Discrete Mathematics 33 (1981) 163–170

SOME REMARKS ON THE REPRESENTATION THEOREM OF MOISIL

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Received 28 *February* 1980

In this paper, using the representation theorem of Moisil (see [2]) the author introduces and examines the concept of representability of Lukasiewicz algebras. The results and notations used are from [1], [2].

1. Preliminaries

Let θ be the order type of a chain J with a first element 0 and a last element 1.

Definition 1. (a) A distributive lattice L with a first and last element is called a θ-valued Lukasiewicz algebra (θ.v.L.a.) if two families of maps

$$(\varphi_\alpha : L \to L \mid \alpha \in J), \qquad (\bar{\varphi}_\alpha : L \to L \mid \alpha \in J)$$

are given which satisfy the following conditions:

(1) $\forall x, y \in L$, $\forall \alpha \in J$, $\varphi_\alpha(x \cap y) = \varphi_\alpha(x) \cap \varphi_\alpha(y)$,
(2) $\forall x, y \in L$, $\forall \alpha \in J$, $\varphi_\alpha(x \cup y) = \varphi_\alpha(x) \cup \varphi_\alpha(y)$,
(3) $\forall \alpha \in J$, $\varphi_\alpha(0) = 0$, $\varphi_\alpha(1) = 1$,
(4) $\forall x \in L$, $\forall \alpha \in J$, $\varphi_\alpha(x) \cap \bar{\varphi}_\alpha(x) = 0$, $\varphi_\alpha(x) \cup \bar{\varphi}_\alpha(x) = 1$,
(5) $\forall \alpha, \beta \in J$, $\alpha \leqslant \beta$, $\forall x \in L$, $\varphi_\beta(x) \leqslant \varphi_\alpha(x)$,
(6) $\forall \alpha, \beta \in J$, $\varphi_\alpha \circ \varphi_\beta = \varphi_\beta$,
(7) if $\forall \alpha \in J$, $\varphi_\alpha(x) = \varphi_\alpha(y)$, then $x = y$.

(b) If L and L' are θ.v.L.a., a map $f : L \to L'$ is called homomorphism if:

$$\forall x, y \in L, \quad f(x \cap y) = f(x) \cap f(y),\ f(x \cup y) = f(x) \cup f(y),$$

$$f(0) = 0, \qquad f(1) = 1,$$

$$\forall x \in L,\ \forall \alpha \in J, \quad f(\varphi_\alpha(x)) = \varphi'_\alpha(f(x)).$$

(c) If L is a θ.v.L.a., then the set $C(L) = \{x \in L \mid \varphi_\alpha(x) = x, \forall \alpha \in J\}$ is called the *centre* of L ($C(L)$ is a Boolean algebra, see [2]).

(d) If B is a Boolean algebra, then the set $D(B) = \{f : J \to B \mid \forall \alpha, \beta \in J, \alpha \leqslant \beta, f(\beta) \leqslant f(\alpha)\}$ is a θ.v.L.a., if $[\varphi_\alpha(f)](\beta) = f(\alpha)$ and $[\bar{\varphi}_\alpha(f)](\beta) = \overline{f(\alpha)}$. For every B there exists a natural isomorphism of Boolean algebras $B \Rightarrow C(D(B))$.

The θ-valued Lukasiewicz algebras and these notions were introduced by Gr.C.

Moisil in 1968, who pointed out the special role of DL_2 in the category of θ.v.L.a. in the following representation theorem:

Theorem 1 (Moisil [2]). *For every θ.v.L.a. L, there exists a set X and a monomorphism of θ.v.L.a.,*

$$\phi: L \to (DL_2)^X$$

which is obtained in the following manner: for the Boolean algebra $C(L)$ there exists a set X and a Boolean-algebra monomorphism $\psi: C(L) \to L_2^X$ (L_2 is the two-element Boolean algebra); and there exists the monomorphism of θ.v.L.a. $F: L \to D(C(L))$ given by $F(x)(\alpha) = \varphi_\alpha(x)$. Then ϕ is given by the composition:

$$\phi = [L \xrightarrow{F} D(C(L))] \circ [D(C(L)) \xrightarrow{D(\psi)} D(L_2^X)] \circ [D(L_2^X) \xrightarrow{\sim} (DL_2)^X].$$

2. The representability of Lukasiewicz algebras

Using Theorem 1 and the role of DL_2 in the category of θ.v.L.a. we shall introduce new concepts and we give some theorems which will extend some known theorems for Boolean algebras. First we introduce a few definitions.

