On the Regularization of Index 2
Differential-Algebraic Equations

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Boundary value problems for linear differential-algebraic equations (DAEs) with
time-varying coefficients $A(t)x'(t) + B(t)x(t) = q(t)$ tractable with index 2 are
considered. These DAEs contain differentiation problems and lead, therefore, to
essentially ill-posed problems. We show that a parametrization proposed by März
is a regularization in the sense of Tikhonov. Convergence rates for noisy data
are derived. Moreover, for the so-called pencil regularization, analogous results
are derived in the case of a time-independent nullspace $N(A(t))$.

1. INTRODUCTION

In recent years, considerable effort has been spent in the investigation of
numerical methods for solving differential-algebraic equations (DAEs). Such systems arise naturally in many applications. While in the past the
simulation of electrical networks was in the centre of attention, currently
the solution of models describing dynamical processes with constraints
(e.g., constrained systems of rigid bodies and chemical reactions subject to
balance invariants) is the main stimulation of discussion. In the present
paper we are only concerned with linear DAEs

$$A(t)x'(t) + B(t)x(t) = q(t), \quad t \in [a, b]$$

subject to the boundary conditions

$$D_a x(a) + D_b x(b) = y.$$  (1.2)

Here, $x$ is a vector-valued real function, $x(t) \in \mathbb{R}^m$. $A$ is a continuous function whose values are $m \times m$ matrices. For every $t \in [a, b]$, the nullspace of
$A(t)$ is assumed to be nontrivial. Under this assumption, (1.1) is a coupled system of differential and algebraic equations. Furthermore, we assume that the coefficients $\{A, B\}$ belong to the class $\mathcal{N} := \{\{\bar{A}, \bar{B}\} | \bar{A}, \bar{B} \in C([a, b], B(\mathbb{R}^m)); N(\bar{A}) \text{ is smooth}\}$. Thereby, the nullspace $N(A)$ is said to be smooth iff there is a matrix function $Q \in C^1([a, b], B(\mathbb{R}^m))$ so that, for each $t \in [a, b]$, $Q(t)$ projects $\mathbb{R}^m$ onto $N(A(t))$.

Note that a smooth nullspace $N(A)$ has constant dimension on $[a, b]$ and $A(t)$ has constant rank. Denote $P := I - Q$. We introduce

$$H^1_A(a, b) := \{x \in L^2(a, b)^m | Px \in H^1(a, b)^m\},$$

$$\|x\|_A^2 = \|x\|^2 + \|(Px)\|^2, \quad x \in H^1_A(a, b).$$

Here, $H^1(a, b)$ is the usual Sobolev space (cf., e.g., [1]) and $\|\cdot\|$ denotes the norm in $L^2(a, b)^m$. Then $(H^1_A(a, b), \|\cdot\|_A)$ is a Hilbert space. Neither $H^1_A(a, b)$ nor the topology induced by the norm $\|\cdot\|_A$ depends on the choice of the projector function $Q$ [10]. $H^1_A(a, b)$ is continuously imbedded into $H^1(a, b)^m$. Define the operator $S: H^1_A(a, b) \rightarrow L^2(a, b)^m$ by

$$Sx = A((Px)' - P'x) + Bx, \quad x \in H^1_A(a, b).$$

$S$ is a continuous linear operator. Taking into account the identity $x' = (Px)' + ((I - P)x)'$ it can be seen that $Sx = q, x \in H^1_A(a, b)$ is a more precise formulation of (1.1). Let

$$B_0 = B - AP',$$

$$A_1 = A + B_0Q,$$

$$B_1 = B_0P.$$

**Definition 1** [8]. The DAE (1.1) is said to be transferable iff $\{A, B\} \in \mathcal{N}$ and $A_1(t)$ is nonsingular for every $t \in [a, b]$.

This definition is independent of the special choice of $Q$ [8]. If (1.1) is transferable, $S$ is surjective and Fredholmian. Using appropriate boundary conditions (1.2) the problem $Sx = q$, (1.2) is well posed in Hadamard's sense [10]. The transferable DAEs are well understood. For transferable DAEs, suitably modified numerical ODE-methods work well [7, 8].

