$. Thus $C/C_C(V_i)$ is a 3-group for each i and C is a 3-group. Hence $2^2 \nmid |G|$. Since $O^{2'}(G) = G$, an involution of G acts fixed-point-freely on F/Z, but not on the non-abelian group F. Hence, as $|Z| = 3$, $Z \leqslant Z(G)$. The conclusion of the theorem is satisfied, once it is established that F has exponent 3. But this follows from a theorem [4, III.13.10] of P. Hall, since every characteristic abelian subgroup is cyclic.

Step 5. $e > 3$.

Proof. Since $F > Z$, $e > 1$. If $e = 2$, $\mathrm{Aut}(F/Z) \cong S_3$. It follows that $|G/F| = 3$ and $F = K$. We assume that $e = 3$. Since $|GL(2,3)| = 2^4 \cdot 3$ and $K > F$, we have a chain $F \leqslant T \leqslant K \leqslant G$ of characteristic subgroups of G such that $T/F = O_2(G/F)$ and $|K/T| = 3$. Furthermore $T/F \nleqslant Z(K/F)$, since $K = O^2(G)$. Since $K/F \subseteq \mathrm{Sp}(2,3)$, it follows that $T/F \cong Q_8$ and acts irreducibly on F/Z. In particular, $O^{2'}(T)=T$. Applying the induction hypothesis to the action of T on V, we conclude that $|T/F|=2$, a contradiction. This step is complete.

Step 6. *If* N char G *and* $q \mid |N|$, *then* $N=G$. *In particular* $q \nmid |K|$ *and* G/K *is elementary abelian.*

Proof. Without loss of generality, $N=O^{q'}(N)$. If $O^q(N)$ is cyclic, then $O^q(N)$ is central in $K \leqslant G'$ and $O^q(N) \leqslant Z$. Now ZN/Z is a normal q-subgroup of G/Z. Since $q \nmid |F|$, $ZN/Z < C_{G/Z}(F/Z)=F/Z$ by Step 3 and we have a contradiction. Hence $O^q(N)$ is not cyclic.

Since $q \nmid |N\colon C_N(x)|$ for all $x \in V$ and since V_N is homogeneous and faithful, the inductive hypothesis implies that $|V|=9$, $|Z(N)|=2$, $|F(N):Z(N)|=4$ or that $|V|=4^3$, $|Z(N)|=3$, and $|F(N):Z(N)|=9$. Since $Z(N) \leqslant Z$, this implies that $F(G)=F(N)$ and $e \leqslant 3$. Apply Step 5.

Step 7. $q \leqslant 3$.

Proof. Assume that $q \geqslant 5$. Let $Q \in \mathrm{Syl}_q(G)$ and note that Q is elementary abelian. For $X \leqslant Q$ with $|X|=q$, let $M=O^{q'}(XK) \trianglelefteq G$. By Lemma 1.2, V_M is homogeneous. Since $q \nmid |M\colon C_M(v)|$ for all $v \in V$, Theorem 1.6 implies that $O^q(M)=K \cap M$ is a cyclic normal subgroup of G. Note that X acts trivially on K/L, where L is the product of all $O^{q'q}(XK)$ for all $X \leqslant Q$ with $|X|=q$. Since $O^{q'}(G)=G$, $L=K$. But L is a product of cyclic normal subgroups of G and thus nilpotent. Apply Step 4.

Step 8. *There is a characteristic subgroup* L *of* G *such that* $Z \leqslant L \leqslant F$, $C_G(F/L)=F$, *and* F/L *is a direct product of minimal normal subgroups of* G/L.

Proof. We let $L=Z \cdot \Phi(G)$. By a theorem of Gaschütz (see [4, Satz III.4.5]), $F/\Phi(G)$ is a product of minimal normal subgroups of $G/\Phi(G)$. Thus $F/\Phi(G)=Z/\Phi(G) \times Y/\Phi(G)$ for a subgroup Y of G and $Y/\Phi(G)$ is a direct product of minimal normal subgroups of $G/\Phi(G)$. Since F/L is G-isomorphic to $Y/\Phi(G)$, we need just show that $C_G(Y/\Phi(G))=F$. Since $Z \leqslant \mathbb{Z}(K)$, $F \leqslant \mathbb{Z}_K(Y/\Phi(G)) = \mathbb{C}_K(F/\Phi(G)) \leqslant \mathbb{C}_G(F/\Phi(G))$. But $C_G(F/\Phi(G))=F$ (see [4, Satz III.4.2]). Since $K>F$, it follows from Step 6 that $C_G(Y/\Phi(G))=C_K(Y/\Phi(G))=F$.

