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On monotonic solutions of an integral equation of Volterra type with supremum ☆

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Abstract

Using a technique associated with measures of noncompactness, we prove the existence of nondecreasing solutions of an integral equation of Volterra type in $C[0,1]$.

Keywords: Measure of noncompactness; Fixed point theorem; Nondecreasing solutions

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1. Introduction

Integral equations arise naturally in applications of real world problems [1,2,6,7,9,11]. The theory of integral equations has been well-developed with the help of various tools from functional analysis, topology and fixed-point theory.

The aim of this paper is to investigate the existence of nondecreasing solutions of an integral equation of Volterra type with supremum. Equations of such kind have been studied

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in other papers ([8,10], among others) and in the monograph [3]. These equations can be considered with connection to the following Cauchy problem:

$$x'(t) = f(t) \cdot \max_{[0,t]} \left|x(\tau)\right|, \qquad x(0) = 0.$$

2. Notation and auxiliary facts

Assume E is a real Banach space with norm $\|\cdot\|$ and zero element 0. Denote by $B(x,r)$ the closed ball centered at x and with radius r and by B_r the ball $B(0,r)$. If X is a nonempty subset of E we denote by $\bar{X}$, $\operatorname{Conv} X$ the closure and the closed convex closure of X, respectively. The symbols λX and $X+Y$ denote the usual algebraic operations on sets. Finally, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

Definition 1 (see [4]). A function $\mu : \mathfrak{M}_E \to [0,\infty)$ is said to be a *measure of noncompactness* in the space E if it satisfies the following conditions:

(1) The family $\ker\mu = \{X \in \mathfrak{M}_E\colon\ \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.
(2) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.
(3) $\mu(\bar{X}) = \mu(\operatorname{Conv} X) = \mu(X)$.
(4) $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
(5) If $\{X_n\}_n$ is a sequence of closed sets of $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ described above is called *the kernel of the measure of noncompactness* μ. Further facts concerning measures of noncompactness and their properties may be found in [4].

Now, let us suppose that M is a nonempty subset of a Banach space E and the operator $T : M \to E$ is continuous and transforms bounded sets onto bounded ones. We say that T satisfies the Darbo condition (with constant $k \geqslant 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M we have

$$\mu(TX) \leqslant k\mu(X).$$

If T satisfies the Darbo condition with $k < 1$, then it is called a contraction with respect to μ.

For our purpose we will only need the following fixed point theorem [4].

Theorem 2. *Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and μ a measure of noncompactness in E. Let $F : Q \to Q$ be a contraction with respect to μ. Then F has a fixed point in the set Q.*

Remark 3. Under the assumptions of the above theorem it can be shown that the set $\operatorname{Fix} F$ of fixed points of F belonging to Q is a member of $\ker\mu$.

Proof. $\mu(\operatorname{Fix} F) = \mu(F(\operatorname{Fix} F)) < k\mu(\operatorname{Fix} F)$ and as $k < 1$, we deduce that $\mu(\operatorname{Fix} F) = 0$. □

Let $C[0,1]$ denote the space of all real functions defined and continuous on the interval $[0,1]$. For convenience, we write $I = [0,1]$ and $C(I) = C[0,1]$. The space $C(I)$ is furnished with standard norm

$$\|x\| = \max\{|x(t)|:\ t \in I\}.$$

Next, we recall the definition of a measure of noncompactness in $C(I)$ which will be used in Section 3. This measure was introduced and studied in [5].