Nontransferable DAEs are essentially more complex. Tractability with index 2 characterizes an important class of nontransferable DAEs. Especially, some equations modelling chemical reactions subject to balance invariants or constrained mechanical systems of rigid bodies belong to this class [5, 6, 13].
**Definition 2** [15]. Let $Q_1(t)$ be a projector function onto $N(A_1(t))$. Define

$$A_2 = A_1 + B_1 Q_1.$$ 

The DAE (1.1) is said to be tractable with index 2 iff $\{A, B\} \in \mathcal{N}$, $A_1(t)$ is singular for every $t \in [a, b]$, and $A_2(t)$ is nonsingular for every $t \in [a, b]$.

Again, tractability with index 2 is independent of the special choice of the projector functions $Q$ and $Q_1$ [15]. In [8, 14] the notion of tractability with index 2 is defined in another way. But it turns out that both definitions are equivalent [15]. Therefore, for a detailed discussion, we refer to [8, 14]. Let us only mention here that DAEs (1.1) having the global index 2 [7] are tractable with index 2.

In Definition 2 there is no need for $A_1$ to have constant rank. But later on we assume that $Q_1$ is even continuously differentiable which implies that $A_1$ has constant rank.

**Example** [7].

$$A(t) = \begin{pmatrix} 0 & 0 \\ 1 & \eta t \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 \\ 0 \\ \eta t \\ 1 + \eta \end{pmatrix}, \eta \in \mathbb{R}$$

(1.3)

The DAE (1.1) with the coefficients (1.3) is tractable with index 2. The solution of (1.1) is

$$x_1 = q_1 - \eta t(q_2 - q'_1),$$

$$x_2 = q_2 - q'_1.$$ 

(1.4)

Consequently, we have a differentiation problem rather than an ordinary differential equation. But differentiation problems are known to be ill posed [9]. For (1.3), a projection onto $N(A(t))$ is given by

$$Q(t) = \begin{pmatrix} 0 & -\eta t \\ 0 & 1 \end{pmatrix}.$$ 

yielding $H^1_d(a, b) = \{x = (x_1, x_2) \in L^2(a, b)^2 \mid x_1 + \eta tx_2 \in H^1_d(a, b)\}$. This shows that $x$ does not depend continuously on the data $q$ in the $(H^1_d(a, b), L^2(a, b)^2)$-setting.

The behaviour demonstrated above characterizes so-called higher-index equations (i.e., index greater than or equal to 2). One can show that, if (1.1) is tractable with index 2, the range of $S$ is a dense, nonclosed subset of $L^2(a, b)^m$. Thus $S$ is not a Fredholm map and all boundary-value problems become essentially ill posed. Therefore, we look for regularization.
methods. In this paper we consider two parametrizations of (1.1). We show that these parametrizations lead to regularization methods in the sense of Tikhonov [18]. In [2, 3, 4] parametrizations of (1.1) into regular implicit differential equations

\[(A + \varepsilon B) x' + Bx = q\]  

are discussed for some special DAEs. Equation (1.5) is called the "pencil regularization" of (1.1). Note that, in the case of a variable nullspace \(N(t) := N(A(t))\), the local pencil \((A(t), B(t))\) may become singular even if (1.1) is tractable with index 2. Then, the differential equation is no more regular; i.e., this parametrization fails.

\textbf{Example.} Let (1.1) be given with the coefficients (1.3). If \(\eta = -1\), we obtain

\[A + \varepsilon B = \begin{pmatrix} \varepsilon & -\varepsilon t \\ -1 & -t \end{pmatrix}\]

which is singular for all \(t\). It is easy to show that (1.5) is tractable with index 2 for all \(\varepsilon\).

Therefore, we consider (1.5) for DAEs with time-independent nullspace \(N(t)\) only.

A further parametrization of (1.1) is proposed by März [14]. She considers the equation

\[(A + \varepsilon BP) x' + (B + \varepsilon BPP') x = q.\]

This parametrization aims at obtaining transferable DAEs. In [14] it is shown that, under weak assumptions, (1.6) is a transferable DAE for sufficiently small \(\varepsilon > 0\) for all transferable DAEs and all DAEs being tractable with index 2. From the viewpoint of numerical methods, (1.6) seems to be preferable. Whereas (1.5) leads to singularly perturbed boundary-value problems in any case (if it works well theoretically), (1.6) is, for transferable DAEs, a regular perturbation only. A detailed discussion of the relative merits of both parametrizations is contained in [12].