Definition 2. (a) Let m be a cardinal; the θ.v.L.a. L will be called m-*complete* if the lattice L is m-complete.

(b) A θ.v.L.a. L will be called m-*completely chrysippian*, if for every family $(x_i)_{i \in I}$ in L, with card$(I) \leq$ m,

$$\bigcup_{i \in I} x_i \text{ exists} \iff \forall \alpha \in J,\ \bigcup_{i \in I} \varphi_\alpha(x_i) \text{ exists},$$

$$\bigcap_{i \in I} X_i \text{ exists} \iff \forall \alpha \in J,\ \bigcap_{i \in I} \varphi_\alpha(x_i) \text{ exists},$$

$$\forall \alpha \in J,\quad \varphi_\alpha\Big(\bigcup_{i \in I} x_i\Big) = \bigcup_{i \in I} \varphi_\alpha(x_i),$$

$$\forall \alpha \in J,\quad \varphi_\alpha\Big(\bigcap_{i \in I} x_i\Big) = \bigcap_{i \in I} \varphi_\alpha(x_i).$$

Lemma 1. (a) *If B is an* m-*complete Boolean algebra, then $D(B)$ is an* m-*complete θ.v.L.a..*

(b) *If L is an* m-*completely chrysippian θ.v.L.a., then for every $\alpha \in J$, $\bar{\varphi}_\alpha$ is an* m-*antiendomorphism, that is $\bar{\varphi}_\alpha(\bigcup_{i \in I} x_i) = \bigcap_{i \in I} \bar{\varphi}_\alpha(x_i)$ and $\bar{\varphi}_\alpha(\bigcap_{i \in I} x_i) = \bigcup_{i \in I} \bar{\varphi}_\alpha(x_i)$ if $\bigcup_{i \in I} x_i$, $\bigcap_{i \in I} x_i$, with* card$(I) \leq$ m *exists.*

Proof. (a) If B is m-complete, then the set $B^J = \{f: J \to B\}$ is an m-complete lattice if we define

$$\left(\bigcup_{i\in I} f_i\right)(\alpha)=\bigcup_{i\in I} f_i(\alpha),\qquad \left(\bigcap_{i\in I} F_i\right)(\alpha)=\bigcap_{i\in I} f_i(\alpha),$$

for every family $(f_i)_{i\in I}$ in B^J, with card$(I)\leq$m. If $f_i\in D(B)$ it is easy to prove that $\bigcup_{i\in I} f_i$, $\bigcap_{i\in I} f_i\in D(B)$.

(b) For every $\alpha\in J$ we have

$$\left[\bigcap_{i\in I}\bar{\varphi}_\alpha(x_i)\right]\cup\left[\varphi_\alpha\left(\bigcup_{i\in I} x_i\right)\right]=1,$$

$$\left[\bigcap_{i\in I}\bar{\varphi}_\alpha(x_i)\right]\cap\left[\varphi_\alpha\left(\bigcup_{i\in I} x_i\right)\right]=0$$

therefore

$\bar{\varphi}_\alpha\left(\bigcup_{i\in I} x_i\right)=\bigcap_{i\in I}\bar{\varphi}_\alpha(x_i)$. Similarly for the dual condition.

Definition 3. (a) A subalgebra L' of the θ.v.L.a. L, will be called an m-*subalgebra* of L, if L' is an m-sublattice of L.

(b) A homomorphism $f:L\to L'$ of θ.v.L.a. will be called m-*homomorphism*, if f preserves the meets and joins of all the families $(x_i)_{i\in I}$ in L, with card$(I)\leq$m.