Fix a nonempty and bounded subset X of $C(I)$. For $\varepsilon > 0$ and $x \in X$ denote by $w(x,\varepsilon)$ the modulus of continuity of x defined by

$$w(x,\varepsilon) = \sup\{|x(t) - x(s)|:\ t, s \in I,\ |t - s| \leqslant \varepsilon\}.$$

Furthermore, put

$$w(X,\varepsilon) = \sup\{w(x,\varepsilon):\ x \in X\} \quad \text{and} \quad w_0(X) = \lim_{\varepsilon \to 0} w(X,\varepsilon).$$

Next, let us define the following quantities:

$$i(x) = \sup\{|x(s) - x(t)| - [x(s) - x(t)]:\ t, s \in I,\ t \leqslant s\} \quad \text{and}$$
$$i(X) = \sup\{i(x):\ x \in X\}.$$

Observe that $i(X) = 0$ if and only if all functions belonging to X are nondecreasing on I. Finally, let

$$\mu(X) = w_0(X) + i(X). \tag{1}$$

It can be shown [5] that the function μ is a measure of noncompactness in the space $C(I)$. Moreover, the kernel $\ker \mu$ consists of all sets X belonging to $\mathfrak{M}_{C(I)}$ such that all functions from X are equicontinuous and nondecreasing on the interval I.

3. Main result

In this section we consider the following nonlinear integral equation of Volterra type:

$$x(t) = a(t) + (Tx)(t) \int_0^t \phi(t,s) \max_{[0,r(s)]} |x(\tau)|\, ds, \quad t \in I. \tag{2}$$

The functions $a(t)$, $r(s)$, $\phi(t,s)$ and $(Tx)(t)$ are given while $x = x(t)$ is an unknown function.

We will study this equation under the following assumptions:

(i) $a \in C(I)$ and it is nondecreasing and nonnegative on the interval I.
(ii) $\phi : I \times I \to \mathbb{R}_+$ is continuous on $I \times I$ and the function $t \to \phi(t,s)$ is nondecreasing for each $s \in I$.

(iii) $r : I \to I$ is a continuous and nondecreasing function.

(iv) The operator $T : C(I) \to C(I)$ is continuous and satisfies the Darbo condition for the measure of noncompactness μ (defined in (1)) with a constant Q. Moreover, T is a positive operator, i.e., $Tx \geqslant 0$ if $x \geqslant 0$.

(v) There exist nonnegative constants c and d such that

$$\|Tx\| \leqslant c + d\|x\|$$

for each $x \in C(I)$ and $t \in I$.

(vi) There exists $r_0 > 0$ such that $\|a\| + (c + dr_0) \cdot \|\phi\| \cdot r_0 \leqslant r_0$ and $Q\|\phi\|r_0 < 1$.

Before we formulate our main result we will prove the following lemmas which be needed further on.

Lemma 1. *Suppose that* $x \in C(I)$ *and we define*

$$(Gx)(t) = \max_{[0,r(t)]} \left|x(\tau)\right| \quad \text{for } t \in I.$$

Then $Gx \in C(I)$.

Proof. Without loss of generality we can assume that $x \geqslant 0$. We will prove that for $\varepsilon > 0$,

$$w(Gx, \varepsilon) \leqslant w(x \circ r\ , \varepsilon).$$

Suppose contrary. This means that there exist $t_1, t_2 \in I$, $t_1 \leqslant t_2$, $t_2 - t_1 \leqslant \varepsilon$ such that

$$w(x \circ r, \varepsilon) < \left|(Gx)(t_2) - (Gx)(t_1)\right|. \tag{3}$$

As Gx is a nondecreasing function, we have

$$0 < (Gx)(t_2) - (Gx)(t_1). \tag{4}$$

Further, let us find $0 \leqslant \tau_2 \leqslant r(t_2)$ with the property $(Gx)(t_2) = x(\tau_2)$.

Taking into account the inequality (4), $r(t_1) \leqslant \tau_2$. In virtue of the continuity of the function r, $\tau_2 = r(p_2)$, and we can deduce that

$$\begin{aligned}(Gx)(t_2) - (Gx)(t_1) &= x(\tau_2) - (Gx)(t_1) \leqslant x\bigl(r(p_2)\bigr) - x\bigl(r(t_1)\bigr)\\ &= (x \circ r)(p_2) - (x \circ r)(t_1)\end{aligned}$$

and as $p_2 - t_1 \leqslant t_2 - t_1 \leqslant \varepsilon$,

$$(Gx)(t_2) - (Gx)(t_1) \leqslant (x \circ r)(p_2) - (x \circ r)(t_1) \leqslant w(x \circ r, \varepsilon).$$

Thus we arrive at a contradiction.

