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Noetherian $\mathcal{D}$-Bimodules

S. C. COUTINHO*

Instituto de Matematica, Universidade Federal do Rio de Janeiro, P.O. Box 68530, 21945 Rio RJ, Brazil

AND

M. P. HOLLAND†

Department of Pure Mathematics, Sheffield University, Sheffield, S3 7RH, United Kingdom

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INTRODUCTION

Let X be a smooth, affine variety defined over $\mathbb{C}$. Write $\mathcal{D}(X)$ for the ring of differential operators on X. This paper is concerned with the structure of $\mathcal{D}(X)$-bimodules, specifically those which are finitely generated as left modules and as right modules and on which $\mathbb{C}$ acts centrally. Such $\mathcal{D}(X)$-bimodules are said to be *noetherian*. As $\mathcal{D}(X)$ is a simple, noetherian doamin a noetherian bimodule is automatically projective, as a right module and as a left module. However, consider as bimodules, noetherian bimodules have finite length. Much $\mathcal{D}$-module theory concerns holonomic modules, which have finite length. Thus we attempt in this paper to connect the study of bimodules to the work on holonomic modules. In Section 2 we show that the category of noetherian $\mathcal{D}(X)$-bimodules is equivalent to a full subcategory of the category of holonomic $\mathcal{D}(X \times X)$-modules. This result allows one to plug in to the holonomic theory. In particular, we are able to use Kashiwara's direct image functor to prove our main result in Section 3.

* *E-mail address:* COLLIER@IMPA.BR.
† *E-mail address:* M.HOLLAND@UK.AC.SHEFFIELD.

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THEOREM. *There is an exact, fully faithful functor $\mathcal{F}$ from the category of holonomic $\mathfrak{D}(X)$-modules with singular support equal to $T^*_X X$ to the category of noetherian $\mathfrak{D}(X)$-bimodules. Further, if M is a holonomic $\mathfrak{D}(X)$-module with $SS(M) = T^*_X X$ then*

$$\text{rank } \mathcal{F}(M)_{\mathfrak{D}(X)} = \text{rank}_{\mathfrak{D}(X)} F(M) = \text{rank } M_{\mathcal{O}(X)}.$$

The holonomic $\mathfrak{D}(X)$-modules with $\text{SS}(M) = T^*_X X$ are reasonably tractable; they are precisely the finitely generated, projective $\mathcal{O}(X)$-modules equipped with a flat connection. Thus the theorem gives a useful way of producing examples of noetherian $\mathfrak{D}(X)$-bimodules. We illustrate this with a number of examples in Section 4, in the special case of the Weyl algebra A_1 (which occurs when X is the affine line). In particular we prove the following.

THEOREM. (a) *For each $k \geq 1$ there exists a simple noetherian A_1-bimodule M with* $\text{rank}_{A_1} M = \text{rank}\ _{A_1} = k$.

(b) *For each $k \geq 0$ there exist simple noetherian A_1-bimodules M, N such that*

$$\dim_{\mathbb{C}} \text{Ext}^1 (M, N) = k.$$

We complete Section 4 by raising some open problems that have arisen during the course of this work. In particular we conjecture that a noetherian $\mathfrak{D}(X)$-module has the same rank as a left $\mathfrak{D}(X)$-module as it has as a right $\mathfrak{D}(X)$-module.

1. Rings of Differential Operators

1.1. The following notation is used throughout the paper: X is a smooth, connected, affine algebraic variety defined over $\mathbb{C}$ of dimension n. The ring of regular functions on X is denoted $\mathcal{O}(X)$ and its ring of differential operators is denoted by $\mathfrak{D}(X)$. The latter is generated, as a subalgebra of $\text{End}_{\mathbb{C}}\mathcal{O}(X)$ by $\mathcal{O}(X)$ (acting by multiplication) and $\text{Der}_{\mathbb{C}}(X)$, the $\mathbb{C}$-linear derivations of $\mathcal{O}(X)$.

In this section we collect some basic results on $\mathfrak{D}(X)$ and its modules that are used in the paper. More details and proofs of these results can be found in [Bo].

1.2. The ring $\mathfrak{D}(X)$ has a natural filtration given by the order of a differential operator. The ith term of the filtration is defined inductively by $\mathfrak{D}^0(X) = \mathcal{O}(X)$ and, for $i \geq 1$,

$$\mathfrak{D}^i(X) = \{\theta \in \mathfrak{D}(X) : [\theta, f] \in \mathfrak{D}^{i-1}(X), \quad \text{for all} \quad f \in \mathcal{O}(X)\}.$$

One immediately checks that $\mathcal{D}^1(X) = \mathrm{Der}_{\mathbb{C}}(X) + \mathcal{O}(X)$. The graded ring $\mathrm{gr}\, \mathcal{D}(X)$ associated to this filtration is isomorphic to the symmetric algebra of the $\mathcal{O}(X)$-module $\mathrm{Der}_{\mathbb{C}}(X)$, which is denoted $S(\mathrm{Der}_{\mathbb{C}}(X))$. Geometrically, $\mathrm{gr}\, \mathcal{D}(X)$ is isomorphic to the ring of regular functions on the contangent bundle of $X : T^* X \simeq \mathrm{Spec}\, S\, (\mathrm{Der}_{\mathbb{C}}(X))$. The inclusion $\mathcal{O}(X) \hookrightarrow S(\mathrm{Der}_{\mathbb{C}}(X))$ induces the projection $p : T^* X \to X$.

