An Inequality for Positive Definite Matrices
With Applications to Combinatorial Matrices

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ABSTRACT

If $A \in M_n(\mathbb{C})$ is a positive definite Hermitian matrix, $d$ the average of the diagonal entries of $A$, and $f$ the average of the absolute values of the off-diagonal entries of $A$, then $\det A \leq (d - f)^{n-1}[d + (n - 1)f]$. As a corollary we obtain a strengthening of Hadamard's inequality for positive definite matrices. The results can be used to prove inequalities for the determinants of $(\pm 1)$ matrices, $(0, 1)$ matrices, positive matrices, stochastic matrices, and constant-column-sum matrices. © 1997 Elsevier Science Inc.

1. THE INEQUALITY

The purpose of this section is to prove the following theorem:

**Theorem 1.1.** Let $A = (a_{ij}) \in M_n(\mathbb{C})$ be a positive definite Hermitian matrix, $d = (1/n)\sum_{i=1}^{n} a_{ii}$, and $f = [1/n(n-1)]\sum_{1 \leq i < j \leq n} |a_{ij}|$. Then

$$|\det A| \leq (d - f)^{n-1}[d + (n - 1)f].$$

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We remark that equality is attained when \( A = aI_n + bJ_n \), \( a > b \), \( I_n \) is the \( n \times n \) identity matrix, and \( J_n \) is the \( n \times n \) all-1 matrix.

The inequality (1) is stronger when the diagonal entries of \( A \) are close together. This can always be achieved by rescaling \( A \). If \( A = (a_{ij}) \) and \( D = \text{diag}(a_{11}, \ldots, a_{nn})^{-1/2} \), then the diagonal entries of the positive definite matrix \( DAD^T \) are equal to 1. Hence we obtain the following variation of (1).

**Theorem 1.2.** Let \( A = (a_{ij}) \) be a positive definite Hermitian matrix, and let

\[
f' = \frac{1}{n(n-1)} \sum_{1 \leq i < j \leq n} \frac{|a_{ij}|}{\sqrt{a_{ii}a_{jj}}}.\]

Then

\[
|\det A| \leq (1 - f')^{n-1}[1 + (n-1)f'] \prod_{i=1}^{n} a_{ii}. \tag{2}
\]

**Proof.** Apply Theorem 1.1 to the matrix \( DAD^T \), where \( D \) is the diagonal matrix \( \text{diag}(a_{11}, \ldots, a_{nn})^{-1/2} \).

These theorems fit squarely into the classical theory of determinantal inequalities starting with the seminal result of Hadamard that for positive definite Hermitian matrices \( A \in M_n(\mathbb{C}) \) we have

\[
\det A \leq \prod_{i=1}^{n} a_{ii}. \tag{3}
\]

Theorem 1.1 can be thought of as a variation of Hadamard's inequality, while Theorem 1.2 is a strengthening of Hadamard's inequality by taking the contribution of the off-diagonal entries of \( A \) into account. Since \( A = (a_{ij}) \) is positive definite, \( |a_{ij}|/\sqrt{a_{ii}a_{jj}} < 1 \), and hence the quantity \( f' \) of Theorem 1.2 satisfies \( 0 \leq f' < 1 \). Thus the inequality (2) is a strengthening of (3), since \((d - f')^{n-1}[d + (n-1)f']\) is a decreasing function of \( f \) for \( f \in [0, d] \). This also shows that (3) is implied by Theorem 1.2.
Proof. Theorem 1.1. Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( A \). Then

\[
\sum_{i=1}^{n} \lambda_i = nd,
\]

\[
\sum_{i=1}^{n} \lambda_i^2 = \text{trace } A^2
\]

\[
= \text{trace } AA^*
\]

\[
= \sum_{1 \leq i, j \leq n} a_{ij} \bar{a}_{ij}
\]

\[
= \sum_{1 \leq i, j \leq n} |a_{ij}|^2
\]

\[
= \sum_{i=1}^{n} a_{ii}^2 + \sum_{1 \leq i \neq j \leq n} |a_{ij}|^2
\]

\[
\geq nd^2 + n(n - 1)f^2. \tag{4}
\]