Thus, for $\varepsilon > 0$,

$$w(Gx, \varepsilon) \leqslant w(x \circ r, \varepsilon)$$

and as $x \circ r \in C(I)$, the proof is complete. □

Lemma 2. *Let* $(x_n), x \in C(I)$. *Suppose that* $x_n \to x$ *in* $C(I)$. *Then* $Gx_n \to Gx$ *uniformly on I.*

Proof. Note that for $t \in I$ and $y \in C(I)$,

$$(Gy)(t) = \|y|_{[0,r(t)]}\|,$$

where $y|_{[0,r(t)]}$ denotes the restriction of the function y on the interval $[0, r(t)]$ and the norm is considered in the space $C([0, r(t)])$. In view of this fact, we can deduce

$$\begin{aligned}\|Gx_n - Gx\| &= \sup_{t\in I}\bigl|(Gx_n)(t) - (Gx)(t)\bigr| = \sup_{t\in I}\bigl|\|xn|_{[0,r(t)]}\| - \|x|_{[0,r(t)]}\|\bigr| \\ &\leqslant \sup_{t\in I}\bigl\|(x_n - x)|_{[0,r(t)]}\bigr\| \leqslant \|x_n - x\|.\end{aligned}$$

As $x_n \to x$ in $C(I)$, we obtain the desired result. □

Now we present our main result.

Theorem 3. *Under assumptions* (i)–(vi), *Eq.* (2) *has at least one solution $x = x(t)$ which belongs to the space $C(I)$ and is nondecreasing on the interval I.*

Proof. Let us consider two operators A, B defined on the space $C(I)$ by

$$(Ax)(t) = a(t) + (Tx)(t)\int_0^t \phi(t,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds \quad \text{and}$$

$$(Bx)(t) = \int_0^t \phi(t,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds.$$

Firstly, we prove that if $x \in C(I)$, then $Ax \in C(I)$. To do this it is sufficient to show that if $x \in C(I)$, then $Bx \in C(I)$. Fix $\varepsilon > 0$, let $x \in C(I)$ and $t_1, t_2 \in I$ such that $t_1 \leqslant t_2$ and $t_2 - t_1 \leqslant \varepsilon$. Then

$$\begin{aligned}\bigl|(Bx)(t_2) - (Bx)(t_1)\bigr| &= \left|\int_0^{t_2} \phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - \int_0^{t_1} \phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right| \\ &\leqslant \left|\int_0^{t_2} \phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - \int_0^{t_2} \phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right| \\ &\quad + \left|\int_0^{t_2} \phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - \int_0^{t_1} \phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right| \\ &\leqslant \int_0^{t_2} \bigl|\phi(t_2,s) - \phi(t_1,s)\bigr|\cdot\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds \\ &\quad + \int_{t_1}^{t_2} \bigl|\phi(t_1,s)\bigr|\cdot\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds.\end{aligned}$$

Therefore, if we denote

$$w_\phi(\varepsilon,\cdot)=\sup\bigl\{\bigl|\phi(t,s)-\phi(t',s)\bigr|\colon\ t,t',s\in I \text{ and } |t-t'|\leqslant\varepsilon\bigr\},$$

we obtain that

$$\begin{aligned}\bigl|(Bx)(t_2)-(Bx)(t_1)\bigr| &\leqslant w_\phi(\varepsilon,\cdot)\cdot\|x\|\cdot t_2+\|\phi\|\cdot\|x\|\cdot(t_2-t_1)\\ &\leqslant w_\phi(\varepsilon,\cdot)\cdot\|x\|+\|\phi\|\cdot\|x\|\cdot\varepsilon.\end{aligned}$$

Now, in virtue of the uniform continuity of the function ϕ on $I\times I$ we have that $w_\phi(\varepsilon,\cdot)\to 0$ as $\varepsilon\to 0$. Thus $Bx\in C(I)$, and consequently, $Ax\in C(I)$.