Let M be a finitely generated right $\mathcal{D}(X)$-module and let F_0 be a finitely generated $\mathcal{O}(X)$-submodule of M that generates M. Now set $F_j = \mathcal{D}^j(X)M_0$, for each $j \geq 1$. Then $\mathcal{F} = \{F_j\}$ is a filtration for M and $\mathrm{gr}^{\mathcal{F}} M$ is finitely generated as a $\mathrm{gr}\, \mathcal{D}(X) = S(\mathrm{Der}_{\mathbb{C}}(X))$-module. Let $I = \mathrm{Ann}_{S(\mathrm{Der}_{\mathbb{C}}(X))}$ $(gr^{\mathcal{F}} M)$. It turns out that the radical $\sqrt{I}$ is independent of the particular choice of generating set F_0. Thus the subvariety $Z(\sqrt{I})$ of $T^* X$ is an invariant of M. It is called the *characteristic variety* or *singular support* of M, and is denoted by $\mathrm{SS}(M)$. The *dimension* $d(M)$ of M is defined to be $\dim \mathrm{SS}(M)$. It turns out that $d(M) = \mathrm{GK}\, M$. To prove this, note that by the main result of [MS] we have $\mathrm{GK}\, M = \mathrm{GK}\, \mathrm{gr}\, M$. Now the latter is evidently the Krull dimension of $\mathrm{gr}\, \mathcal{D}(X)/\mathrm{Ann}(\mathrm{gr}\, M)$, which equals $d(M)$.

Since $\mathrm{SS}(M)$ is an involutive subvariety of $T^* X$, we have that $d(M) = \dim \mathrm{SS}(M) \geq n$ (if $M \neq 0$). A finitely generated $\mathcal{D}(X)$-module M is called *holonomic* if $M = 0$ or $d(M) = n$. These modules play an important rôle in the theory of $\mathcal{D}$-modules.

1.3. Many calculations in $\mathcal{D}(X)$ are simplified by a good choice of local coordinates. Let us explain precisely what this term means. Let $\Omega^1(X)$ denote the $\mathcal{O}(X)$-module of Kähler differentials and let $d : \mathcal{O}(X) \to \Omega^1(X)$ denote the universal derivation. Since X is nonsingular, the $\mathcal{O}(X)$-module $\Omega^1(X)$ is locally free. Given a point $p \in X$ we may choose $f \in \mathcal{O}(X)$ such that $p \in D(f))$ and $\Omega^1\, D(f)) \cong \Omega^1(X)_f$ has a free basis of the form $dx_1, \ldots, dx_n$, for some $x_1, \ldots\, x_n \in \mathcal{O}(D(f)) = \mathcal{O}(X)_f$ (see [MR, Theorem 15.2.13]). Generally, if U is an open affine neighbourhood of p and $\Omega^1(U)$ is free on a basis $dx_1, \ldots, dx_n$, for some $x_1, \ldots, x_n \in \mathcal{O}(U)$ we say that $(x_1, \ldots, x_n)$ is a *system of local coordinates* in the neighbourhood U of p. Now $\mathrm{Der}_{\mathbb{C}}(U) = \mathrm{Hom}_{\mathcal{O}(U)}(\Omega^1(U), \mathcal{O}(U))$ and so we can choose a basis $\partial_1, \ldots, \partial_n$ of $\mathrm{Der}_{\mathbb{C}}(U)$ dual to $dx_1, \ldots, dx_n$. It follows that $\mathcal{D}(U)$ is generated by $\mathcal{O}(U)$ and $\partial_1, \ldots, \partial_n$. For short we write

$$\mathcal{D}(U) = \mathcal{O}(U)[\partial_1, \ldots, \partial_n].$$

1.4. It is traditional in $\mathcal{D}$-modules to work with left rather than right modules (because $\mathcal{O}(X)$ is a natural example of a left module). However, in this paper it will be far more convenient to deal with right modules. It is well known that such a choice is quite arbitrary, as the categories of right and left $\mathcal{D}(X)$-modules are equivalent. As we shall need the details of this equivalence, we sketch it here. Let $\omega(X)$ be the nth exterior power of $\Omega^1(X)$. Then $\omega(X)$ is a rank one projective $\mathcal{O}(X)$-module. It is also a right

$\mathcal{D}(X)$-module. Here $\mathcal{O}(X)$ acts naturally and derivations act via the Lie derivative: if $d \in \mathrm{Der}_{\mathbb{C}}(X)$ and $\omega \in \omega(X)$ then $\omega \cdot d = -\mathrm{Lie}_d(\omega)$. For the details see [Bj, 6.2.4].

The ring of differential operators on $\omega(X)$ is defined inductively, as follows. First, $\mathcal{D}^0(\omega X)) = \mathrm{End}_{\mathcal{O}(X)}(\omega(X))$. Then

$$\mathcal{D}^n(\omega(X)) = \{\theta \in \mathrm{End}_{\mathbb{C}}(\omega(X)) : [\theta, f] \in \mathcal{D}^{n-1}(\omega(X)) \text{ for all } f \in \mathcal{O}(X)\},$$

for $n \geq 1$, and finally $\mathcal{D}(\omega(X)) = \bigcup \mathcal{D}^n(\omega(X))$. There exists a natural map

$$\Psi : \mathcal{D}(\omega(X)) \to \mathrm{End}_{\mathcal{D}(X)}(\mathcal{D}(X)\omega^{-1}(X))$$

defined as follows. If $\theta \in \mathcal{D}(\omega(X))$ then $\Psi(\theta)$ is the endomorphism given by right multiplication by θ. Here, $\omega^{-1}(X) = \mathrm{Hom}_{\mathcal{O}(X)}(\omega(X), \mathcal{O}(X))$ is the dual of $\omega(X)$. An easy calculation with local coordinates shows that Ψ is an isomorphism of $\mathbb{C}$-algebras. Note that $\mathrm{End}_{\mathcal{D}(X)}(\mathcal{D}(X)\omega^{-1}(\mathrm{X})) \cong \omega(X)\mathcal{D}(X)\omega^{-1}(X)$ as shown in [CH2, Proposition 3.8]. We summarise the information we need concerning the equivalence between right and left modules in the next proposition. First, though, we need some notation. If R is any ring $\mu_r(R)$ denotes the category of finitely generated right R-modules and $\mu_l(R)$ denotes the category of finitely generated left R-modules.