Step 9. *Assume* $q^2 \mid |G|$. *Then*

(a) $Z=Z(G)$; *and*

(b) *If* $e<64$, *then* $e=25$ *or* 49.

Proof. If S is a Sylow-subgroup of Z, then $K \leqslant \mathbb{C}_G(S)$ and $G/\mathbb{C}_G(S)$ is cyclic. Since $q^2 \mid |G|$, it follows from Step 6 that $S \leqslant \mathbb{Z}(G)$. Part (a) follows.

Let $P/Z \in \mathrm{Syl}_p(F/Z)$ for a prime p and then $|P/Z| = p^{2n}$ for an integer n. Since $q \neq p$, a Sylow-q-subgroup of $\mathrm{Sp}(2, p) \cong \mathrm{SL}(2, p)$ is cyclic or generalized quaternion (see [4, Satz II.8.10]) and has no non-cyclic abelian subgroup. If $n = 1$, it then follows from Step 6 as $q^2 \mid |G|$ that $C_G(P/Z) = G$. But then a Sylow-q-subgroup of G centralizes P and $P \leqslant Z(G)$ by Step 6, a contradiction. Hence $n \neq 1$. Whenever $p \mid e$, $p^2 \mid e$.

To prove (b), we may thus assume that $e = 3^2$ or 3^3 and $q = 2$ or that $e = 2^2$, 2^3, 2^4, or 2^5 and $q = 3$. By Steps 6 and 8, the prime divisors of $F(K/F)$ are all at least 5. By calculating $\mathrm{Sp}(2n, p)$ for the appropriate values and applying Step 1(c), we see that a Sylow-subgroup B/F of $F(K/F)$ is cyclic or $q = 3$ and $|B/F| = 5^2$. In all cases, $KC_G(B/F)$ has index 1 or q. By Step 6, B/F and thus $F(K/F)$ are central in G/F. Thus $F(G/F) > F(K/F)$ and then, by Step 6, G/F is nilpotent. Thus $K = F$, contradicting Step 4. This step is complete.

Step 10. $q = 2$.

Proof. We assume that $q = 3$ and arrive at a contradiction. By Step 8 and repeated applications of Lemma 1.5, we have that $|K/F| \leqslant |F : Z|^{9/4}/2$ and thus $|K : Z| \leqslant e^{13/2}/2$. By Step 2, $|Z|\, e^{13/2}/2 > |W|^{te/2}$. Since $|Z| < |W|$, we have that

$$e^{13} \geqslant |W|^{te-2}. \qquad (*)$$

Since each prime divisor of e divides $|Z|$ and $|Z| \mid |W| - 1$, it follows from $(*)$ that $e \neq 25, 40, 49, 56$. Furthermore, since $e > 3$ and $|W| \geqslant 3$, $t < 8$ and $e < 64$.

By Step 9, $3^2 \nmid |G|$. We apply Theorem 1.6 and Lemma 1.2 to conclude that there exists $C \trianglelefteq G$ such that $G/C \cong J$, $V_C = V_1 \oplus \cdots V_8$ and G/C transitively permutes the V_i. Since $t < 8$, $F \not\leqslant C$ (see Step 1(b)). Since $C \leqslant K$ and $Z \leqslant \mathbb{Z}(K)$, $Z \leqslant C$. It follows that G/CF is non-abelian of order 21, $|F/F \cap C| = 8$, and a Sylow-7-subgroup of K/F acts non-trivially on the Sylow-2-subgroup of F/Z. By Step 1(c), $8 \mid e$. By the first paragraph, we may assume that $e = 8$, 16, or 32.

Now, by Steps 6 and 8, $F(K/F)$ is a $\{2, 3\}'$-group whose order must simultaneously divide $|\mathrm{Sp}(10, 2)|$ and $|\mathrm{GL}(3, 2)| \mid |\mathrm{GL}(7, 2)|$. Thus $F(K/F)$ is cyclic with order dividing $5 \cdot 7 \cdot 31$ and thus is central in K/F, as $K \leqslant G'$. Since K/F is solvable and $O^{3'}(G) = G$, $|K/F| \mid 7 \cdot 31$. Hence $|K| \leqslant 2^8 \cdot e^2 \cdot |Z|$ and by Step 2, $2^{16} e^4 \geqslant |W|^{e-2} \geqslant 3^{e-2}$. Hence $e \leqslant 16$ and $|W| < 27$. Hence $31 \nmid |K/F|$ and $Z \leqslant Z(G)$, as $|Z| \mid |W| - 1$. Thus $7e^2 \geqslant |W|^{e/2}$ and $e = 8$.