(c) A θ-ideal $\mathfrak{a}$ of the θ.v.L.a. L is said to be an m-ideal, if for every family $(x_i)_{i\in I}$, card$(I)\leq$m, $x_i\in\mathfrak{a}\Rightarrow\bigcup_{i\in I} x_i\in\mathfrak{a}$ (a θ-ideal $\mathfrak{a}$ of L is an ideal of L such that if $x\in\mathfrak{a}$ then $\varphi_\alpha(x)\in\mathfrak{a}$ for every $\alpha\in J$).

(d) A subalgebra of the θ.v.L.a. $(DL_2)^X$, where X is a set, is said to be a θ-*valued Lukasiewicz field on* X. An m-subalgebra of $(DL_2)^X$ is said to be an m-θ-*valued Lukasiewicz field on* X (m-θ.v.L.f.).

Definition 4. A θ.v.L.a. L will be called m-*representable*, if there exists a set X, an m-θ.v.L.f. on X, $\mathcal{F}\subseteq(DL_2)^X$, an m-ideal $\mathfrak{a}$ of $\mathcal{F}$ and an m-monomorphism

$$\varphi:L\to\mathcal{F}/\mathfrak{a}.$$

Proposition 1. *If L is a θ.v.L.a., m-completely chrysippian, then $C(L)$ is an m-representable Boolean algebra.*

Proof. Firstly we prove that for every θ.v.L.a. L and for every set X there exists a natural isomorphism

$$P:C(L^X)\xrightarrow{\sim}(C(L))^X$$

defined by: if $f\in C(L^X)$, then $P(f):X\to C(L)$ is given by $P(f)(x)=f(x)$, which is well defined because $f\in C(L^X)\Leftrightarrow\varphi_\alpha(f)=f\Leftrightarrow[\varphi_\alpha(f)](x)=f(x)\ \forall x\in X\Leftrightarrow\varphi_\alpha(f(x))=f(x)$, therefore $f(x)\in C(L)$ for every $x\in X$.

Using this observation we have the monomorphism G,

$$G=[C((DL_2)^X)\xrightarrow{\sim}(C(DL_2))^X]\circ[(C(DL_2))^X\to L_2^X];$$

if $\mathcal{F}\subseteq(DL_2)^X$ is an m-θ.v.L.f. on X, then $C(\mathcal{F})\subseteq C((DL_2)^X)$ and $G(C(\mathcal{F}))\subseteq L_2^X$ is an m-field of subsets on X.

We now show that there exists an isomorphism

$$H\colon C(\mathcal{F}/\mathfrak{a})\to C(\mathcal{F})/\mathfrak{a}\cap C(\mathcal{F}).$$

Let $[x]_\mathfrak{a}\in C(\mathcal{F}/\mathfrak{a})$, that is $\varphi_\alpha[x]_\mathfrak{a}=[x]_\mathfrak{a}$ for every $\alpha\in J$, or $[\varphi_\alpha(x)]_\mathfrak{a}=[x]_\mathfrak{a}$, where $\varphi_\alpha(x)\in C(\mathcal{F})$ and $\varphi_\alpha(x)\equiv x(\mathfrak{a})$. We define $H([x]_\mathfrak{a})=[\varphi_\alpha(x)]_{\mathfrak{a}\cap C(\mathcal{F})}$ and we show that the definition is correct; it suffices to show the implication: if $y, y'\in C(\mathcal{F})$, $[y]_\mathfrak{a}=[y']_\mathfrak{a}$, then $[y]_{\mathfrak{a}\cap C(\mathcal{F})}=[y']_{\mathfrak{a}\cap C(\mathcal{F})}$. If $[y]_\mathfrak{a}=[y']_\mathfrak{a}$ then there exists $z\in\mathfrak{a}$ such that $y\cup z=y'\cup z$, therefore $\varphi_\alpha(y\cup z)=\varphi_\alpha(y'\cup z)$, or $y\cup\varphi_\alpha(z)=y'\cup\varphi_\alpha(z)$; but $\varphi_\alpha(z)\in\mathfrak{a}\cap C(\mathcal{F})$ and finally $[y]_{\mathfrak{a}\cap C(\mathcal{F})}=[y']_{\mathfrak{a}\cap C(\mathcal{F})}$. The map H is injective because