In [11] we proved convergence properties of (1.6) for a special DAE which arises in many practical problems. Among other things, [12] contains a convergence proof of (1.5) for the same problem. It turns out that the essential ingredients used in [11] can be carried over to the general case using more sophisticated arguments.

An alternative way to treat higher-index DAEs numerically is the immediate application of finite difference methods to (1.1). In [13, 16] the convergence of the BDF is studied. It turns out that this method becomes
unstable. This fact is not surprising in the context of the ill posedness of (1.1). But, for certain DAEs, the instability is not very severe. Whereas, for stable methods, the norms of the inverse discrete operators are bounded if the stepsize $h$ tends to zero, these norms grow polynomially in $h^{-1}$ if $h$ tends to zero for some higher index DAEs. Sometimes this behaviour is called weak instability. If the order of consistency of a method is large enough, convergence can be obtained even if the method is weakly unstable. For certain DAE's being tractable with index 2 the BDF is only weakly unstable and converges with the same order as for explicit ordinary differential equations [13, 16]. In our simple example (1.3) the implicit Euler method is weakly unstable and convergent for $\eta \geq -\frac{1}{2}$ but unstable and not convergent for $\eta < -\frac{1}{2}$. For $\eta = -1$, the method is even not feasible.

For Banach spaces $X$ and $Y$, $B(X, Y)$ denotes the Banach space of all bounded linear operators defined on $X$ mapping into $Y$. Moreover, $B(X) := B(X, X)$. For $T \in B(X, Y)$, $R(T)$ denotes the range of $T$.

2. **Main Theorems**

Let (1.1) be tractable with index 2. Let $P_1 = I - Q_1$. Then $Q_{1,s} := Q_1 A_2^{-1} B_0 P$ is also a projector function onto $N(A_1)$ [15].

**Assumption (P).** $Q_{1,s}, PP_1 A_2^{-1} B_0 P \in C^1([a, b], B(\mathbb{R}^m))$, where $P_1 = I - Q_{1,s}$.

In particular, $Q_{1,s} \in C^1([a, b], B(\mathbb{R}^m))$ for DAEs having the global index 2 [15]. Note that (P) implies $A_1$ to have constant rank.

We choose in the following $Q_1 = Q_{1,s}$. Now we give a precise formulation of the boundary condition (1.2). For ordinary differential equations, it is possible to impose boundary conditions for all components of the solution $x$; i.e., $\dim R(D_a, D_b) = m$ should hold in order to obtain appropriate boundary conditions. When considering transferable DAEs it is only possible to formulate boundary conditions for the $Px$-component of $x$ since the $Qx$-components are uniquely determined via algebraic relations [10]. Thus, for transferable DAEs, $D_a = D_a P(a)$, $D_b = D_b P(b)$, and $\dim R(_{-P(A)} D_a P(a) - D_b P(b)) = m$ is necessary. If the DAE is tractable with index 2, the situation changes again. Now, $x$ consists of three components of different types in general. Some components are determined by differential equations, some by algebraic relations, and others by differentiation problems. Boundary conditions are only allowed for components given by differential equations. This consideration leads to

**Assumption (BC).** Let $D_a, D_b \in B(\mathbb{R}^m)$ such that $D_a = D_a PP_1(a)$, $D_b = $
Moreover, suppose that the homogeneous boundary-value problem
\[ w' + (PP_1 A_2^{-1} B_0 P - (PP_1)' ) w = 0, \]
\[ D_\alpha w(a) + D_\beta w(b) = 0, \quad (I - PP_1 ) w(a) = 0 \]
(2.1)
has only the trivial solution.

EXAMPLE. Consider the problem with the coefficients (1.3). The solution (1.4) is uniquely determined without any boundary conditions. Choosing \( Q = (0 \quad -i^m) \) we obtain \( PP_1 = 0 \). Hence, (BC) leads to \( D_a = D_b = 0 \) and (2.1) reads \( w' = 0, \ w(a) = 0 \); i.e., no boundary conditions are allowed.