Moreover, for each $t\in I$ we have

$$\begin{aligned}\bigl|(Ax)(t)\bigr| &= \left|a(t)+(Tx)(t)\int_0^t\phi(t,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right|\\ &\leqslant \|a\|+\bigl(c+d\|x\|\bigr)\int_0^t\bigl|\phi(t,s)\bigr|\cdot\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\\ &\leqslant \|a\|+\bigl(c+d\|x\|\bigr)\cdot\|\phi\|\cdot\|x\|.\end{aligned}$$

Hence,

$$\|Ax\|\leqslant\|a\|+\bigl(c+d\|x\|\bigr)\cdot\|\phi\|\cdot\|x\|.$$

Thus, if $\|x\|\leqslant r_0$ we obtain from assumption (vi) that

$$\|Ax\|\leqslant\|a\|+(c+dr_0)\cdot\|\phi\|\cdot r_0\leqslant r_0.$$

Consequently, the operator A transforms the ball $B_{r_0}=B(0,r_0)$ into itself.

In the sequel we consider the operator A on the subset $B_{r_0}^+$ of the ball B_{r_0} defined by

$$B_{r_0}^+=\bigl\{x\in B_{r_0}\colon\ x(t)\geqslant 0 \text{ for } t\in I\bigr\}.$$

Obviously, the set $B_{r_0}^+$ is nonempty, bounded, closed and convex. On the other hand, in view of our assumptions (i), (iii) and (v) if $x\in B_{r_0}^+$, then $Ax\in B_{r_0}^+$.

Next, we prove that A is continuous on $B_{r_0}^+$. To do this, let $\{x_n\}$ be a sequence in $B_{r_0}^+$ such that $x_n\to x$ and we will prove that $Ax_n\to Ax$.

In fact, for each $t\in I$ we have

$$\begin{aligned}&\bigl|(Ax_n)(t)-(Ax)(t)\bigr|\\ &\quad=\left|(Tx_n)(t)\int_0^t\phi(t,s)\max_{[0,r(s)]}\bigl|x_n(\tau)\bigr|\,ds-(Tx)(t)\int_0^t\phi(t,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right|\\ &\quad\leqslant\left|(Tx_n)(t)\int_0^t\phi(t,s)\max_{[0,r(s)]}\bigl|x_n(\tau)\bigr|\,ds-(Tx)(t)\int_0^t\phi(t,s)\max_{[0,r(s)]}\bigl|x_n(\tau)\bigr|\,ds\right|\end{aligned}$$

$$+\left|(Tx)(t)\int_0^t \phi(t,s)\max_{[0,r(s)]}\left|x_n(\tau)\right|ds-(Tx)(t)\int_0^t \phi(t,s)\max_{[0,r(s)]}\left|x(\tau)\right|ds\right|$$

$$\leqslant\left|(Tx_n)(t)-(Tx)(t)\right|\int_0^t\left|\phi(t,s)\right|\cdot\max_{[0,r(s)]}\left|x_n(\tau)\right|ds$$

$$+\left|(Tx)(t)\right|\int_0^t\left|\phi(t,s)\right|\cdot\left|\max_{[0,r(s)]}\left|x_n(\tau)\right|-\max_{[0,r(s)]}\left|x(\tau)\right|\right|ds.$$

In virtue of Lemma 2:

$$\|Ax_n-Ax\|\leqslant\|Tx_n-Tx\|\cdot\|\phi\|\cdot r_0+(c+dr_0)\cdot\|\phi\|\cdot\|x_n-x\|. \tag{5}$$

As T is a continuous operator, there exists $n_1\in\mathbb{N}$ such that for $n\geqslant n_1$ we have

$$\|Tx_n-Tx\|\leqslant\frac{\varepsilon}{2\|\phi\|\cdot r_0}.$$

Moreover, we can find $n_2\in\mathbb{N}$ such that for all $n\geqslant n_2$ we have that $\|x_n-x\|\leqslant\frac{\varepsilon}{2\|\phi\|\cdot(c+dr_0)}$. Finally, if we take $n\geqslant\max\{n_1,n_2\}$, from (5) we get

$$\|Ax_n-Ax\|\leqslant\varepsilon.$$

This fact proves that A is continuous in $B_{r_0}^+$.