Define \( z_i = -1 + \lambda_i / d \), and observe that \( z_i > -1 \), as \( A \) is positive definite. It is easy to see that \( \sum_{i=1}^{n} z_i = 0 \). Also

\[
\sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n} \left( -1 + \frac{\lambda_i}{d} \right)^2
\]

\[
= \sum_{i=1}^{n} \left[ 1 - \frac{2\lambda_i}{d} + \left( \frac{\lambda_i}{d} \right)^2 \right]
\]

\[
\geq n - 2n + \frac{1}{d^2} \left[ nd^2 + n(n - 1)f^2 \right]
\]

\[
= \left( \frac{fn}{d} \right)^2 \left( 1 - \frac{1}{n} \right). \tag{5}
\]

We can now apply the lemma of [3], which is stated below, with \( c = fn / d \), which yields

\[
\sum_{i=1}^{n} \ln(1 + z_i) \leq \ln(1 + c - cn^{-1}) + (n - 1) \ln(1 - cn^{-1}). \tag{6}
\]
Exponentiating both sides, we get
\[ \prod_{i=1}^{n} \frac{\lambda_i}{d} \leq \left(1 + \frac{(n-1)f}{d}\right)\left(1 - \frac{f}{d}\right)^{n-1}. \] (7)

The result of the theorem follows on multiplying both sides by \(d^{n}\).

**Lemma 1.3** (J. H. E. Cohn [3]). If \(n > 1\), \(z_r > -1\), for all \(r \in \{1, \ldots, n\}\), \(\Sigma_{r=1}^{n} z_r = 0\), and \(\Sigma_{r=1}^{n} z_r^2 \geq c^2(1 - n^{-1})\), then
\[ \sum_{r=1}^{n} \ln(1 + z_r) \leq \ln(1 + c - cn^{-1}) + (n-1)\ln(1 - cn^{-1}). \] (8)

2. NEW PROOFS FOR OLD RESULTS ON \((\pm 1)\) MATRICES

As an application we present short proofs of two determinantal inequalities on \((\pm 1)\) matrices. If \(n \equiv 0 \mod 4\), then Hadamard's original inequality applied to the matrix \(AA^T\) yields that \(\det AA^T \leq n^n\). However, when \(n \not\equiv 0 \mod 4\), \(n \neq 2\), this inequality can never be sharp, and improvements have been given, starting with Barba [2] and culminating with the results of Ehlich [5, 6] and Wojitas [17]. Though the results below are not new (see [5, 17, 1] for the \(n \equiv 1 \mod 4\) case and [5, 17, 13, 4] for the \(n \equiv 2 \mod 4\) case), the proofs using Theorem 1.1 are new. In addition they are shorter than the original proofs.

**Proposition 2.1** (Ehlich, Wojitas, Cheng). If \(A \in M_{m,n}(\pm 1)\) is a \(\pm 1\) matrix with \(n\) odd, then
\[ \det AA^T \leq (n + m - 1)(n - 1)^{n-1}. \] (9)

**Proof.** Since \(A\) is a \(\pm 1\) matrix, the diagonal entries of \(B = AA^T\) are equal to \(n\), and since \(n\) is odd, none of the off-diagonal entries of \(B = (b_{ij})\) are 0. Hence \(|b_{ij}| \geq 1\) for all \(1 \leq i \neq j \leq m\). Thus \(f \geq 1\). We can now apply Theorem 1.1 to \(B\), using the fact that \((d - f)^{n-1}[d + (n - 1)f]\) is a decreasing function of \(f\) for \(0 \leq f \leq d\).
The corresponding result for $n \equiv 2 \pmod{4}$ follows in a similar fashion.

**Proposition 2.2** (Ehlich, Wojitas, Payne). If $A \in M_{m,n}(\pm 1)$ is a $\pm 1$ matrix with $n \equiv 2 \pmod{4}$, then

$$
\det AA^T \leq \begin{cases} 
(n - 2)^2(n + m - 2)^{m-2} & \text{if } m \text{ is even,} \\
(n - 2)^9(n + m - 3)^{(m-3)/2}(n + m - 1)^{(m-1)/2} & \text{if } m \text{ is odd}
\end{cases}
$$

(10)

The proof follows the proof given in [4] and is included here for completeness only.

**Proof.** Let $(v_i, v_j)$ denote the inner product of rows $i$ and $j$ of $A$. Observe that $(v_i + v_j, v_i - v_k) \equiv 0 \pmod{4}$, as both $v_i + v_j$ and $v_i - v_k$ are $(-2, 0, 2)$ vectors. If we assume that $(v_i, v_j) \equiv (v_i, v_k) \pmod{4}$, then $(v_i, v_j - v_k) \equiv 0 \pmod{4}$ and we have $(v_i, v_i) - (v_j, v_k) = n + (v_j, v_k) \equiv 0 \pmod{4}$, i.e. $(v_i, v_j) \equiv 2 \pmod{4}$.