Throughout the paper we assume that (P) and (BC) hold. Denote \( M = R(D_a, D_b) \subseteq \mathbb{R}^m \). Define
\[ T_x := \left( \begin{array}{c} Sx \\ D_\alpha x(a) + D_\beta x(b) \end{array} \right), \quad x \in H^1_a(a, b). \]
Recall \( S = A(Px)' + B_0 x \) with \( B_0 = B - AP' \).

THEOREM 1. (i) \( R(S) = \{ q \in L^2(a, b)^m \mid PQ_1 A_2^{-1} q \in H^1(a, b)^m \} \).

(ii) \( T \in B(H^1_a(a, b), L^2(a, b)^m \times \mathbb{R}^m) \) is injective and \( R(T) = R(S) \times M \).

This theorem is in another form already contained in [15]. Since \( PQ_1 \neq 0 \), \( R(S) \) is a dense, nonclosed subset of \( L^2(a, b)^m \). Therefore, the problem \( T_x = (q, \gamma) \) is ill posed.

A simple calculation shows that (1.6) is equivalent to \( S_\epsilon x = q \) with \( S_\epsilon \) given by
\[ S_\epsilon x = (A + \epsilon B_0 P)(Px)' + B_0 x, \quad x \in H^1_a(a, b). \]
(2.2)
The nullspace of \( S_\epsilon \) has a greater dimension than the nullspace of \( S \). In order to obtain an injective operator we must add, besides (1.2), additional boundary conditions. We choose this condition to be
\[ PQ_1 x(a) = PQ_1 A_2^{-1} q(a). \]
(2.3)
This condition is in some sense natural since every solution of (1.1) fulfills (2.3). Denote \( L = R(PQ_1 (a)) \). Now, let
\[ T_\epsilon x := \left( \begin{array}{c} S_\epsilon x \\ D_\alpha x(a) + D_\beta x(b) \\ PQ_1 x(a) \end{array} \right), \quad x \in H^1_a(a, b). \]
(2.4)
THEOREM 2. For sufficiently small $\varepsilon > 0$, (2.2) is a transferable DAE. Moreover, $T_\varepsilon \in B(H^1_a(a, b), L^2(a, b)^m \times M \times L)$ is bijective.

The first part of Theorem 2 is already proved in [14] under slightly modified assumptions. Here, we give a new proof. Let now $\varepsilon_* > 0$ be fixed such that the assertions of Theorem 2 are true for $\varepsilon \in (0, \varepsilon_*)$. Let $(q, \gamma) \in R(T)$. Then there exist solutions of the equations $Tx = (q, \gamma)$ and $T_\varepsilon x = (q, \gamma, PQ_1 A^{-1}_2 q(a))$. Moreover, by Banach's theorem, the latter problem is well posed in the $(H^1_a(a, b), L^2(a, b)^m \times M \times L)$-setting. We obtain

THEOREM 3. For $(q, \gamma) \in R(T)$ and $\varepsilon \in (0, \varepsilon_*)$, let $x$ and $x_\varepsilon$ be the solutions of the equations $Tx = (q, \gamma)$ and $T_\varepsilon x = (q, \gamma, PQ_1 A^{-1}_2 q(a))$, respectively. Then:

(i) $\|x_\varepsilon - x\|_A \to 0$ for $\varepsilon \to 0$.

(ii) If $PQ_1, PP_1 \in C^2([a, b], B(\mathbb{R}^m))$ and $PQ_1 A^{-1}_2 q \in H^2(a, b)^m$, $\|x_\varepsilon - x\|_A = O(\varepsilon^{1/2})$.

Unfortunately, the order of convergence $O(\varepsilon)$ is only obtained under rather restrictive assumptions; namely, if the hypotheses of Theorem 3(ii) are true and, moreover, $(PQ_1)' P Q_1 A^{-1}_2 q(a) + (PQ_1)' P P_1 x(a) - (PQ_1 A^{-1}_2 q)'(a) = 0$, then $\|x_\varepsilon - x\|_A = O(\varepsilon)$. However, in the special case considered in [11, 12] this assumption leads to a simple condition on the right-hand side $q$. On the other hand, a rigorous analysis of the error shows that the had convergence order is due to the components $Qx$ and $Qx_\varepsilon$, respectively. In the proofs below we obtain $\|Px - Px_\varepsilon\| = O(\varepsilon)$ under the hypotheses of Theorem 3(i). In applications $Px$ often represents the state variables or a control, whereas $Qx$ is a kind of Lagrange multiplier (cf., e.g., [5, 13]). If we are only interested in the state or control variable, the convergence order in $L^2(a, b)^m$ is the best we could expect.