Now G/F, which is non-abelian of order 21, acts on both F/C and C/Z. Thus $|W|^{4t} \leqslant |\mathrm{Syl}_3(G)| \leqslant 7 \cdot 2^4$, which yields that $|W| = 3$, $t = 1$, and

$|V| = 3^8$. In the action of G/C on $\{V_1, ..., V_8\}$, a Sylow-3-subgroup has orbit sizes 3, 3, 1, and 1. Let $\Delta = \{(v_1, ..., v_8) \mid \text{exactly six } v_i \text{ are non-zero}\}$ and let $Q \in \mathrm{Syl}_3(G)$. Then Q fixes precisely four elements of Δ. Since $|\mathrm{Syl}_3(G)| = 2^4 \cdot 7$ and $|\Delta| = 2^6 \cdot 28$, the hypotheses of the theorem imply that $2^4 \cdot 7 \cdot 4 \geqslant 2^6 \cdot 28$. This contradiction completes this step.

Step 11. *Conclusion.*

Proof. By Step 10 and Lemma 1.4, $q = 2 = \mathrm{char}(W)$. First assume $2^2 \mid |G|$, so that $Z \leqslant Z(G)$ by Step 9. Steps 6 and 8 and repeated applications of Lemma 1.5 imply that $|K/F| \leqslant e^4/2$. By Step 2, $e^{18} \geqslant 2^3 \cdot |W|^e$. Since $|W| \geqslant 4$, $e < 64$ and Step 9 implies that $5^{36} \geqslant 2^3 \cdot 16^{25}$ or that $7^{36} \geqslant 2^3 \cdot 8^{49}$. Both inequalities are false. Hence $2^2 \nmid |G|$.

We may apply Theorem 1.6 and Lemma 1.2 to conclude that there exists $C \trianglelefteq G$ and $r = 3$ or 5, such that $G/C \cong D_{2r}$, $V_C = V_1 \oplus \cdots \oplus V_r$ for C-modules V_i whose non-zero elements are transitively permuted by C. By Corollary 1.3, C is metabelian. Thus $K''' = 1$, $K'' \leqslant Z \leqslant Z(K)$ and hence K' is nilpotent, whence $K' \leqslant F$. Since K/F is abelian and $|K|$ is odd, it follows from Step 8 and Lemma 1.5 that $|K/F| \leqslant e^2/2$. Since $|Z| \mid |W| - 1$, we have by Step 2 that, if $Q \in \mathrm{Syl}_2(G)$, then

$$\begin{aligned} e^8 &\geqslant 2^2\, |W|^{e-2} && \text{if } Q \nleqslant C_G(Z) \\ e^{12} &\geqslant 2^3\, |W|^e && \text{if } Q \leqslant C_G(Z). \end{aligned} \tag{$**$}$$

First suppose that $e = 27$. Since $|W| \geqslant 4$, $(**)$ implies that $Q \leqslant C_G(Z)$. Since K/F is abelian, it follows from Steps 6 and 8 that K/F is a $\{2, 3\}'$-group and from Step 1(c) that $|K/F| \mid 5 \cdot 7 \cdot 13$. A Sylow-5-subgroup of K/F has a non-trivial centralizer in F/Z, and a non-trivial Sylow-7-subgroup of K/F would act irreducibly on F/Z. Hence $35 \nmid |K/F|$. Thus $|\mathrm{Syl}_2(G)| \leqslant 7 \cdot 13 \cdot 3^6$, contradicting Step 2. Thus $e \neq 27$. If $e = 15$ or 25, then $5 \mid e$ and $|W| \geqslant 16$, contradicting $(**)$. Similarly $e \neq 21$. Since $|W| \geqslant 4$, $(**)$ implies that $e < 30$. Since $e \neq 15$, 21, 25, or 27, e is prime or $e = 9$.

Since $K > F$ and $O^2(G) = K$, it follows that G/F is non-abelian, and furthermore, via Step 1(c), that $5 \mid |G/F|$ if $e = 9$. In either case, we must have that G/F acts irreducibly on F/Z. Thus every normal abelian subgroup of G is cyclic. We apply Theorem 1.6 to conclude the proof. ∎

Note, in the conclusion of Theorem 1.8(c), that V_N is not homogeneous for all $N \trianglelefteq G$.