$$\begin{aligned}H([x]_\mathfrak{a})=H([y]_\mathfrak{a})&\Rightarrow[\varphi_\alpha(x)]_{\mathfrak{a}\cap C(\mathcal{F})}=[\varphi_\alpha(y)]_{\mathfrak{a}\cap C(\mathcal{F})}\\&\Rightarrow[\varphi_\alpha(x)]_\mathfrak{a}=[\varphi_\alpha(y)]_\mathfrak{a}\\&\Rightarrow x\equiv\varphi_\alpha(x)\equiv\varphi_\alpha(y)\equiv y(\mathfrak{a})\\&\Rightarrow[x]_\mathfrak{a}=[y]_\mathfrak{a}.\end{aligned}$$

H is surjective because for $[y]_{\mathfrak{a}\cap C(\mathcal{F})}$ with $y\in C(\mathcal{F})$ we have $H([y]_\mathfrak{a})=[y]_{\mathfrak{a}\cap C(\mathcal{F})}$. H is a homomorphism of lattices with 0 and 1; finally H is an isomorphism of Boolean algebras.

Now consider the map

$$\varphi_1=[C(L)\xrightarrow{\varphi\mid C(L)}C(\mathcal{F}/\mathfrak{a})\xrightarrow{H}C(\mathcal{F})/\mathfrak{a}\cap C(\mathcal{F})\to G(C(\mathcal{F})/G(\mathfrak{a}\cap C(\mathcal{F})]$$

where $\varphi\mid C(L)$ is an m-monomorphism, and consequently

$$\varphi_1\colon C(L)\to G(C(\mathcal{F}))/G(\mathfrak{a}\cap C(\mathcal{F}))$$

is an m-monomorphism, where $G(C(\mathcal{F}))\subseteq L_2^X$ is a boolean m-field on X and $G(\mathfrak{a}\cap C(\mathcal{F}))$ is an m-ideal of $G(C(\mathcal{F}))$, thus the boolean algebra $C(L)$ is m-representable.

Remarks. (1) In the proof we have used the fact that $\mathfrak{a}\cap C(\mathcal{F})$ is an m-ideal of Boolean algebra $C(\mathcal{F})$; in fact, if $x_i\in\mathfrak{a}\cap C(\mathcal{F})$, $\forall i\in I$, card$(I)\leqslant\mathfrak{m}$, then $\bigcup_{i\in I}x_i\in\mathfrak{a}$ and $\bigcup_{i\in I}x_i\in C(\mathcal{F})$ because $\varphi_\alpha(\bigcup_{i\in I}x_i)=\bigcup_{i\in I}\varphi_\alpha(x_i)=\bigcup_{i\in I}x_i$.

(2) *If* $y,y'\in C(\mathcal{F})$, $[y]_{\mathfrak{a}\cap C(\mathcal{F})}=[y']_{\mathfrak{a}\cap C(\mathcal{F})}$ then $[y]_\mathfrak{a}=[y']_\mathfrak{a}$. Let be $x\in[y]_\mathfrak{a}$, there exists $a\in\mathfrak{a}$ such that $x\cup a=y\cup a$, and there exists $c\in\mathfrak{a}\cap C(\mathcal{F})$ such that $y\cup c=y'\cup c$; thus $x\cup a\cup c=y\cup a\cup c=y'\cup a\cup c$, where $a\cup c\in\mathfrak{a}$, therefore $x\in[y']_\mathfrak{a}$.

Proposition 2. *If the θ.v.L.a. L is* m*-completely chrysippian,* card $J\leqslant\mathfrak{m}$ *and* $C(L)$ *is* m*-representable, then L is* m*-representable.*

Proof. Firstly we prove that for every set X there exists a natural isomorphism

$$S: D(L_2^X) \xrightarrow{\sim} (DL_2)^X$$

defined by: if $f \in D(L_2^X)$ then $S(f): X \to DL_2$ is given by $[S(f)(x)](\alpha) = [f(\alpha)](x)$.