THEOREM 4. Let $(q, \gamma, \beta) \in L^2(a, b)^m \times M \times L$ such that $(q, \gamma) \notin R(T)$. Then $\|x_\varepsilon\|_A \to \infty$ for $\varepsilon \to 0$.

Theorems 3 and 4 show that the parametrization (2.2) has a convergence behaviour which is very similar to that of well-known regularization methods for integral equations of the first kind, e.g., Tikhonov regularization (cf.[9, Chap. II]). Let us now consider the case where, instead of the exact data $(q, \gamma, PQ_1 A^{-1}_2 q(a))$ of (2.4) only perturbed data $(q^\delta, \gamma^\delta, \beta^\delta) \in L^2(a, b)^m \times M \times L$ are available, where

$$\|q - q^\delta\| \leq \delta_1, \quad |\gamma - \gamma^\delta| \leq \delta_2, \quad |PQ_1 A^{-1}_2 q(a) - \beta^\delta| \leq \delta_3,$$

$$\delta = (\delta_1, \delta_2, \delta_3). \quad (2.5)$$
Here, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$. By $x_\delta$ we denote the solution of $T_\delta x = (q_\delta, \gamma_\delta, \beta_\delta)$. Then the following theorem shows how the regularization parameter $\varepsilon$ must be chosen in dependence on $\delta$ in order to obtain the convergence of $x_{\varepsilon(\delta)}$ toward $x$ for $\delta \to 0$.

**Theorem 5.** Let $(q, \gamma) \in R(T)$ and $(q_\delta, \gamma_\delta, \beta_\delta) \in L^2(a, b)^m \times M \times L$ with $\delta = (\delta_1, \delta_2, \delta_3) \in \mathbb{R}^3$ such that (2.5) holds. If $\varepsilon = \varepsilon(\delta)$ is chosen such that

$$\lim_{\delta \to 0} \varepsilon(\delta) = 0, \quad \lim_{\delta \to 0} \varepsilon(\delta) = 0, \quad \lim_{\delta \to 0} \varepsilon(\delta) = 0,$$

(2.6)

then

$$\| x_{\varepsilon(\delta)} - x \|_A \to 0 \quad \text{for} \quad \delta \to 0.$$  

If the exact data possess more smoothness properties, order results are obtained.

**Theorem 6.** Let the hypotheses of Theorem 3(ii) be fulfilled. Let $(q_\delta, \gamma_\delta, \beta_\delta) \in L^2(a, b)^m \times M \times L$ such that (2.5) holds. If $\varepsilon = \max(\delta_1^{1/3}, \delta_2, \delta_3)$, then

$$\| x_{\varepsilon} - x \|_A = O(\delta_1^{1/3} + \delta_2 + \delta_3^{1/2}).$$

Now we turn to the pencil regularization (1.5). As mentioned earlier we consider only the case when $A(t)$ has constant nullspace. For time-dependent $N(A(t))$, (1.5) may fail to give correctly posed problems. Since (1.5) is an implicit ordinary differential equation for $\varepsilon > 0$ sufficiently small we must impose boundary conditions for the whole vector function $x$. Using (1.2) and (2.3) we obtain boundary conditions for $Px$, only. As opposed to (2.2), a natural condition like (2.3) could not be derived from (1.1)–(1.2) for $Qx$ since this component of $x$ is only contained in $L^2(a, b)^m$. Even if all functions involved are sufficiently smooth, the expression for $Qx(a)$ is so expensive (cf. Eq. (3.1) below) that it cannot be evaluated practically. Therefore, we impose the initial condition

$$Qx(a) = v_a$$  

(2.7)

with $v_a \in N := N(A(a))$ fixed. Now, (1.9) is equivalent to $S_\varepsilon x = q$, where

$$S_\varepsilon x := A(Px)' + \varepsilon Bx' + Bx, \quad x \in H^1(a, b)^m.$$  

(2.8)