Now let $r$ be the maximum number of entries $\equiv 2 \pmod{4}$ in any row of $AA^T$. Using the above observation, we can then assume, after a suitable permutation of the rows of $A$, that

$$
AA^T = \begin{pmatrix}
G_1 & G_2 \\
G_2^T & G_4
\end{pmatrix}
$$

(11)

where $G_1 \in M_r(\pm 1)$ and $G_4 \in M_{m-r}(\pm 1)$ are such that all entries are $\equiv 2 \pmod{4}$ while all entries of $G_2 \in M_{r, m-r}(\pm 1)$ are $\equiv 0 \pmod{4}$.

Theorem 1.1 implies that

$$
\det G_1 \leq (n - 2)^{r-1}[n + 2(r - 1)],
$$

(12)

$$
\det G_4 \leq (n - 2)^{m-r-1}[n + 2(m - r - 1)].
$$

(13)

Since $\det AA^T \leq \det G_1 \det G_4$, we have

$$
\det AA^T \leq (n - 2)^{m-2}[n + 2(r - 1)][n + 2(m - r - 1)].
$$

(14)

Assuming that $r$ is a positive integer, we can maximize the right-hand side by setting $r = m/2$ is $m$ is even and $r = (m - 1)/2$ if $m$ is odd. 

\[\blacksquare\]
Ehlich [5] and Wojitas [17] prove the results by proving different inequalities first. It is known that the inequalities (9) and (10) are both sharp for infinitely many values of \( n \). For the square case see [10]; for the nonsquare case, [1] and [7]. For the inequality (9) to be sharp in the square case it is obviously necessary that \( 2n - 1 \) be a square. It was shown in [10] that when \( n = 2(q^2 + q) + 1 \), \( q \) an odd prime power, then there exists \( A \in M_n(\pm 1) \) such that \( AA^T = (n - 1)I_n + J_n \) and hence \( \det AA^T = (n - 1)^{n-1}(2n - 1) \).

It can also be shown that if \( \det AA^T = (n - 1)^{n-1}(2n - 1) \) then we can multiply suitable rows and columns of \( A \) by \(-1\) so that the resulting matrix \( B \) satisfies \( BB^T = (n - 1)I_n + J_n \) (see [4] or [10]). In the nonsquare case optimal examples of size \( j \times n \), \( j < n = 1 \mod 4 \), can be obtained by adjoining any \( j \) rows of a suitable Hadamard matrix with the all-1 vector of length \( j \) (see [7]).

If \( n = 3 \mod 4 \), the inequality (9) has been improved by Ehlich (see [6]). It is not known if the inequality given in [6] for \( n = 3 \mod 4 \) is sharp for infinitely many values of \( n \). The best results seem to be obtained in [10].

3. \((0, 1)\) MATRICES AND AN INEQUALITY OF RYSER

Theorem 1.1 above can also be applied to the following situation. Let \( A \in M_{m,n}(0, 1) \), \( m \leq n \), such that the \( i \)th column of \( A \) contains \( s_i \) 1's. Let \( s' = (s_1, \ldots, s_n) \) and let \( e_n = (1, \ldots, 1) \). We call \( s' \) the column sum vector of \( A \), and likewise we define the row sum vector of \( A \). Also let \( \langle \vec{v}, \vec{u} \rangle \) denote the standard inner product of two vectors. It was Ryser [15] who first studied \((0, 1)\) matrices with given column sum and row sum vectors.

**Proposition 3.1.** If \( A \in M_{m,n}(0, 1) \), \( m \leq n \), and \( s' = (s_1, \ldots, s_n) \) is the column sum vector, then

\[
\det AA^T \leq \left( \frac{1}{m} \right)^m \left( \frac{1}{m - 1} \right)^{m-1} \langle s', me - s \rangle^{m-1} \langle s', s \rangle .
\]  

(15)

In particular, if \( s_i = s \) is constant for all \( 1 \leq i \leq n \), then

\[
\det AA^T \leq s \left( \frac{sn}{m} \right)^m \left( \frac{m - s}{m - 1} \right)^{m-1} .
\]  

(16)
Proof. The $i, i$ entry of $AA^T$ is the total number of 1's in row $i$ of $A$. Thus $\sum_{i=1}^m (AA^T)_{i,i} = \sum_{i=1}^n s_i = (\bar{s}, \bar{e}_n)$, the total number of 1's in $A$. Hence the average of the diagonal entries of $AA^T$ is $d = (1/m)(\bar{s}, \bar{e})$.