Let φ be the m-monomorphism $\varphi: C(L) \to \mathscr{F}_1/\mathfrak{a}$ given by the m-representability of the Boolean algebra $C(L)$, $\mathscr{F}_1 \subseteq L_2^X$. If L is m-completely chrysippian the canonical monomorphism $F: L \to D(C(L))$ is an m-monomorphism of θ.v.L.a., and φ may be extended to an m-monomorphism

$$D(\varphi): D(C(L)) \to D(\mathscr{F}_1/\mathfrak{a}).$$

In the θ.v.L.a. $D(\mathscr{F}_1) = \{f: J \to \mathscr{F}_1 \mid f \text{ is decreasing}\}$ we consider the subset $\mathfrak{a}_1 \subset D(\mathscr{F}_1)$ defined as follows:

$$f \in \mathfrak{a}_1 \quad \Leftrightarrow \quad \forall \alpha \in J, f(\alpha) \in \mathfrak{a}.$$

We prove that $\mathfrak{a}_1$ is an m-ideal on $D(\mathscr{F}_1)$:

$$\begin{aligned} f, g \in \mathfrak{a}_1 &\Rightarrow \forall \alpha \in J, f(\alpha), g(\alpha) \in \mathfrak{a} \\ &\Rightarrow \forall \alpha \in J, (f \cup g)(\alpha) = f(\alpha) \cup g(\alpha) \in \mathfrak{a} \\ &\Rightarrow f \cup g \in \mathfrak{a}_1; \end{aligned}$$

$$\begin{aligned} f \in \mathfrak{a}_1, g \leq f &\Rightarrow \forall \alpha \in J, g(\alpha) \leq f(\alpha) \in \mathfrak{a} \\ &\Rightarrow \forall \alpha \in J, g(\alpha) \in \mathfrak{a} \Rightarrow g \in \mathfrak{a}_1; \end{aligned}$$

$$\begin{aligned} f \in \mathfrak{a}_1 &\Rightarrow \forall \alpha \in J, f(\alpha) \in \mathfrak{a} \\ &\Rightarrow \forall \alpha, \beta \in J, \varphi_\beta(f(\alpha)) = f(\beta) \in \mathfrak{a} \\ &\Rightarrow \varphi_\beta(f) \in \mathfrak{a}_1, \forall \beta \in J; \end{aligned}$$

$$\begin{aligned} f_i \in \mathfrak{a}_1, i \in I, \operatorname{card}(I) \leq \mathrm{m} &\Rightarrow \forall \alpha \in J, f_i(\alpha) \in \mathfrak{a}, i \in I \\ &\Rightarrow \forall \alpha \in J, \left(\bigcup_{i \in I} f_i\right)(\alpha) = \bigcup_{i \in I} f_i(\alpha) \in \mathfrak{a} \\ &\Rightarrow \bigcup_{i \in I} f_i \in \mathfrak{a}_1. \end{aligned}$$

Now we consider the map

$$G: D(\mathscr{F}_1)/\mathfrak{a}_1 \to D(\mathscr{F}_1/\mathfrak{a})$$

defined by $G([f]_{\mathfrak{a}_1}): J \to \mathscr{F}_1/\mathfrak{a}$ is $G([f]_{\mathfrak{a}_1})(\alpha) = [f(\alpha)]_{\mathfrak{a}}$ for every $\alpha \in J$ (the definition is correct because from $f \equiv g (\operatorname{mod} \mathfrak{a}_1)$ it follows that $f(\alpha) \equiv g(\alpha) (\operatorname{mod} \mathfrak{a})$ for every $\alpha \in J$).