Define

$$T_\varepsilon x := \begin{pmatrix} S_\varepsilon x \\ D_a x(a) + D_{\beta} x(b) \\ PQ_x x(a) \\ Qx(a) \end{pmatrix}.$$
THEOREM 7. Let $N(A(t)) = \text{const.}$ For sufficiently small $\varepsilon > 0$, (2.8) is an implicit ODE. Moreover, $T_\varepsilon \in B(H^1(a,b)^m, L^2(a,b)^m \times M \times L \times N)$ is bijective.

Obviously, $H^1(a,b)^m$ is continuously imbedded into $H^1_a(a,b)$. Let now $\varepsilon_{**} > 0$ be fixed such that the assertion of Theorem 7 is true for $\varepsilon \in (0, \varepsilon_{**})$. Let $(q, \gamma) \in R(T)$ and $v_a \in N$. Then there exist unique solutions of the equations $Tx = (q, \gamma)$ and $T_\varepsilon x = (q, \gamma, PQ_1 A^{-1}_2 q(a), v_a)$. Again, the latter equation is well posed in the $(H^1(a,b)^m, L^2(a,b)^m \times M \times L \times N)$-setting and, consequently, also in the $(H^1_a(a,b)^m, L^2(a,b)^m \times M \times L \times N)$-setting.

THEOREM 8. Let $N(A(t)) = \text{const.}$ for $(q, \gamma) \in R(T)$, $v_a \in N$, and $\varepsilon \in (0, \varepsilon_{**})$, and let $x$ and $\tilde{x}_\varepsilon$ be solutions of the equations $Tx = (q, \gamma)$ and $T_\varepsilon x = (q, \gamma, PQ_1 A^{-1}_2 q(a), v_a)$, respectively. Then:

(i) $\|\tilde{x}_\varepsilon - x\|_A \to 0$ for $\varepsilon \to 0$.

(ii) If $PQ_1, PP_1 \in C^2([a,b], B(\mathbb{R}^m))$, $Qx \in H^1(a,b)^m$, and $PQ_1 A^{-1}_2 q \in H^2(a,b)^m$, $\|\tilde{x}_\varepsilon - x\|_A = O(\varepsilon^{1/2})$.

For the pencil regularization, the convergence rate cannot be expected to be better than $O(\varepsilon^{1/2})$. This is due to the initial layer which is introduced by (2.7). But the remark concerning the convergence properties of the state variables $Px$ following Theorem 3 is valid here, too. Analogously to the parametrization (2.2) we obtain the following theorems.

THEOREM 9. Let $N(A(t)) = \text{const.}$ Let $(q, \gamma, \beta, v_a) \in L^2(a,b)^m \times M \times L \times N$ such that $(q, \gamma) \notin R(T)$. Then $\|\tilde{x}_\varepsilon\|_A \to \infty$.

THEOREM 10. Let $N(A(t)) = \text{const.}$ Let $(q, \gamma) \in R(T)$, $v_a \in N$, and $(q^*, \gamma^*, \beta^*) \in L^2(a,b)^m \times M \times L$ with $\delta = (\delta_1, \delta_2, \delta_3) \in \mathbb{R}^3$ such that (2.5) holds. Denote by $\tilde{x}_\varepsilon$ the solution of $T_\varepsilon x = (q^*, \gamma^*, \beta^*, v_a)$. If $\varepsilon = \varepsilon(\delta)$ is chosen such that (2.6) holds, then

$$\|\tilde{x}_\varepsilon - x\|_A \to 0 \quad \text{for} \quad \delta \to 0.$$ 

THEOREM 11. Let the hypotheses of Theorem 8(ii) be fulfilled. Let $(q^*, \gamma^*, \beta^*) \in L^2(a,b)^m \times M \times L$ and $v_a \in N$ such that (2.5) holds. If $\varepsilon = \max(\delta_2^{1/2}, \delta_3)$, then $\|\tilde{x}_\varepsilon - x\|_A = O(\delta_1^{1/2} + \delta_2 + \delta_3^{1/2})$.