PROPOSITION. (a) *There exists an anti-isomorphism* $\Phi : \mathcal{D}(X) \to \mathcal{D}(\omega(X))$.

(b) $\mathcal{D}(X)$ *and* $\mathcal{D}(\omega(X))$ *are Morita equivalent via the progenerator*

$${}_{\mathcal{D}(X)}\mathcal{D}(X)\omega^{-1}(X)_{\mathcal{D}(\omega(X))}.$$

(c) *There is an equivalence of categories between* $\mu_r(\mathcal{D}(X))$ *and* $\mu_l(\mathcal{D}(X))$.

Proof. (a) Define $\Phi : \mathcal{D}(X) \to \mathcal{D}(\omega(X))$ by $f \mapsto f$, for $f \in \mathcal{O}(X)$ and $\theta \mapsto -\mathrm{Lie}_\theta$, for $\theta \in \mathrm{Der}_{\mathbb{C}}(X)$. Now let $p \in X$ and choose $(x_1, \ldots, x_n)$ a system of local coordinates in an open affine neighbourhood U of p. Then $\omega(U) \cong \mathcal{O}(U)$ and $\mathcal{D}(\omega(U) \cong \mathcal{D}(U)$. Thus the right action of $\mathcal{D}(X)$ on $\omega(X)$ induces a right action of $\mathcal{D}(U)$ on $\mathcal{O}(U)$. This is defined as follows: if $\theta \in \mathcal{D}(U)$ and $g \in \mathcal{O}(U)$ then $g \cdot \theta = \theta^t(g)$. Here, as usual, the transpose of $\theta = \sum_\alpha f_\alpha \partial^\alpha$ (written in multi-index notation) is $\theta^t = \sum_\alpha (-1)^{|\alpha|}\partial^\alpha f_\alpha$. Because $\theta \mapsto \theta^t$ evidently gives an anti-automorphism of $\mathcal{D}(U)$, we see that Φ is bijective, as required.

(b) This is routine.

(c) Part (a) shows that $\mu_l(\mathcal{D}(X))$ and $\mu_r(\mathcal{D}(\omega(X))$ are equivalent categories. By part (b) the categories $\mu_r(\mathcal{D}(X))$ and $\mu_r(\mathcal{D}(\omega(X)))$ are equivalent. Hence the result.

1.5. We complete this section by describing the version of Kashiwara's theorem that we will need. Let $\Delta(X) \subseteq X \times X$ be the diagonal subvariety and let $\iota : \Delta(X) \to X \times X$ be the natural inclusion. Let I be the ideal of $\Delta(X)$ in $\mathcal{O}(X \times X) \approx \mathcal{O}(X) \otimes_{\mathbb{C}} \mathcal{O}(X)$. Note that $\mathcal{D}(X \times X) \cong \mathcal{D}(X) \otimes_{\mathbb{C}} \mathcal{D}(X)$. We always identify $\mathcal{O}(X \times X) = \mathcal{O}(X) \otimes \mathcal{O}(X)$ and $\mathcal{D}(X \times X) = \mathcal{D}(X) \otimes \mathcal{D}(X)$. Now the idealiser

$$\mathbb{I}_{\mathcal{D}(X\times X)}(I\mathcal{D}(X \times X)) = \{\theta \in \mathcal{D}(X \times X) : \theta(I) \subseteq I\}$$

is a subalgebra of $\mathcal{D}(X \times X)$ that contains $I\mathcal{D}(X \times X)$ as a two-sided ideal. Further, as is shown in [SS, Proposition 1.6], the natural homomorphism induces an isomorphism:

$$\mathcal{D}(\Delta(X)) \cong \mathbb{I}(I\mathcal{D}(X \times X))/I\mathcal{D}(X \times X).$$

Now define the direct image functor $\iota_+ : \mu_r(\mathcal{D}\Delta(X))) \to \mu_r(\mathcal{D}(X \times X))$ by

$$M \to M \otimes_{\mathcal{D}(\Delta(X))} \mathcal{D}(X \times X)/I\mathcal{D}(X \times X).$$

Kashiwara's theorem [Bo, Theorem VI.7.11] shows that ι_+ is well defined, exact, full, and faithful. Since there is an isomorphism $j : X \to \Delta(X)$ there is an induced category equivalence $j_+ : \mu_r(\mathcal{D}(X)) \to \mu_r(\mathcal{D}(\Delta(X)))$. Write $\Delta_+ = j_+ \circ \iota_+$.

THEOREM (Kashiwara). *The functor* $\Delta_+ : \mu_r(\mathcal{D}(X)) \to \mu_r(\mathcal{D}(X \times X))$ *is exact, full, and faithful.*

2. Noetherian Bimodules

2.1. Let M be a $\mathcal{D}(X)$-bimodule. M is said to be *noetherian* if it is finitely generated as a right $\mathcal{D}(X)$-module and as left $\mathcal{D}(X)$-module and $\mathbb{C}$ acts centrally on M. That $\mathbb{C}$ acts centrally means that $\alpha m = m\alpha$, for all $m \in M$ and $\alpha \in \mathbb{C}$. Let $\mu_b(\mathcal{D}(X))$ denote the category of noetherian bimodules. We note that by Goodearl's result in [Br, Theorem 10] a noetherian $\mathcal{D}(X)$-bimodule is projective as a left $\mathcal{D}(X)$-module and as a right $\mathcal{D}(X)$-module.