In the sequel we prove that the operator A satisfies the Darbo condition with respect to the measure of noncompactness introduced in Section 2.

Let X be a nonempty subset of $B_{r_0}^+$. Fix $\varepsilon>0$ and $t_1,t_2\in I$ with $|t_2-t_1|\leqslant\varepsilon$. Without loss of generality we may assume that $t_1\leqslant t_2$, then

$$\left|(Ax)(t_2)-(Ax)(t_1)\right|$$

$$=\left|a(t_2)+(Tx)(t_2)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\left|x(\tau)\right|ds\right.$$

$$\left.-a(t_1)-(Tx)(t_1)\int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\left|x(\tau)\right|ds\right|$$

$$\leqslant\left|a(t_2)-a(t_1)\right|+\left|(Tx)(t_2)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\left|x(\tau)\right|ds\right.$$

$$\left.-(Tx)(t_1)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\left|x(\tau)\right|ds\right|$$

$$+\left|(Tx)(t_1)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\left|x(\tau)\right|ds-(Tx)(t_1)\int_0^{t_2}\phi(t_1,s)\max_{[0,r(s)]}\left|x(\tau)\right|ds\right|$$

$$
\begin{aligned}
&+\left|(Tx)(t_1)\int_0^{t_2}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds-(Tx)(t_1)\int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right|\\
&\leqslant w(a,\varepsilon)+\bigl|(Tx)(t_2)-(Tx)(t_1)\bigr|\int_0^{t_2}\bigl|\phi(t_2,s)\bigr|\cdot\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\\
&\quad+\bigl|(Tx)(t_1)\bigr|\int_0^{t_2}\bigl|\phi(t_2,s)-\phi(t_1,s)\bigr|\cdot\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\\
&\quad+\bigl|(Tx)(t_1)\bigr|\int_{t_1}^{t_2}\bigl|\phi(t_1,s)\bigr|\cdot\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\\
&\leqslant w(a,\varepsilon)+w(Tx,\varepsilon)\cdot\|\phi\|\cdot r_0\cdot t_2+(c+dr_0)\cdot w_\phi(\varepsilon,\cdot)\cdot r_0\cdot t_2\\
&\quad+(c+dr_0)\cdot\|\phi\|\cdot r_0\cdot(t_2-t_1)\\
&\leqslant w(a,\varepsilon)+w(Tx,\varepsilon)\cdot\|\phi\|\cdot r_0+(c+dr_0)\cdot r_0\cdot\bigl(w_\phi(\varepsilon,\cdot)+\varepsilon\cdot\|\phi\|\bigr).
\end{aligned}
$$

Hence,

$$w(Ax,\varepsilon)\leqslant w(a,\varepsilon)+w(Tx,\varepsilon)\cdot\|\phi\|\cdot r_0+(c+dr_0)\cdot r_0\cdot\bigl(w_\phi(\varepsilon,\cdot)+\varepsilon\cdot\|\phi\|\bigr).$$

Consequently,

$$w(AX,\varepsilon)\leqslant w(a,\varepsilon)+w(TX,\varepsilon)\cdot\|\phi\|\cdot r_0+(c+dr_0)r_0\bigl(w_\phi(\varepsilon,\cdot)+\varepsilon\cdot\|\phi\|\bigr).$$

From the uniform continuity of the function ϕ on the set $I\times I$ and the continuity of the function a on I we have that $w_\phi(\varepsilon,\cdot)\to 0$ and $w(a,\varepsilon)\to 0$ as $\varepsilon\to 0$. So, applying limit when $\varepsilon\to 0$, we obtain

$$w_0(AX)\leqslant\|\phi\|\cdot r_0\cdot w_0(TX). \tag{6}$$

Now, we study the term related to the monotonicity.