Note that $\bar{e}_n A = \bar{s}$ and hence $AA^T \bar{e}_n = (\bar{s}, \bar{s})$. On the other hand, $\bar{e}_n AA^T \bar{e}_m$ is the sum of the entries of $AA^T$. Thus the average of the off-diagonal entries of $AA^T$ is

$$f = \frac{1}{m(m-1)} \left[ (\bar{s}, \bar{s}) - (\bar{e}_n, \bar{e}_n) \right] = \frac{1}{m(m-1)} \left( \bar{s}, \bar{s} - \bar{e}_n \right).$$

The result follows now from Theorem 1.1.

The same method used in the last proof also allows for a new proof of a generalized inequality of Ryser [14].

PROPOSITION 3.2. If $A \in M_{m,n}(0, 1)$ with a total of $t \geq n$ 1's then

$$\det AA^T \leq \left( \frac{t}{n} \right) \left( \frac{t}{m} \right)^m \left[ \left( m - \frac{t}{n} \right) \left( \frac{1}{m-1} \right) \right]^{m-1}. \quad (17)$$

Proof. If $t < n$, then $A$ contains a column of 0's. If $A'$ is the matrix obtained from $A$ by deleting this column, we have $\det AA'^T = \det AA^T$. Hence the assumption that $t \geq n$. It is needed in the proof below.

The average of the diagonal entries of $AA^T$ is $d = t/m$, while the average of the off-diagonal entries of $AA^T$ is

$$f = \frac{1}{m(m-1)} \left[ (\bar{s}, \bar{s}) - (\bar{e}_n, \bar{e}_n) \right] \quad (18)$$

$$\geq \frac{1}{m(m-1)} \left( \frac{1}{n} (\bar{s}, \bar{e}_n)^2 - (\bar{s}, \bar{e}_n) \right) \quad (19)$$

$$= \frac{1}{m(m-1)} \left( \frac{t^2}{n} - t \right). \quad (20)$$

We have use the fact that $(d - f)^n - [d + (n - 1)f]$ is a decreasing function of $f$ for $0 \leq f \leq d$. Since we assumed $t \geq n$, we have $t^2/n - t \geq 0$.

The result follows from Theorem 1.1.
Ryser's inequality [14, Theorem 1] for square matrices follows now as a corollary to Proposition 3.2.

**Corollary 3.3 (Ryser).** If $A \in M_n(0, 1)$ (i.e., $A$ is a square matrix) containing exactly $t$ 1's, then

$$\det A < k(k - \lambda)^{(n-1)/2},$$

where $k = t/n$ and $\lambda = k(k - 1)/(n - 1)$.

**Proof.** The inequality (17) specializes to (21).

4. **INEQUALITIES FOR NONNEGATIVE MATRICES**

Assume that $A = (a_{ij}) \in M_{m, n}(R)$ is a nonnegative matrix, i.e. $a_{ij} \geq 0$. The proof of Proposition 3.1 can be applied to yield the following. As above, $\bar{s}^*$ denotes the column sum vector of $A$.

**Proposition 4.1.** If $A \in M_{m, n}(R)$ is a nonnegative matrix, then

$$\det AA^T \leq \frac{\langle \bar{s}^*, \bar{s}^* \rangle}{m^m(m - 1)^{m-1}} \left[ m \text{ trace } AA^T - (\bar{s}^*, \bar{s}^*) \right]^{m-1}. \quad (22)$$

In particular, if $A$ is a nonnegative and has constant column sums $s$, then

$$\det AA^T \leq \frac{ns^2}{m^m(m - 1)^{m-1}} (m \text{ trace } AA^T - ns^2)^{m-1}. \quad (23)$$

**Proof.** The proof follows the lines of the proof of Proposition 3.1. In fact, Proposition 4.1 could have been stated and proved before Proposition 3.1.

This, of course, can be applied to the special case of stochastic matrices.

**Corollary 4.2.** Assume that $A \in M_n(R)$ is a row or column stochastic matrix, i.e. the row sums of $A$ or the column sums are constant and equal to
1. Then

\[ |\det A| \leq \left( \frac{\text{trace } AA^T}{n-1} \right)^{(n-1)/2}. \]  

(24)

We remark here that by the Cauchy-Schwarz inequality \( \text{trace } AA^T = \text{trace } A^T A \geq 1 \).

If \( A \) is a matrix all of whose column sums are nonzero (or all of whose row sums are nonzero), then we can extend the result of the previous corollary.