The map G is injective; if $G([f]_{\mathfrak{a}_1}) = G([g]_{\mathfrak{a}_1})$, then $[f(\alpha)]_{\mathfrak{a}} = [g(\alpha)]_{\mathfrak{a}}$ for every $\alpha \in J$, or for every $\alpha \in J$ there exists $t_\alpha \in \mathfrak{a}$ such that $f(\alpha) \cup t_\alpha = g(\alpha) \cup t_\alpha$; if $t'_\alpha = \bigcup_{\lambda \geq \alpha} t_\lambda$, then $t'_\alpha \in \mathfrak{a}$ for every $\alpha \in J$ (using the assumption $\operatorname{card}(J) \leq \mathrm{m}$). We conclude that $f \cup t' = g \cup t'$, where $t'(\alpha) = t'_\alpha$ and $t' \in \mathfrak{a}_1$, or $f \equiv g (\operatorname{mod} \mathfrak{a}_1)$, or $[f]_{\mathfrak{a}_1} = [g]_{\mathfrak{a}_1}$.

The map G is surjective; let be $f_1 \in D(\mathcal{F}_1/\mathfrak{a})$, or $f_1 : J \to \mathcal{F}_1/\mathfrak{a}$ decreasing and we shall show that there exists $f : J \to \mathcal{F}_1$ decreasing such that $G([f]_{\mathfrak{a}_1}) = f_1$, or $[f(\alpha)]_{\mathfrak{a}} = f_1(\alpha)$ for every $\alpha \in J$. Since $f_1(\alpha) \in \mathcal{F}_1/\mathfrak{a}$, there exists $f_\alpha \in \mathcal{F}_1$ such that $f_1(\alpha) = [f_\alpha]_{\mathfrak{a}}$. We define $f : J \to \mathcal{F}_1$ by $f(\alpha) = \bigcup_{\lambda \geq \alpha} f_\lambda \in \mathcal{F}_1$ and we have

$$G([f]_{\mathfrak{a}_1})(\alpha) = [f(\alpha)]_{\mathfrak{a}} = \Big[\bigcup_{\lambda \geq \alpha} f_\lambda\Big]_{\mathfrak{a}} = \bigcup_{\lambda \geq \alpha} [f_\lambda]_{\mathfrak{a}} = \bigcup_{\lambda \geq \alpha} f_1(\lambda) = f_1(\alpha).$$

The map G is an homomorphism of θ.v.L.a.:

$$\begin{aligned} G([f]_{\mathfrak{a}_1} \cup [g]_{\mathfrak{a}_1}) &= G([f \cup g]_{\mathfrak{a}_1})(\alpha) = [(f \cup g)(\alpha)]_{\mathfrak{a}} \\ &= [f(\alpha) \cup g(\alpha)]_{\mathfrak{a}} - [f(\alpha)]_{\mathfrak{a}} \cup [g(\alpha)]_{\mathfrak{a}} \\ &= G([f]_{\mathfrak{a}_1})(\alpha) \cup G([g]_{\mathfrak{a}_1})(\alpha) \\ &= (G([f]_{\mathfrak{a}_1}) \cup G([g]_{\mathfrak{a}_1}))(\alpha). \end{aligned}$$

Similarly for "$\cap$".

$$G(\varphi_\beta([f]_{\mathfrak{a}_1}))(\alpha) = G([\varphi_\beta(f)]_{\mathfrak{a}_1})(\alpha) = [(\varphi_\beta f)(\alpha)]_{\mathfrak{a}} = [f(\beta)]_{\mathfrak{a}} = (\varphi_\beta(G[f]_{\mathfrak{a}_1}))(\alpha)$$

Finally G is an isomorphism of θ.v.L.a.,

$$G : D(\mathcal{F}_1)/\mathfrak{a}_1 \to D(\mathcal{F}/\mathfrak{a}).$$

Consider now the composite:

$$\varphi_1 = (L \xrightarrow{F} D(C(L)) \xrightarrow{D(\varphi)} D(\mathcal{F}_1/\mathfrak{a}) \xrightarrow{G^{-1}} D(\mathcal{F}_1)/\mathfrak{a}_1);$$

thus φ_1 is an m-monomorphism $\varphi_1 : L \to D(\mathcal{F}_1)/\mathfrak{a}_1$, $\mathfrak{a}_1$ is an m-ideal on $D(\mathcal{F}_1)$, $D(\mathcal{F}_1) \subseteq D(L_2^X) \xrightarrow{S} (DL_2)^X$, and consequently L is m-representable.