1.9. Corollary. *Assume the hypotheses of Theorem* 1.8 *and that* $O^q(G)$ *is a p-group for a prime* $p \neq q$. *Then*

(a) $q^2 \nmid |G|$;

(b) *If $q=2$, then p is a Fermat prime; and*

(c) *If $q=2$ and $O^q(G)$ is cyclic, then $G \cong D_{2p}$.*

Proof. We may assume that $O^q(G)$ is cyclic. The hypotheses imply $O_q(G)=1$. Since $O^q(G)$ is a cyclic p-group, G is a Frobenius group. Choose $Q \leqslant Q_0 \in \mathrm{Syl}_q(G)$ with $|Q|=q$ and note that Q_0 is the only Sylow-q-subgroup of G containing Q. Hence the hypothesis that $q \nmid |G: C_G(x)|$ for all $x \in V$ implies that $C_V(Q)=C_V(Q_0)$. Since G is a Frobenius group and $\mathrm{char}(V) \nmid O^q(G)$, it follows (see [7, Theorem 15.16]) that $Q=Q_0$ and $\dim(C_V(Q))=\dim(V)/q$. We thus have $q^2 \nmid |G|$ and we assume that $q=2$. Using Lemma 1.4, we have that $|C_V(Q)|=2^a$ and $|V|=2^{2a}$ for an integer a. Since each non-zero vector of V is centralized by a unique Sylow-2-subgroup of G, we have that $2^a+1=|\mathrm{Syl}_q(G)|=|O^q(G)|$. Thus $p=|O^q(G)|$ is a Fermat prime. ∎

2. Brauer Characters

Our purpose here is to investigate the structure of G when $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}_p(G)$, $p \neq q$. We will first prove the results for solvable G and then show that they extend to p-solvable G via the classification of finite simple groups. Given the nature of the results of the first section, it is not surprising that the cases $q=2$, $q=3$, and $q \geqslant 5$ are done differently. For $q=2$, the next result is quite helpful. It was originally proved by R. Gow (in correspondence) using the idea of Lemma 1.4.

2.1. Corollary. *Let p be an odd prime and assume that $2 \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}_p(G)$. If $O^{2'}(G)$ is solvable, then $O^{2'}(G)$ is a $\{2, p\}$-group.*

Proof. The hypothesis on character degrees is inherited by normal subgroups and factor groups. Arguing by induction on $|G|$, we may assume that $O^{2'}(G)=G$, that G has a unique minimal normal subgroup M, that G/M is a $\{2, p\}$-group and M is an elementary abelian r-group for an odd prime $r \neq p$. In particular, $M=\mathbb{C}_G(M)$. Then $2 \nmid |G: I_G(\lambda)|$ for all $\lambda \in \mathrm{Irr}(M)$. Since $\mathrm{Irr}(M)$ is a faithful G/M-module, Lemma 1.4 implies that $r=2$, a contradiction. ∎

2.2. Lemma. *Suppose that $O^{2'}(G)=G$ and that G is a $\{2, p\}$-group for an odd prime p. Assume G acts faithfully and irreducibly on a vector space V such that $2 \nmid |G: C_G(x)|$ for all $x \in V$. If $C \leqslant G$ is maximal such that $C \trianglelefteq G$ and V_C is not homogeneous, then*

(i) *$G/C \cong D_6$ and $p=3$;*

(ii) $V = V_1 \oplus V_2 \oplus V_3$ *for subspaces* V_i *that are faithfully permuted by* G/C;

(iii) $|V_i| = 4$ *and* C *acts transitively on* $V_i^\#$; *and*

(iv) *There is a non-zero vector* $v \in V$ *such that* $C_G(v)$ *has a normal-2-complement.*

Proof. Since $O^{2'}(G) = G$, it follows from Lemma 1.2 that for $n = 3$ or 5, V is a direct sum $V_1 \oplus \cdots \oplus V_n$ of subspaces V_i that are faithfully and transitively permuted by $G/C \cong D_{2n}$, and that C acts transitively on each $V_i^\#$. By Lemma 1.4, $|V_i| = 2^j$ for some $j \geq 2$. Since C is a $\{2, p\}$-group, we have that $2^{j-1} = p^a$ for some a. Since $p = 3$ or 5, it follows that $n = p = 3$ and $|V_i| = 4$. This proves (i), (ii), and (iii). For (iv), just let $v = (v_1, v_2, v_3)$ be a non-zero vector with at least one component zero. ∎

If $M \trianglelefteq G$ and $p \nmid |M|$, then $\mathrm{Irr}(G)$ and $\mathrm{IBr}_p(G)$ coincide. For $\rho \in \mathrm{Irr}(M)$, the inertia group of ρ as a Brauer character and the inertia group of ρ as an ordinary character are the same, say I. By Clifford's Theorem [4, V.17.3], $\zeta \to \zeta^G$ is a bijection from $\mathrm{IBr}_p(I|\rho)$ onto $\mathrm{IBr}_p(G|\rho)$. In particular, if $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}_p(G)$, then $q \nmid |G : I|$. This will be used repeatedly in the remainder of this section.