3. PROOFS

It is convenient to use a splitting of $x$ into $x = Qx + PP_1 x + PQ_1 x$. This leads to equivalent formulations of the various boundary-value problems. In the following we always use $Q_1 = Q_{1,s} = Q_1 A^{-1}_2 B_0 P$. 

**Lemma 1.** The boundary-value problem $Tx = (q, y)$ is equivalent to the system

$$ y = PQ_1 A_2^{-1} q, $$

$$ z' + (PP_1 A_2^{-1} B_0 P - (PP_1)' ) z - (PP_1)' y = PP_1 A_2^{-1} q, $$

$$ v = QP_1 A_2^{-1} q - QP_1 A_2^{-1} B_0 z + (QP_1 P)' z - QP_1 P y', $$

$$ D_a z(a) + D_b z(b) = y, $$

$$ (I - PP_1) z(a) = 0, $$

where $y = PQ_1 x, z = PP_1 x, v = Qx.$

A proof of this lemma can be found in [15]. Moreover, it can be proved along the same lines as Lemma 2 below.

**Proof of Theorem 1.** The theorem is an immediate consequence of Lemma 1.

**Lemma 2.** The boundary-value problem $T_\gamma x = (q, y, PQ_1 A_2^{-1} q(a))$ is equivalent to the system

$$ \varepsilon y' + (I - \varepsilon (PQ_1)') y - \varepsilon (PQ_1)' PP_1 z = PQ_1 A_2^{-1} q, $$

$$ (I + \varepsilon PP_1 A_2^{-1} B_0 P) z' + ((P_1 A_2^{-1} B_0 P - (PP_1)') z $$

$$ - ((PP_1)' + \varepsilon (PP_1 A_2^{-1} B_0 P)') y = PP_1 A_2^{-1} q, $$

$$ v = QP_1 A_2^{-1} q - QP_1 A_2^{-1} B_0 z + (QP_1 P)' z - QP_1 P y' $$

$$ - \varepsilon QP_1 A_2^{-1} B_0 (z' + y'), $$

$$ D_a z(a) + D_b z(b) = y, $$

$$ (I - PP_1) z(a) = 0, $$

$$ y(a) = PQ_1 A_2^{-1} q(a), $$

where $y = PQ_1 x, z = PP_1 x, v = Qx.$

**Proof.** Let $x$ be a solution of $T_\gamma x = (q, y, PQ_1 A_2^{-1} q(a)).$ From (2.2) we obtain

$$ A_2 \{ P_1 P( Px)' + P_1 Qx + Q_1 x \} + B_0 PP_1 x + B_0 P(Px)' = q. $$

Using $Q_1 = Q_{1,\gamma} = Q_1 A_2^{-1} B_0 P$ we have $Q_1 Q = 0.$ Therefore, $PP_1 P = PP_1,$ $PP_1 Q = 0,$ $QP_1 Q = Q,$ and $P_1 PP_1 = PP_1.$ Hence, (3.3) is equivalent to

$$ PP_1 (Px)' + \varepsilon PP_1 A_2^{-1} B_0 PP_1 x + \varepsilon PP_1 A_2^{-1} B_0 P( Px)' = PP_1 A_2^{-1} q, $$

$$ QP_1 P( Px)' + Qx + QP_1 A_2^{-1} B_0 PP_1 x + \varepsilon QP_1 A_2^{-1} B_0 P( Px)' = QP_1 A_2^{-1} q, $$

$$ Q_1 x + \varepsilon Q_1 (Px)' = Q_1 A_2^{-1} q. $$

(3.4)
Furthermore, it holds that

\[ PP_1(Px)' = PP_1(PP_1x + PQ_1x)' \]

\[ = (PP_1x)' - (PP_1)' PP_1x = (PP_1)' PQ_1x, \]

\[ PQ_1(PP_1x)' = -(PQ_1)' PP_1x, \]

\[ QP_1 P(PP_1x)' = -(QP_1 P)' PP_1x, \]

\[ A_2^{-1} B_0 PQ_1 = A_2^{-1}(A_1 + B_0 PQ_1) Q_1 \]

\[ = A_2^{-1}(A_1 + B_0 Q_1) Q_1 = Q_1, \]

\[ PP_1 A_2^{-1} B_0 P(PQ_1x)' = -(PP_1 A_2^{-1} B_0 P)' PQ_1x. \]

After multiplying the third equation of (3.4) by \( P \) we obtain, therefore, the first three equations of (3.2). The boundary and initial conditions follow immediately from the definition of \( y, z, \) and (BC).