**Corollary 4.3.** Assume that \( A = (a_{ij}) \in M_n(\mathbb{R}) \) is a nonnegative matrix. Let \( \vec{s} = (s_1, \ldots, s_n) \) be the column sum vector of \( A \). If \( s_i \neq 0 \) for all \( 1 \leq i \leq n \), then with \( AA^T = (b_{ij}) \) we have

\[ |\det A| \leq \left( \frac{1}{n-1} \right)^{(n-1)/2} \prod_{i=1}^{n} s_i \left( -1 + \sum_{i=1}^{n} \frac{b_{ii}}{s_i^2} \right)^{(n-1)/2}. \]  

(25)

**Proof.** Apply Corollary 4.2 to the matrix \( A' = A \text{ diag}(s_1, \ldots, s_n)^{-1} \).

Even if \( A \in M_{m,n}(\mathbb{R}) \) has negative entries we can estimate the determinant of \( AA^T \) under the assumption that the column sums of \( A \) are constant.

**Proposition 4.4.** Assume that \( A = (a_{ij}) \in M_{m,n}(\mathbb{R}) \) with constant column sum \( s \). Set \( l = \min\{a_{ij}\} \). Then

\[ \det AA^T \leq \frac{ns^2}{m^m(m-1)^{m-1}} (m \text{ trace } AA^T - ns^2). \]  

(26)

**Proof.** Observe that \( A - I_{m,n} \) is a positive matrix with column sums \( s' = s - ml \). By assumption, \( J_m A = s J_{m,n} \) and hence

\[ A - I_{m,n} = A - (l/s) J_m A = (I_m - l/s J_m) A. \]  

(27)

Hence

\[ \det (A - I_{m,n}) (A - I_{m,n})^T = \frac{s'^2}{s^2} \det AA^T. \]  

(28)
Since $A - I_{m,n}$ is nonnegative, we can apply Proposition 4.1 to the left-hand side, which yields

$$\det (A - I_{m,n})(A - I_{m,n})^T$$

$$\leq \frac{ns^{r^2}}{m^m(m-1)^{m-1}} \left[ m \trace (A - I_{m,n})(A - I_{m,n})^T - ns^{r^2} \right]$$

$$= \frac{ns^{r^2}}{m^m(m-1)^{m-1}} \left[ m(\trace AA^T - 2ns + t^2 mn) - ns^{r^2} \right]$$

$$= \frac{ns^{r^2}}{m^m(m-1)^{m-1}} (m \trace AA^T - ns^{r^2}).$$

This implies then that

$$\det AA^T \leq \frac{ns^{r^2}}{m^m(m-1)^{m-1}} (m \trace AA^T - ns^{r^2}).$$

In the square case the previous proposition holds for constant-row-sum matrices as well.

5. EXAMPLES

We provide a few examples for which equality is attained in the inequalities above.

If $A = aI_n + bJ_n$ with $a > b \geq 0$, then equality is attained in Theorem 1.1.

For instances of equality in the results of Section 2 the reader is referred to \[10, 16, 9\].

Let $B \in M_{5,10}(0,1)$ be the matrix whose columns are the 10 distinct $(0,1)$ vectors of of length 5 with exactly three 1's. Then $BB^T = 3(I_5 + J_5)$ and hence $\det BB^T = 1458$ and thus $B$ provides an example of equality for Proposition 3.1. In \[18\], \[11\], and \[12\] more examples are discussed which attain equality in Proposition 3.1.
As pointed out by Ryser (see [14]), equality is attained in Corollary 3.3 if and only if \( C \) is the incidence matrix of an \((n, k, \lambda)\)-design. In particular, if

\[
C = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1
\end{bmatrix},
\]

i.e., \( C \) is the incidence matrix of a \((7, 4, 2)\)-design, then \( \det A = 32 \) and equality is attained in Corollary 3.3, as \( t = 28, k = 4, \) and \( \lambda = 2. \)

If \( D = \frac{1}{4}C \), then \( D \) is a doubly stochastic matrix with \( \det D = 2^{-9} \) and hence \( D \) is an example of equality in Corollary 4.2. Of course, any permutation matrix also provides an example of equality.

Let \( E = f_{5,10} - 2B. \) Then \( E \in M_{5,10}(\pm 1), \) and all column sums are \(-1.\) Furthermore, \( EE^T = 12I_5 - 2f_5, \) and hence \( \det EE^T = 2^93^4 \) and \( E \) provides an example of equality in Proposition 4.4.

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