2.3. COROLLARY. *Let* M *be a minimal normal subgroup of a solvable group* G, *let* $D = C_G(M) \geq M$, *let* p *be an odd prime and suppose that* $O^{2'}(G) = G$. *Assume that* $2 \nmid \beta(1)$ *for all* $\beta \in \mathrm{IBr}_p(G)$ *and* $p \nmid |M|$. *Then* G/D *has two-length at most one and p-length at most one.*

Proof. Let $V = \mathrm{Irr}(M)$. The hypotheses imply that $2 \nmid |G: C_G(v)|$ for all $v \in V$ (see comments preceding this corollary). We may assume that $D < G$ and V is an irreducible G-module. If V_N is homogeneous for all N char G, we finish by applying Theorem 1.8. By Corollary 2.1 and Lemma 2.2, we may assume that G is a $\{2, 3\}$-group, M is a 2-group, and there exists a $\lambda \in \mathrm{Irr}(M)$ such that $I_G(\lambda)/D$ has a normal-2-complement L/D. Let $I = I_G(\lambda)$. We have that $\zeta \to \zeta^G$ is a bijection from $\mathrm{IBr}_3(I|\lambda)$ onto $\mathrm{IBr}_3(G|\lambda)$. Let $\sigma \in \mathrm{IBr}_3(I|\lambda)$, so that the hypotheses imply that $\sigma(1)$ is odd. In particular, σ_L is irreducible. Thus, as I/L is a 2-group, every $\mu \in \mathrm{IBr}(I/L)$ is linear (see [5, Theorem VII.9.12]). Consequently I/L, which is isomorphic to a Sylow-2-subgroup of G/D, is abelian. Since G/D has abelian Sylow-2-subgroups, G/D has two length at most one.

We now show that G has 3-length at most one. By Lemma 2.2, we may assume that there exists $D \leq B \leq C \leq G$ with $B, C \trianglelefteq G$, B/D a 3-group, C/B a 2-group, and G/C isomorphic to S_3. We are done if $O_3(G/D) \in \mathrm{Syl}_3(G/D)$. Without loss of generality $B/D = O_3(G/D)$. By the last paragraph, $O_2(G/B) \in \mathrm{Syl}_2(G/B)$. This yields a contradiction, since G/B has a factor group isomorphic to S_3. Hence G has 3-length one. ∎

2.4. LEMMA. *Assume* $O^{3'}(G)=G$ *for a solvable group* $G \neq 1$ *that acts faithfully and irreducibly on a finite vector space* V *and* $3 \nmid |G: C_G(v)|$ *for all* $v \in V$. *Suppose that* $p \neq 3$ *is prime and* $3 \nmid \beta(1)$ *for all* $\beta \in \mathrm{IBr}_p(G)$. *Then*

(a) *G has* 3-*length* 1;

(b) *G has p-length at most two*;

(c) *If G has p-length two, then* $p=7$ *and there exists* $0 \neq v \in V$ *such that* $2 \mid |G: C_G(v)|$.

Proof. Applying Theorem 1.8 and Lemma 1.2, we may assume that there exists $C \trianglelefteq G$ such that $G/C \cong J$, that $V_C = V_1 \oplus \cdots \oplus V_8$ for subspaces V_i that are transitively and faithfully permuted by G/C, and such that $C/C_C(V_i)$ acts transitively on each $V_i^\#$. The hypothesis on Brauer characters implies that $p=7$. Since $GL(2, 7)$ does not have a solvable subgroup whose order is divisible by 48.7 and since $\bigcap C_C(V_i)=1$, it follows from Corollary 1.3 that C is metabelian or $7 \nmid |C|$. In particular, C has 7-length at most one and G has 7-length at most two. Note that $2 \mid |G: C_C(v)|$ whenever v is a non-zero vector of the form $(x, 0, ..., 0)$.