Using the anti-isomorphism of the previous section we can turn a noetherian bimodule M into a finitely generated right $\mathcal{D}(X) \otimes \mathcal{D}(\omega(X))$-module. In detail, if $m \in M$, $a \in \mathcal{D}(X)$, and $b \in \mathcal{D}(\omega(X))$ define $m \cdot (a \otimes b) = \Phi^{-1}(b)ma$. Denote M, with this right $\mathcal{D}(X) \otimes \mathcal{D}(\omega(X))$-module structure by $\mathcal{F}_1(M)$. Evidently,

$$\mathcal{F}_1 : \mu_b(\mathcal{D}(X)) \to \mu_r(\mathcal{D}(X) \otimes \mathcal{D}(\omega(X)))$$

is exact, full, and faithful. It induces an equivalence of $\mu_b(\mathcal{D}(X))$ with the full subcategory of $\mu_r(\mathcal{D}(X) \otimes \mathcal{D}(\omega(X)))$ whose objects are finitely generated as right $\mathcal{D}(X) \otimes \mathbb{C}$-modules and as right $\mathbb{C} \otimes \mathcal{D}(\omega(X))$-modules.

Now define a functor

$$\mathcal{F}_2 : \mu_r(\mathcal{D}(X) \otimes \mathcal{D}(\omega(X))) \to \mu_r(\mathcal{D}(X) \otimes \mathcal{D}(X))$$

as follows. If M is an object of $\mu_r(\mathcal{D}(X) \otimes \mathcal{D}(\omega(X)))$ then

$$\mathcal{F}_2(M) = M \otimes_{\mathcal{D}(X) \otimes \mathcal{D}(\omega(X))} (\mathcal{D}(X) \otimes_{\mathbb{C}} \omega(X)\mathcal{D}(X)).$$

By Proposition 1.4(b), we see that $\mathcal{F}_2$ is an equivalence of categories.

Finally, we define a functor

$$\mathcal{H} = \mathcal{F}_2 \circ \mathcal{F}_1 : \mu_b(\mathcal{D}(X)) \to \mu_r(\mathcal{D}(X) \otimes \mathcal{D}(X)).$$

THEOREM. *$\mathcal{H}$ induces an equivalence of categories between $\mu_b(\mathcal{D}(X))$ and the full subcategory of $\mu_r(\mathcal{D}(X) \otimes \mathcal{D}(X)$ whose objects are finitely generated modules over $\mathbb{C} \otimes \mathcal{D}(X)$ and $\mathcal{D}(X) \otimes \mathbb{C}$. The objects in this subcategory are holonomic $\mathcal{D}(X \times X)$-modules.*

Proof. As we observed above, $\mathcal{F}_1$ induces an equivalence of $\mu_r(\mathcal{D}(X))$ with the full subcategory of $\mu_r(\mathcal{D}(X) \otimes \mathcal{D}(\omega(X)))$ whose objects are finitely generated as right $\mathcal{D}(X) \otimes \mathbb{C}$-modules and as right $\mathbb{C} \otimes \mathcal{D}(\omega(X))$-modules. Finally, if M is an object of $\mu_r(\mathcal{D}(X) \otimes \mathcal{D}(\omega(X)))$ it is apparent that $\mathcal{F}_2(M)$ is finitely generated over $\mathcal{D}(X) \otimes \mathbb{C}$ and $\mathbb{C} \otimes \mathcal{D}(X)$ if and only if M is finitely generated over $\mathcal{D}(X) \otimes \mathbb{C}$ and $\mathbb{C} \otimes \mathcal{D}(\omega(X))$.

Now, let M be an object of $\mu_b(\mathcal{D}(X))$. By [Lo, Proposition I.3.1] we have

$$\mathrm{GK}\ \mathcal{H}(M)_{\mathcal{D}(X) \otimes \mathcal{D}(X)} \leq \mathrm{GK}\ \mathcal{D}(X) \otimes \mathbb{C} = 2n.$$

But, $X \times X$ has dimension $2n$, so $\mathcal{H}(M)$ is a holonomic $\mathcal{D}(X \times X) = \mathcal{D}(X) \otimes \mathcal{D}(X)$-module.

2.2. *Remark.* If M is an object of $\mu_r(\mathcal{D}(X \times X))$ that is finitely generated over $\mathcal{D}(X) \otimes \mathbb{C}$ and over $\mathbb{C} \otimes \mathcal{D}(X)$ we denote the corresponding noetherian bimodule by $\mathcal{B}(M)$.

2.3. If M is a noetherian $\mathcal{D}(X)$-bimodule we define $\mathrm{Ext}^j_{\mu_b(\mathcal{D}(X))}(M, \_)$ to be the jth right derived functor of $\mathrm{Hom}(M, \_)$ in the category of $\mathcal{D}(X)$-bimodules. This makes sense, because the category of $\mathcal{D}(X)$-bimodules is equivalent to the category of $\mathcal{D}(X) \otimes \mathcal{D}(X)$-modules (via a functor which restricts to $\mathcal{H}$).

COROLLARY. *If M and N are noetherian $\mathcal{D}(X)$-bimodules then*

$$\mathrm{Ext}^j_{\mu_b(\mathcal{D}(X))}(M, N)$$

is a finite-dimensional vector space, for $j \geq 0$. It is zero if $j > 2n$.

Proof. By the category equivalence,

$$\mathrm{Ext}^j_{\mu_b(\mathcal{D}(X))}(M, N) = \mathrm{Ext}^j_{\mathcal{D}(X)\otimes\mathcal{D}(X)}(\mathcal{H}(M), \mathcal{H}(N)).$$

Since $\mathcal{H}(M)$ and $\mathcal{H}(N)$ are holonomic and $\mathcal{D}(X \times X)$ has global dimension $2n$, by [MR, Theorem 15.3.7], the result follows from [Bj, Theorem 3.2.7, p. 95].