Fix $x\in X$ and $t_1,t_2\in I$ with $t_1<t_2$. Then, taking into account our assumptions, we have

$$
\begin{aligned}
&\bigl|(Ax)(t_2)-(Ax)(t_1)\bigr|-\bigl((Ax)(t_2)-(Ax)(t_1)\bigr)\\
&=\left|a(t_2)+(Tx)(t_2)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds-a(t_1)\right.\\
&\quad\left.-(Tx)(t_1)\int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right|\\
&\quad-\Biggl(\Biggl(a(t_2)+(Tx)(t_2)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds-a(t_1)
\end{aligned}
$$

$$\begin{aligned}
&\quad - (Tx)(t_1)\int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\Biggr)\Biggr)\\
&\leqslant \bigl[\bigl|a(t_2)-a(t_1)\bigr| - \bigl(a(t_2)-a(t_1)\bigr)\bigr]\\
&\quad + \left|(Tx)(t_2)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - (Tx)(t_1)\int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right|\\
&\quad - \left((Tx)(t_2)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - (Tx)(t_1)\int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right)\\
&\leqslant \left|(Tx)(t_2)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - (Tx)(t_1)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right|\\
&\quad + \left|(Tx)(t_1)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - (Tx)(t_1)\int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right|\\
&\quad - \left((Tx)(t_2)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - (Tx)(t_1)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right)\\
&\quad - \left((Tx)(t_1)\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - (Tx)(t_1)\int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right)\\
&\leqslant \bigl[\bigl|(Tx)(t_2)-(Tx)(t_1)\bigr| - \bigl((Tx)(t_2)-(Tx)(t_1)\bigr)\bigr]\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\\
&\quad + (Tx)(t_1)\left[\left|\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - \int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right|\right.\\
&\quad \left. - \left(\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - \int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds\right)\right]. \qquad (7)
\end{aligned}$$

Now, we will prove that

$$\int_0^{t_2}\phi(t_2,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds - \int_0^{t_1}\phi(t_1,s)\max_{[0,r(s)]}\bigl|x(\tau)\bigr|\,ds \geqslant 0.$$

In fact, notice

$$\begin{aligned}
&\int_0^{t_2} \phi(t_2,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds - \int_0^{t_1} \phi(t_1,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds \\
&\quad = \int_0^{t_2} \phi(t_2,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds - \int_0^{t_2} \phi(t_1,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds \\
&\qquad + \int_0^{t_2} \phi(t_1,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds - \int_0^{t_1} \phi(t_1,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds \\
&\quad = \int_0^{t_2} \bigl(\phi(t_2,s) - \phi(t_1,s)\bigr) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds + \int_{t_1}^{t_2} \phi(t_1,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds.
\end{aligned}$$

Since $t \to \phi(t,s)$ is nondecreasing, we have that $\phi(t_2,s) \geqslant \phi(t_1,s)$, then

$$\int_0^{t_2} \bigl(\phi(t_2,s) - \phi(t_1,s)\bigr) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds \geqslant 0. \tag{8}$$

On the other hand, as $\phi \geqslant 0$ and $\max_{[0,r(s)]} |x(\tau)| \geqslant 0$, then

$$\int_{t_1}^{t_2} \phi(t_1,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds \geqslant 0. \tag{9}$$

Finally, (8) and (9) imply

$$\int_0^{t_2} \phi(t_2,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds - \int_0^{t_1} \phi(t_1,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds \geqslant 0.$$