It suffices to show that $3 \nmid |C|$. Since $3 \nmid |G: C_G(x)|$ for all $x \in V$, $O_3(C)=O_3(G)=1$.

If $7 \nmid |C|$, then the condition on Brauer characters implies that every $\rho \in \mathrm{Irr}(O_{3'}(C))$ is invariant in $O_{3'3}(C)$ and thus (see, e.g., [12, Proposition 1.5]) $O_{3'3}(C)=O_{3'}(C) \times O_3(C)$ and $3 \nmid |C|$. We thus assume that $7 \mid |C|$. It follows from the first paragraph that C is metabelian and $3 \nmid |C'|$.

We may choose a chief factor C/D of G such that C/D is a 3-group and C/D is not centralized by the maximal normal subgroup K/C of G/C. Let $W=\mathrm{Irr}(C/D)$ so that W is an irreducible G/C-module. The hypotheses imply that $3 \nmid |G: C_G(w)|$ for all $w \in W$. Let L/C be the minimal normal subgroup of G/C. Then $|L/C|=8$ and $L \leqslant K$. By Lemma 1.2, W_L is homogeneous. Since L/C is not cyclic, L centralizes W and thus $L=C_G(W)$. Since K/L is cyclic, each non-zero vector in W is centralized by exactly one of the seven Sylow-3-subgroups of G/L. Let $Q \in \mathrm{Syl}_3(G/L)$. If $|W|=3^a$ and $|C_W(Q)|=3^b$, then $7(3^b-1)=3^a-1$. This is impossible for $a \geqslant 1$. ∎

2.5. PROPOSITION. *Assume that* $O^{q'}(G)=G$, *that* $q \mid |G|$, *that G is solvable and acts faithfully and irreducible on a finite vector space V such that* $q \nmid |G: C_G(x)|$ *for all* $x \in V$. *If* $p \neq q$ *is prime and* $q \nmid \beta(1)$ *for all* $\beta \in \mathrm{IBr}_p(G)$, *then* $q \mid p-1$ *or* $(p, q)=(2, 3)$.

Proof. If G satisfies the conclusion of Lemma 1.2, then $q=2$ or $q=3$ and $p=7$. Hence, by Theorem 1.8, we may assume that $O^q(G)$ is cyclic.

Since $O^{q'}(G)=G$ and $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}_p(G)$, $O^q(G)$ is a p-group and $q \mid p-1$. ∎

2.6. THEOREM. *Suppose that $O^{q'}(G)$ is solvable and that p and q are distinct primes. Assume that $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}_p(G)$. Then*

(a) *G has q-length at most two;*

(b) *If $q \nmid p-1$ and if $(p, q) \neq (2, 3)$, then G has q-length at most one;*

(c) (R. Gow) *If $q=2$ and G has 2-length two, then $O^{2'}(G)$ is a $\{2, p\}$-group for a Fermat prime p; and*

(d) *If $H=O^{q'}(G)$, then $H/O_p(H)$ has p-length at most one, except possibly when $q=3$, $p=7$, and $H/O_7(H)$ has 7-length two. In this exceptional case, there exists $\rho \in \mathrm{IBr}_7(H/O_7(H))$ with $\rho(1)$ even.*

Proof. We argue by induction on $|G|$. The hypotheses are inherited by normal subgroups and factor groups. We may thus assume that $O^{q'}(G)=G$ and $O_p(G)=1$. By the inductive hypothesis, G has a unique minimal normal subgroup. Hence there is a prime r such that $r \neq p$, $O_r(G) \neq 1$, and $O_{r'}(G)=1$. Choose $1=H_0<H_1< \cdots <H_n=O_r(G)$ such that H_i/H_{i-1} is a chief factor of G and let $C_i=C_G(H_i/H_{i-1})$ for $1 \leqslant i \leqslant n$. Since $O_r(G)=F(G)$, it follows via the Hall–Higman Lemma 1.2.3 that $\bigcap C_i=O_r(G)$. Let $V_i=\mathrm{Irr}(H_i/H_{i-1})$, so that V_i is an irreducible and faithful G/C_i-module. The hypothesis on Brauer characters implies $q \nmid |G: C_G(x)|$ for all $x \in V_i$. We apply various results to conclude the following:

(i) If $q \nmid p-1$ and $(p, q) \neq (2, 3)$ then each $C_i=G$ by Proposition 2.5;

(ii) If $q=2$, then Corollaries 2.1 and 2.3 imply that G is a $\{2, p\}$-group, that $r=2$, and that both the two-length and the p-length of G/C_i are at most one;

(iii) If $q=2$ and $C_i<G$, then p is a Fermat prime by Corollary 1.9 and Lemma 2.2;

(iv) If $q \geqslant 5$, then the p-length and q-length of G/C_i are both at most one by Lemma 1.2 and Theorem 1.8; and

(v) If $q=3$, then Lemma 2.4 implies that G/C_i has 3-length at most 1. Furthermore, the p-length of G/C_i is at most one or else it is two, $p=7$, and there exists $x \in V_i$ such that $2 \mid |G: C_G(x)|$. Since $p \neq r$, it follows in this exceptional case that G has a Brauer character of even degree.