3. BIMODULES AND VECTOR BUNDLES

3.1. In this section we show how noetherian $\mathcal{D}(X)$-bimodules can be constructed from certain holonomic $\mathcal{D}(X \times X)$-modules. In fact we show how one can construct $\mathcal{D}(X \times X)$-modules which are finitely generated as $\mathcal{D}(X) \otimes \mathbb{C}$ and $\mathbb{C} \otimes \mathcal{D}(X)$-modules and then use Proposition 2.1. The holonomic modules that we use are very special. Let T^*_XX denote the zero section of the cotangent bundle. In other words,

$$T^*_XX = Z(\mathrm{Der}_{\mathbb{C}}(X)S(\mathrm{Der}_{\mathbb{C}}(X))) \cong X.$$

The holonomic modules that we need are those with singular support equal to T^*_XX. Another characterisation of these modules is that they are finitely generated as $\mathcal{O}(X)$-modules. In fact, this is enough to force them to be projective over $\mathcal{O}(X)$, by [Bo, Proposition VI.1.7].

THEOREM. *Let M be a finitely generated right $\mathcal{D}(X)$-module with $SS(M) = T^*_XX$. Then $\Delta_+(M)$ is finitely generated and locally free as a right $\mathcal{D}(X) \otimes \mathbb{C}$-module and as a right $\mathbb{C} \otimes \mathcal{D}(X)$-module. Further,*

$$\mathrm{rank}\ \Delta_+M_{\mathcal{D}(X)\otimes\mathbb{C}} = \mathrm{rank}\ \Delta_+M_{\mathbb{C}\otimes\mathcal{D}(X)} = \mathrm{rank}\ M_{\mathcal{O}(X)}.$$

Proof. We prove only that $\Delta_+(M)$ is finitely generated and locally free with rank equal to rank $M_{\mathcal{O}(X)}$ when considered as a $\mathcal{D}(X) \otimes \mathbb{C}$-module. That it is also finitely generated and locally free of this rank, as a $\mathbb{C} \otimes \mathcal{D}(X)$-module, follows symmetrically. We begin by establishing the notation that will be used in the proof. We keep the notation of 1.5. In particular, $\mathcal{O}(\Delta(X)) = \mathcal{O}(X) \otimes \mathcal{O}(X)/I$, where I is ideal of $\mathcal{O}(X) \otimes \mathcal{O}(X)$ generated by $f \otimes 1 - 1 \otimes f$, for $f \in \mathcal{O}(X)$. Note that, by 1.5, it is enough to prove that if M is a finitely generated right $\mathcal{D}(\Delta(X))$-module with $SS(M) = T^*_{\Delta(X)}\Delta(X)$ then $\iota_+(M)$ is finitely generated and locally free with rank equal to rank $M_{\mathbb{C}(\Delta(X))}$.

We shall prove this statement locally. In order to do this, suppose that there exist $f_1, \ldots, f_t \in \mathcal{O}(X)$ and a positive integer k such that:

(1) if we set $U_j = D(f_j)$ then $\{U_j : 1 \leq j \leq t\}$ is an open affine cover of X, and

(2) for each $1 \leq j \leq t$,

$$M_{\overline{f_j \otimes 1}} \otimes_{\mathcal{D}(\Delta(U_j))} \mathcal{D}(U_j \times U_j)/I\mathcal{D}(U_j \times U_j)$$

is a free $\mathcal{D}(U_j) \otimes \mathbb{C}$-module of rank k. (Here, $\bar{\ }: \mathcal{O}(U_j) \otimes \mathcal{O}(U_j) \to \mathcal{O}(\Delta(U_j))$ is the canonical map.)

Now, using the fact that $1 \otimes f - f \otimes 1 \in I$, whenever $f \in \mathcal{O}(X)$, one obtains that

$$\mathcal{D}(X \times X)/I\mathcal{D}(X \times X) \otimes_{\mathcal{D}(X) \otimes \mathbb{C}} (\mathcal{D}(U_j) \otimes \mathbb{C}) \cong \mathcal{D}(U_j \times U_j)/I\mathcal{D}(U_j \times U_j).$$

Since $\mathcal{D}(\Delta(U_j)) = \mathbb{I}(I\mathcal{D}(U_j \times U_j))/I\mathcal{D}(U_j \times U_j)$ and $\mathcal{O}(U_j \times U_j) \subseteq \mathbb{I}(I\mathcal{D}(U_j \times U_j))$ it follows that

$$\iota_+(M) \otimes_{\mathcal{D}(X) \otimes \mathbb{C}} (\mathcal{D}(U_j) \otimes \mathbb{C}) \cong M_{\overline{f_j \otimes 1}} \otimes_{\mathcal{D}(\Delta(U_j))} \mathcal{D}(U_j \times U_j)/I\mathcal{D}(U_j \times U_j). \quad (3)$$

It is clear that Eq. (3) will prove the theorem, provided that there exist f_j and k such that (1) and (2) hold. From 1.3 above, together with the quasi-compactness of X, we can certainly find that $f_j \in \mathcal{O}(X)$, for $1 \leq j \leq t$, such that $\{D(f_j)\}$ is a cover of X and each $D(f_j)$ admits a system of local coordinates. Thus, the theorem will follow if we can prove in the special case when X admits a system of local coordinates.