This together with (7) yields

$$\begin{aligned}
&\bigl|(Ax)(t_2) - (Ax)(t_1)\bigr| - \bigl((Ax)(t_2) - (Ax)(t_1)\bigr) \\
&\quad \leqslant \Bigl[\bigl|(Tx)(t_2) - (Tx)(t_1)\bigr| - \bigl((Tx)(t_2) - (Tx)(t_1)\bigr)\Bigr] \int_0^{t_2} \phi(t_2,s) \max_{[0,r(s)]} \bigl|x(\tau)\bigr|\,ds \\
&\quad \leqslant \|\phi\| \cdot r_0 \cdot i(Tx).
\end{aligned}$$

Therefore,

$$i(Ax) \leqslant \|\phi\| \cdot r_0 \cdot i(Tx),$$

consequently,

$$i(AX) \leqslant \|\phi\| \cdot r_0 \cdot i(TX). \tag{10}$$

Finally, combining (6) and (10), we get

$$\mu(AX) = w_0(AX) + i(AX) \leqslant \|\phi\| \cdot r_0 \cdot \mu(TX) \leqslant \|\phi\| \cdot r_0 \cdot Q \cdot \mu(X).$$

Since $\|\phi\| \cdot r_0 \cdot Q < 1$ (assumption (vi)), Theorem 1 guarantees the existence of a solution of (2). $\square$

4. Examples

In this section we present examples where existence can be established using Theorem 2.

Example 1. Consider

$$x(t) = t^2 + \frac{1}{2e} \int_0^t \frac{e^t}{1+s^2} \cdot \max_{[0,r(s)]} \left|x(\tau)\right| ds. \tag{11}$$

Let $a(t) = t^2$. This function satisfies assumption (i) and $\|a\| = 1$. In this case $\phi(t,s) = \frac{e^t}{1+s^2}$ which satisfies assumption (ii) and $\|\phi\| = e$. Let $r : I \to I$ be given by $r(s) = \sqrt{s}$ and it satisfies assumption (iii). Let $(Tx)(t) = \frac{1}{2e}$ and this operator satisfies (iv) and (v) with $c = \frac{1}{2e}$, $d = 0$ and $Q = 0$.

In this case the first inequality of assumption (vi) has the form

$$1 + \frac{1}{2e} \cdot e \cdot r \leqslant r$$

and it admits $r_0 = 2$ as a positive solution. Moreover, as $Q = 0$, $Q\|\phi\|r_0 < 1$.

Theorem 2 guarantees that (11) has a nondecreasing solution.

Example 2. Consider the integral equation

$$x(t) = t^3 + \frac{1}{\alpha} x(t) \int_0^t \ln(1 + \sqrt{t+s}) \max_{[0,\sqrt{s}]} \left|x(\tau)\right| ds,$$

where $\alpha > 0$.

In this example $a(t) = t^3$ and this function verifies assumption (i) and $\|a\| = 1$. Moreover, $\phi(t,s) = \ln(1+\sqrt{t+s})$ satisfies (ii) and $\|\phi\| = \ln(1+\sqrt{2})$. The function r is defined by $r(s) = \sqrt{s}$ and satisfies hypothesis (iii). The operator T is defined by $(Tx)(t) = \frac{1}{\alpha}x(t)$ and satisfies (iv) with $Q = \frac{1}{\alpha}$ and $c = 0$ and $d = \frac{1}{\alpha}$. In this case the first inequality of assumption (vi) has the form

$$1 + \frac{1}{\alpha} r^2 \ln(1+\sqrt{2}) \leqslant r.$$

This inequality admits to

$$r_0 = \frac{\alpha - \sqrt{\alpha^2 - 4\alpha \ln(1+\sqrt{2})}}{2\ln(1+\sqrt{2})}$$

as a positive solution if $\alpha > 4 \cdot \ln(1+\sqrt{2})$. Moreover, as

$$\frac{1}{\alpha}\ln(1+\sqrt{2}) \cdot \frac{\alpha - \sqrt{\alpha^2 - 4\alpha\ln(1+\sqrt{2})}}{2\ln(1+\sqrt{2})} < \frac{1}{2} < 1,$$

Theorem 2 guarantees that our equation has a nondecreasing solution.

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