Since $\bigcap C_i=O_r(G)$ for a prime $r \neq p$, the theorem follows. ∎

2.7. COROLLARY. *Assume the hypotheses of Theorem* 2.6 *and let* $Q \in \mathrm{Syl}_q(G)$. *Then* $Q'' = 1$. *Furthermore,* $Q' = 1$ *unless* $q \mid p-1$ *or* $(p, q) = (2, 3)$.

Proof. Let $N \trianglelefteq M \trianglelefteq G$ with M/N a q-group. The hypotheses imply that $q \nmid \rho(1)$ for all $\rho \in \mathrm{Irr}(M/N)$ and thus M/N is abelian. Apply Theorem 2.6. ∎

We next show that Theorem 2.6 and Corollary 2.7 remain valid if we just assume G to be p-solvable. To do so, we invoke the classification of finite simple groups. We first collect some consequences of the classification. Part (a) of the next theorem is due to Michler and shows that Ito's theorem generalizes from p-solvable groups to all groups (see [7, Theorem 12.33]).

2.8. THEOREM. *Let H be a simple non-abelian group and let q be a prime. Then*

(a) *If* $q \mid |H|$, *then* $q \mid \mu(1)$ *for some* $\mu \in \mathrm{Irr}(H)$; *and*

(b) *If* $q \nmid |H|$, *then a Sylow-q-subgroup of* $\mathrm{Out}(H)$ *is central in* $\mathrm{Out}(H)$.

Proof. For (a), see [9, Theorem 2.3]; for (b), see [3, Lemma 1.3]. ∎

2.9. THEOREM. *Assume that G is p-solvable and* $q \nmid \beta(1)$ *for all* $\beta \in \mathrm{IBr}_p(G)$ *for a prime* $q \neq p$. *Then* $O^{q'}(G)$ *is solvable. In particular, G is q-solvable and the conclusion of Theorem* 2.6 *holds.*

Proof. We may assume that $G = O^{q'}(G)$. Choose a non-solvable chief factor M/N of G with M as large as possible. Since $p \nmid |M/N|$, the hypotheses imply that $q \nmid \beta(1)$ for all $\beta \in \mathrm{Irr}(M/N)$. By Theorem 2.8(a), $q \nmid |M/N|$. Since $O^{q'}(G) = G$, it follows that $M < G$ and G/M is solvable. By the maximality of M, G/M is isomorphic to a subgroup of $\mathrm{Out}(M/N)$. If M/N is simple it follows from Theorem 2.8(b) that G/M is a q-group. But then a Sylow-q-subgroup Q of G/N fixes every irreducible character of M/N and thus, as $q \nmid |M/N|$, Q centralizes M/N (see [12, Proposition 1.5]), a contradiction.

We may now write $M/N = S_1 \times \cdots \times S_n$ for $n > 1$ isomorphic non-solvable groups that are faithfully and primitively permuted by G/C, where $M \leqslant C \Delta G$. Let $j \leqslant n$ and let $1 \neq \alpha_i \in \mathrm{Irr}(S_i)$ for $i \leqslant j$. Then $(\alpha_1, ..., \alpha_j, 1, ..., 1)$ is invariant under some Sylow-q-subgroup Q of G/M and note $QC/C \in \mathrm{Syl}_q(G/C)$ stabilizes $\{S_1, ..., S_j\}$. Consequently, every subset of $\Omega = \{S_1, ..., S_n\}$ is stabilized by a Sylow-q-subgroup of G/C. Since G/C is solvable, the conclusion of Lemma 1.1 is valid. Now, choose $1 \neq \delta_i \in \mathrm{Irr}(S_i)$ for $1 \leqslant i \leqslant q$ and $Q_0 \in \mathrm{Syl}_q(G/M)$ that stabilizes $(\delta_1, ..., \delta_q, 1, ..., 1)$. Since Q_0 permutes $\{S_1, ..., S_q\}$ transitively, it follows that $\delta_1 = \cdots = \delta_q(1)$. In particular, S_1 has at most two distinct character degrees. Theorem 12.5 of [7] implies that S_1 is solvable, a contradiction. ∎