So, suppose that there exist $x_1, \ldots x_n \in \mathcal{O}(X)$ such that $dx_1, \ldots, dx_n$ is a free basis of the $\mathcal{O}(X)$-module $\Omega^1(X)$. Clearly, we may also assume that $x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n$ generate I. We show that if M is a right $\mathcal{D}(\Delta(X))$-module that is free as an $\mathcal{O}(\Delta(X))$-module with basis $u_1, \ldots, u_k$ then $\iota_+(M)$ is a free $\mathcal{D}(X) \otimes \mathbb{C}$-module with basis $u_1 \otimes \bar{1}, \ldots, u_k \otimes \bar{1}$.

Let $\partial_1, \ldots, \partial_n$ be the basis of $\mathrm{Der}_{\mathbb{C}}(X)$ dual to $dx_1, \ldots dx_n$. Note that we then have that the $\mathcal{O}(X \times X)$-module $\mathrm{Der}_{\mathbb{C}}(X \times X)$ is free on the basis

$$(\partial_i \otimes 1 + 1 \otimes \partial_i : 1 \leq i \leq n; \qquad \partial_j \otimes 1 : 1 \leq j \leq n).$$

We interrupt the proof of the theorem to establish the following lemma, which describes all the various objects under consideration, in terms of the coordinates $(x_1, \ldots, x_n)$. We write $\bar{\ }: \mathcal{D}(X \times X) \to \mathcal{D}(X \times X)/I\mathcal{D}(X \times X)$, for the canonical map.

3.2. LEMMA. (1) $\mathbb{I}(I\mathcal{D}(X \times X)) = I\mathcal{D}(X \times X) + \mathcal{O}(X \times X)[(\partial_1 \otimes 1 + 1 \otimes \partial_1), \ldots, (\partial_n \otimes 1 + 1 \otimes \partial_n)]$.

(2) $\mathcal{D}(\Delta(X)) = \overline{\mathcal{O}(X \times X)}[\overline{\partial_1 \otimes 1 + 1 \otimes \partial_1}, \ldots, \overline{\partial_n \otimes 1 + 1 \otimes \partial_n}]$.

(3) *$\mathcal{D}(X \times X)/I\mathcal{D}(X \times X)$ is a free left $\mathcal{D}(\Delta(X))$-module with basis*

$$(\overline{(\partial_1 \otimes 1)^{\alpha_1} \cdots (\partial_n \otimes 1)^{\alpha_n}} : \alpha_1, \ldots, \alpha_n \in \mathbb{N}).$$

Proof. Recall that $\mathcal{D}(\Delta(X)) = \overline{\mathbb{I}(I\mathcal{D}(X \times X))}$. Thus, we prove (1) and (2) together. Since $\partial_i \otimes 1 + 1 \otimes \partial_i$ commutes with $x_j \otimes 1 - 1 \otimes x_j$, for $1 \le j \le n$, we see that $\partial_i \otimes 1 + 1 \otimes \partial_i \in \mathbb{I}(I\mathcal{D}(X \times X))$, for $1 \le i \le n$. The isomorphism $X \to \Delta(X)$ makes it clear that $\Omega^1(\Delta(X))$ is free on the basis

$$(d(\overline{x_1 \otimes 1 + 1 \otimes x_1}), \ldots, d(\overline{x_n \otimes 1 + 1 \otimes x_n})).$$

The derivatives $(\overline{\partial_1 \otimes 1 + 1 \otimes \partial_1}, \ldots, \overline{\partial_n \otimes 1 + 1 \otimes \partial_n})$ evidently give a dual basis of $\mathrm{Der}_{\mathbb{C}}(\Delta(X))$. This proves (2) and (1) follows immediately.

Let us prove (3). We shall use multi-index notation. Thus, every element of $\mathcal{D}(X \times X)$ is of the form: $\sum_{\alpha \in Y} \Theta_\alpha(\partial \otimes 1)^\alpha$, with $\Theta_\alpha \in \mathcal{O}(X \times X)[\partial_i \otimes 1 + 1 \otimes \partial_i]$ and Y a finite subset of $\mathbb{N}^n$. Hence $\mathcal{D}(X \times X)/I\mathcal{D}(X \times X)$ is generated as a left $\mathcal{D}(\Delta(X))$-module by the images of $(\partial \otimes 1)^\alpha$, for $\alpha \in \mathbb{N}^n$. It is enough to show that these images are linearly independent in $\mathcal{D}(X \times X)/I\mathcal{D}(X \times X)$. Suppose that they are not. Then there exist $\Theta_\alpha \in \mathcal{O}(X \times X)[\partial_i \otimes 1 + 1 \otimes \partial_i]$, for $\alpha \in Y$ such that

$$\sum_{\alpha \in Y} \Theta_\alpha(\partial \otimes 1)^\alpha \in I\mathcal{D}(X \times X). \tag{4}$$

Since the monomials in $\partial_i \otimes 1 + 1 \otimes \partial_i$ and $\partial_i \otimes 1$, for $1 \le i \le n$, are a free basis for $\mathcal{D}(X \times X)$ as an $\mathcal{O}(X \times X)$-module, (4) implies that each $\Theta_\alpha \in I\mathcal{D}(X \times X)$. This yields the desired independence.

3.3. Let us return to the proof of the theorem. Recall that $(u_1, \ldots, u_k)$ is a basis for M over $\mathcal{O}(\Delta(X))$. We want to show that $(u_1 \otimes \bar{1}, \ldots, u_k \otimes \bar{1})$ is a basis for $\iota_+(M)$ over $\mathcal{D}(X) \otimes \mathbb{C}$. Since

$$\iota_+(M) = M \otimes_{\mathcal{O}(\Delta(X))} \mathcal{D}(X \times X)/I\mathcal{D}(X \times X),$$

it follows that $\iota_+(M)$ is spanned, as a $\mathbb{C}$-vector space, by elements of the form $\overline{u \otimes (\partial \otimes 1)^\alpha}$, where $u \in M$ and $\alpha \in \mathbb{N}^n$. Now, if $f \in \mathcal{O}(\Delta(X))$ and $u \in M$ then

$$uf \otimes \bar{1} = u \otimes f = u \otimes \bar{g} = (u \otimes \bar{1})g,$$

for some $g \in \mathcal{O}(X) \otimes \mathbb{C}$. Hence, as a $\mathcal{D}(X) \otimes \mathbb{C}$-module, $\iota_+(M)$ is generated by $u_j \otimes \bar{1}$, for $1 \leq j \leq k$. It remains only to prove that these elements are independent over $\mathscr{D}(X) \otimes \mathbb{C}$.