From Propositions 1 and 2 we deduce:

Theorem 1. *A θ.v.L.a. L, m-completely chrysippian with* m $\geq$ card J *is m-representable iff $C(L)$ is an m-representable Boolean algebra.*

Corollary. *A θ.v.L.a. L, $\aleph_0$-completely chrysippian, with* $\aleph_0 \geq$ card J, *is $\aleph_0$-representable.*

Proof. Follows from the fact that every Boolean algebra is $\aleph_0$-representable (Loomis–Sikorski theorem) and Theorem .

Finally, using the representation theorem of Moisil, we obtain a representation theorem, where some infinite meets and joins are preserved.

Proposition 3. *Let L be a θ.v.L.a., $\aleph_0$-completely chrysippian with* card$(J) \leq \aleph_0$. *Then for every given countable family of joins and meets in L,*

$$\begin{cases} a_j = \bigcup_{i \in I_j} a_{ij}, \ j \in I, & \text{where } \mathrm{card}(I), \mathrm{card}(I_j) \leq \aleph_0, \\ b_k = \bigcap_{t \in T_k} b_{tk}, \ k \in T, & \text{where } \mathrm{card}(T), \mathrm{card}(T_k) \leq \aleph_0 \end{cases}$$

there exist a set X and a monomorphism $\phi : L \to (DL_2)^X$ of θ.v.L.a. which preserves all these joins and meets:

$$\phi(a_j) = \bigcup_{i\in I_j} \phi(a_{ij}), \quad j \in I,$$

$$\phi(b_k) = \bigcap_{t\in T_k} \phi(b_{tk}), \quad k \in T.$$

Proof. As L is $\aleph_0$-completely chrysippian, we have

$$(*) \qquad \varphi_\alpha(a_j) = \bigcup_{i\in I_j} \varphi_\alpha(a_{ij}), \quad \forall j \in I, \forall \alpha \in J,$$

$$(**) \qquad \varphi_\alpha(b_k) = \bigcap_{t\in T_k} \varphi_\alpha(b_{tk}), \quad \forall k \in T, \forall \alpha \in J.$$

This family is countable because card(J) $\leqslant \aleph_0$; using a theorem of Sikorski (see [5]) for Boolean algebras there exists a convenable set X and a Boolean monomorphism

$$\psi : C(L) \to L_2^X,$$

such that ψ preserves the joins and meets from $(*)$, $(**)$. Let us denote by

$$\phi : L \to (DL_2)^X$$

the morphism defined by $\phi(a)(x)(\alpha) = 1$ iff $\psi(\varphi_\alpha(a))(x) = 1$, we obtain a monomorphism of θ.v.L.a. (see [2]) and we shall show that preserves the given joins and meets in L. We have:

$$\begin{aligned}
\phi(a_j)(x)(\alpha) = 1 &\Leftrightarrow \phi\Big(\bigcup_{i\in I_j} a_{ij}\Big)(x)(\alpha) = 1 \\
&\Leftrightarrow \psi\Big(\varphi_\alpha\Big(\bigcup_{i\in I_j} a_{ij}\Big)\Big)(x) = 1 \\
&\Leftrightarrow \psi\Big(\bigcup_{i\in I_j} \varphi_\alpha(a_{ij})\Big)(x) = 1 \\
&\Leftrightarrow \bigcup_{i\in I_j} \psi(\varphi_\alpha(a_{ij}))(x) = 1 \\
&\Leftrightarrow \bigcup_{i\in I_j} \phi(a_{ij})(x)(\alpha) = 1 \\
&\Leftrightarrow \Big(\bigcup_{i\in I_j} \phi(a_{ij})\Big)(x))(\alpha) = 1,
\end{aligned}$$

thus $\phi(a_j)(x) = \bigcup_{i\in I_j} \phi(a_{ij})(x)$, $\forall x \in X$, or $\phi(a_j) = \bigcup_{i\in I_j} \phi(a_{ij})$. The proof for meets is similar.

Acknowledgements

The author wishes to express his gratitude to Professor Sergiu Rudeanu for stimulating him to investigate the problems discussed and also for valuable remarks.

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