Assume that G is p-solvable and that $q \nmid \beta(1)$ for all $\beta \in \mathrm{IBr}_p(G)$. We have proven that G has q-length at most two. While q-length two can occur (e.g., $q=2$, $p=3$, and $G=S_4$) for some choices of (p, q), this is not the case for many choices of (p, q). An example of q-length two can only occur if $p=2$ and q is Fermat or there exist positive integers a and b such that $(q^{qb}-1)/(q^b-1)=p^a$. Some such examples arise when G is a subgroup of the affine semi-linear group $\Gamma(q^{qb})$ of $GF(q^{qb})$. Here G has a normal series $N<K<G$, where N has order q^b and consists of all translations of $GF(q^{qb})$, and where K/N and G/K are cyclic of orders p^a and q, respectively. If we apply Theorem 2.9 for all $q \neq p$, we get the next corollary.

2.10. Corollary. *Assume that G is p-solvable and $\beta(1)$ is a power of p for all $\beta \in \mathrm{IBr}_p(G)$. Let $K=O^p(G)$. Then G is solvable and $K/O_p(K)$ has p-length at most one.*

Proof. Without loss of generality, $O_p(K)=1$. For $q \neq p$, we have that $O_p(O^{q'}(K))=1$ and thus Theorem 2.9 implies that $O^{q'}(K)$ is solvable with p-length at most one. Let $L=\prod_{q \neq p} O^{q'}(K)$. Then K/L is a q'-group for all $q \neq p$, and hence $K=O^p(K) \leqslant L \leqslant K$. Thus $K=L$ is a product of characteristic solvable subgroups of p-length at most one. Thus K is solvable wtih p-length at most one. ∎

Assume that H has p-length 1, $O_p(H)=1$, $O^p(H)=H$, and every $\rho \in \mathrm{IBr}_p(G)$ has p-power-degree. It is then straightforward to construct a group G of p-length three for which every $\beta \in \mathrm{IBr}(G)$ has p-power-degree. We also mention that the p-solvable hypothesis in Corollary 2.10 cannot be removed. The groups $SL(2, 2^n)$, $Sp(4, 2^n)$, and $Sz(2^{2n+1})$ are examples where all the Brauer characters in characteristic two have degree a power of two. This was brought to our attention by W. Willems.

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References

1. D. Gluck, Trivial set stabilizers in finite permutation groups, *Canad. J. Math.* **35** (1983), 59–67.
2. D. Gluck and T. R. Wolf, Defect groups and character heights in blocks of solvable groups, II, *J. Algebra* **87** (1984), 222–246.

3. D. GLUCK AND T. R. WOLF, Brauer's height conjecture for p-solvable groups, *Trans. Amer. Math. Soc.* **282** (1984), 137–152.
4. B. HUPPERT, "Endliche Gruppen I," Springer-Verlag, Berlin, 1967.
5. B. HUPPERT AND N. BLACKBURN, "Finite Groups II," Springer-Verlag, Berlin, 1982.
6. B. HUPPERT AND N. BLACKBURN, "Finite Groups III," Springer-Verlag, Berlin, 1982.
7. I. M. ISAACS, "Character Theory of Finite Groups," Academic Press, New York, 1976.
8. O. MANZ, On the modular version of Ito's theorem on character degrees for groups of odd order, *Nagoya J. Math.* **105** (1987), 121–128.
9. G. MICHLER, Brauer's conjectures and the classification of finite simple groups, *in* "Representation Theory, II: Groups and Orders," pp. 129–142, Springer-Verlag, Berlin, 1985.
10. G. MICHLER, A finite simple group of Lie-type has p-blocks with different defects, $p \neq 2$, *J. Algebra* **104** (1986), 220–230.
11. T. OKUYAMA, On a problem of Wallace, to appear.
12. T. R. WOLF, Defect groups and character heights in blocks of solvable groups, *J. Algebra* **72** (1981), 183–209.
13. T. R. WOLF, Solvable and nilpotent subgroups of $GL(n, q^m)$, *Canad. J. Math.* **34** (1982), 1097–1111.
14. T. R. WOLF, Sylow-p-subgroups of p-solvable subgroups of $GL(n, p)$, *Archiv. Math.* **43** (1984), 1–10.