Suppose that there exist $\theta_1, \ldots, \theta_k \in \mathscr{D}(X)$ such that

$$\sum_{j=1}^{k} (u_j \otimes \bar{1})(\theta_j \otimes 1) = 0. \tag{5}$$

For each $1 \leq j \leq k$ write $\theta_j = \sum_{\alpha \in \mathbb{N}^n} f^j_\alpha \partial^\alpha$, where $f^j_\alpha \in \mathcal{O}(X)$ and the sum has finitely many non-zero terms. Together with (5) this implies that

$$\sum_{\alpha \in \mathbb{N}^n} \left(\sum_{j=1}^{k} u_j \overline{f^j_\alpha \otimes 1} \right) \otimes_{\mathscr{D}(\Delta(X))} (\overline{\partial^\alpha \otimes 1}) = 0.$$

By Lemma 3.2, part (3), $(\overline{\partial^\alpha \otimes 1} : \alpha \in \mathbb{N}^n)$ is a free basis of $\mathscr{D}(X \times X)/I\mathscr{D}(X \times X)$ as a left $\mathscr{D}(\Delta(X))$-module. Hence $\sum_{j=1}^{k} u_j \overline{f^j_\alpha \otimes 1} = 0$, for all $\alpha \in \mathbb{N}^n$. But $(u_1, \ldots, u_k)$ is a basis for M over $\mathcal{O}(\Delta(X))$, and $I \cap (\mathcal{O}(X) \otimes \mathbb{C}) = 0$, so $f^j_\alpha = 0$, for all $\alpha \in \mathbb{N}^n$ and all $1 \leq j \leq k$. This shows that $(u_1 \otimes \bar{1}, \ldots, u_k \otimes \bar{1})$ is a basis for $\iota_+(M)$ over $\mathscr{D}(X) \otimes \mathbb{C}$, as required to finish the proof.

3.4. Corollary. *$M \mapsto \mathscr{B}(\Delta_+(M))$ defines an exact, full, and faithful functor from the category of holonomic $\mathscr{D}(X)$-modules with singular support equal to $T^*_X X$ to the category of noetherian $\mathscr{D}(X)$-bimodules. Further, if M is a finitely generated $\mathscr{D}(X)$-module with $SS(M) = T^*_X X$ then*

$$\text{rank } M_{\mathcal{O}(X)} = \text{rank } \mathscr{B}(\Delta_+(M))_{\mathscr{D}(X)} = \text{rank}_{\mathscr{D}(X)} \mathscr{B}(\Delta_+(M)).$$

Proof. Combine Theorems 3.1 and 2.1.

4. Examples

4.1. In this section we illustrate the above results in the special case of $X = \mathbb{C}$. Note that the Weyl algebra $A_1 = \mathscr{D}(\mathbb{C}) = \mathbb{C}[x, \partial]$. Let M be a finitely generated right A_1-module. Then it is easy to see that $SS(M) = T^*_{\mathbb{C}}\mathbb{C}$ if and only if $M \cong A_1/J$, where J is a right ideal of A_1 containing an element of the form $\theta = \partial^n + \sum_0^{n-1} a_i \partial^i$, with $a_i \in \mathbb{C}[x]$. On the other hand, $A_1/\theta A_1$ is a free $\mathbb{C}[x]$-module of rank n with basis $1, \partial, \ldots, \partial^{n-1}$. In particular, if $p \in \mathbb{C}[x]$ then $A_1/(\partial + p)A_1$ is free of rank one and there is a corresponding right A_1-module structure on $\mathbb{C}[x]$. Here, $\mathbb{C}[x]$ acts naturally and $f \cdot \partial = -(\partial f/\partial x + pf)$, for $f \in \mathbb{C}[x]$. (In fact, these modules are non-isomorphic for distinct p and give all the right module structures, for which $\mathbb{C}[x]$ acts naturally, on $\mathbb{C}[x]$.)

4.2. Let us compute $\Delta_+(A_1/(\partial + p)A_1)$. Now write $x_1 = x \otimes 1$, $x_2 = 1 \otimes x \in \mathcal{O}(\mathbb{C} \times \mathbb{C})$ and $\partial_1 = \partial \otimes 1$, $\partial_2 = 1 \otimes \partial \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C} \times \mathbb{C})$. Note that $A_2 = \mathcal{D}(\mathbb{C} \times \mathbb{C}) = \mathbb{C}[x_1, x_2, \partial_1, \partial_2]$ is the second Weyl algebra and $\mathcal{D}(\Delta(\mathbb{C})) = \mathbb{C}[\overline{x_1 + x_2}, \overline{\partial_1 + \partial_2}]$. We write $q = p(x_1)$. By definition,

$$\begin{aligned}\Delta_+(A_1/(\partial + p)A_1) &= \iota_+(\mathcal{D}(\Delta(\mathbb{C}))/\overline{\partial_1 + \partial_2} + \bar{q})\mathcal{D}(\Delta(\mathbb{C})) \\ &= A_2/(x_1 - x_2)A_2 + (\partial_1 + \partial_2 + q)A_2.\end{aligned}$$

As $A_1/(\partial + p)A_1$ has rank one, $\Delta_+(A_1/(\partial + p)A_1)$ has rank one over $A_1 \otimes \mathbb{C} = \mathbb{C}[x_1, \partial_1]$ and over $\mathbb{C} \otimes A_1 = \mathbb{C}[x_2, \partial_2]$; in fact, $\bar{1}$ generates $\Delta_+(A_1/(2 + p)A_1)$ over $A_1 \otimes \mathbb{C}$ and $\mathbb{C} \otimes A_1$. Note that $\bar{1}(-\partial_2) = \bar{1}(\partial_1 + q)$. It follows that the corresponding bimodule $\mathcal{B}(\Delta_+(A_1/(\partial + p)A_1)$ is isomorphic to the A_1-bimodule ${}_\sigma A_1$, where $\sigma \in \mathrm{Aut}_k A_1$ is given by $x \mapsto x$ and $\partial \mapsto \partial + p$. Here, ${}_\sigma A_1$ is the bimodule structure on A_1 given by $a \cdot b \cdot c = \sigma(a)bc$.

4.3. Our next result shows that $\mu_b(A_1)$ has simple objects of all ranks.

PROPOSITION. *Let $n \geq 1$. Then $\mathcal{B}(\Delta_+(A_1/(\partial^n + x)A_1))$ is a simple object of $\mu_b(\mathcal{D}(\mathbb{C}))$ and has rank n as a left and right A_1-module.*

Proof. $A_1/(\partial^n + x)A_1$ is evidently a simple A_1-module. Thus, $\mathcal{B}(\Delta_+(A_1/(\partial^n + x)A_1))$ is a simple bimodule. The result follows from Corollary 3.4.

4.4. Set $M(p) = \mathcal{B}(\Delta_+(A_1/(\partial + p)A_1))$, for $p \in \mathbb{C}[x]$. It is a simple bimodule. We have the following result.

PROPOSITION. *Let $p, q \in \mathbb{C}[x]$. Then*

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mu_b(A_1)}(M(p), M(q)) = \deg(p - q).$$

Proof. [MR2, Theorem 5.7] shows that

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{A_1}(A_1/(x + p)A_1, A_1/(x + q)A_1) = \deg(p - q).$$

The result follows, by Corollary 3.4.

4.5. Finally, we show that there are simple noetherian A_1-bimodules with many self-extensions.

PROPOSITION. *Let $p \in \mathbb{C}[x]$ be a polynomial of odd degree. Then $M = \mathcal{B}(\Delta_+(A_1/(\partial^2 + p)A_1$ is a simple noetherian A_1-bimodule of rank two and*

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mu_b(A_1)}(M, M) = \deg(p) - 1.$$

Proof. [MR2, Propositions 5.18, 5.19] combine to show that $N = A_1/(\partial^2 + p)A_1$ is simple and has dim $\mathrm{Ext}(N, N) = \deg(p) - 1$. N is evidently free of rank two as a $\mathbb{C}[x]$-module. Corollary 3.4 applies again and the proof is complete.

4.6. We complete the paper with a couple of plausible conjectures.

Conjecture. If M is a noetherian $\mathcal{D}(X)$-bimodule then

$$\text{rank}_{\mathcal{D}(X)} M = \text{rank}\ M_{\mathcal{D}(X)}.$$

4.7. Let M be a noetherian $\mathcal{D}(X)$-bimodule. If rank $M_{\mathcal{D}(X)} \geq 2$ then $M_{\mathcal{D}(X)}$ is automatically locally free, see [CH], and similarly on the left. If M is an invertible $\mathcal{D}(X)$-bimodule and $n = 1$ then [CaH] shows that M is locally free on either side. Thus there is good evidence to suggest that the following should be true.

Conjecture. Let M be a noetherian $\mathcal{D}(X)$-bimodule. Then $M_{\mathcal{D}(X)}$ and ${}_{\mathcal{D}(X)}M$ are locally free.

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References

[Bo] A. Borel, "Algebraic D-Modules," Perspectives in Math., Vol. 2, Academic Press, Boston, 1987.

[Bj] J.-E. Björk, "Rings of Differential Operators," North-Holland, Amsterdam, 1979.

[Br] K. A. Brown, Noncommutative rings, in "The Representation Theory of Noetherian Rings," (L. W. Small and S. Montgomery, Eds.), pp. 1–25, Springer-Verlag, New York, 1992.

[CaH] R. C. Cannings and M. P. Holland, Etale covers, bimodules and differential operators, *Math. Z.*, to appear.

[CH] S. C. Coutinho and M. P. Holland, Module structure of rings of differential operators, *Proc. London Math. Soc.* **57** (1988), 417–432.

[CH2] S. C. Coutinho and M. P. Holland, Séminaire d'algèbre Malliavin, *in* "Differential Operators on Smooth Varieties," Lect. Notes in Math., Vol. 1404, pp. 201–219, Springer-Verlag, New York/Berlin, 1989.

[Lo] M. Lorenz, "Gelfand–Kirillow Dimension and Poincaré Series," Cuadernos de Algebra, No. 7, University of Granada, Granada, Spain, 1988.

[MR] J. C. McConnell and J. C. Robson, "Noncommutative Noetherian Rings," Wiley, New York, 1987.

[MR2] J. C. McConnell and J. C. Robson, Homomorphisms and extensions of modules over certain differential polynomial rings, *J. Algebra* **26** (1973), 319–342.

[MS] J. C. McConnell and J. T. Stafford, Gelfand–Kirillow dimension and associated graded modules, *J. Algebra* **125** (1989), 197–214.

[SS] S. P. Smith and J. T. Stafford, Differential operators on an affine curve, *Proc. London Math. Soc.* **56** (1988), 229–259.