Conversely, let \( (y, z, u) \) be a solution of (3.2). Multiply the first equation by \( PQ_1 \) and subtract the result from it. Using \( y' = (PQ_1 y)' + ((I - PQ_1) y)' \), we obtain

\[ \varepsilon((I - PQ_1) y' + (I - PQ_1) y + \varepsilon PQ_1(PQ_1)' y - \varepsilon(PQ_1)' PP_1 z \]

\[ + \varepsilon PQ_1(PQ_1)' PP_1 z = 0. \]

But \( PQ_1(PQ_1)' = (PQ_1)' - (PQ_1)' PQ_1 = (PQ_1)'(I - PQ_1); \) hence

\[ \varepsilon((I - PQ_1) y' + (I + \varepsilon(PQ_1)')(I - PQ_1) y = 0. \]

Since, by (3.2) \( (I - PQ_1) y(a) = 0, \)

\[ PQ_1 y = y \]  \hspace{1cm} (3.5)

is true. Analogously, we proceed with the second equation of (3.2). Multiplication by \( PP_1 \) and subtraction imply

\[ ((I - PP_1) z)' + (PP_1)' (I - PP_1) z = 0. \]

The boundary conditions of (3.2) give

\[ PP_1 z = z. \]  \hspace{1cm} (3.6)

Since \( (QP_1 P)' z = (QP_1 P)' PP_1 z = (QP_1 PP_1 z)' - QP_1 Pz' = -QP_1 Pz' \) we obtain from the third equation of (3.2)

\[ Qv = v. \]  \hspace{1cm} (3.7)
Let \( x = y + z + u \). Using the properties (3.5)–(3.7) it is a straightforward calculation to show that indeed \((A + \varepsilon B_0 P)(P x)' + B_0 x = q\) holds.

The system (3.2) clearly shows that the parametrization (2.2) has a singular perturbation behaviour. Our task is, therefore, to study the convergence properties of the solutions of some singular perturbation problems. There is much known about these properties (cf., e.g., [17, 19]). In our setting, however, we must show convergence under very weak smoothness assumptions on the right-hand side.

Let us introduce the following abbreviations:

\[
\begin{align*}
V_{11} &= (P Q_1)', \\
V_{12} &= (P Q_1)' P P_1, \\
V_{21} &= (P P_1 A_2^{-1} B_0 P)', \\
U_{11}(\varepsilon) &= -I + \varepsilon V_{11}, \\
U_{21}(\varepsilon) &= (P P_1)' + \varepsilon V_{21}, \\
U_{22} &= (P P_1)' - P P_1 A_2^{-1} B_0 P, \\
C &= P P_1 A_2^{-1} B_0 P.
\end{align*}
\]

Then, the differential equations of (3.2) are

\[
\begin{align*}
\varepsilon y' &= U_{11}(\varepsilon) y + \varepsilon V_{12} z + P Q_1 A_2^{-1} q, \\
(I + \varepsilon C) z' &= U_{21}(\varepsilon) y + U_{22} z + P P_1 A_2^{-1} q,
\end{align*}
\]

whereas the first two equations of (3.1) read

\[
\begin{align*}
0 &= -y + P Q_1 A_2^{-1} q, \\
z' &= U_{21}(0) y + U_{22} z + P P_1 A_2^{-1} q.
\end{align*}
\]

In [11] we proved the following lemma. It is crucial for the proof of our main results.

**Lemma 3.** Let \( I_\varepsilon w \) be defined by

\[
I_\varepsilon w(t) = \int_a^t \frac{1}{\varepsilon} \exp \left( -\frac{1}{\varepsilon} (t-s) \right) w(s) \, ds, \quad t \in [a, b]
\]

for \( w \in L^2(a, b)^m \) and \( \varepsilon > 0 \). Then \( I_\varepsilon \in B(L^2(a, b)^m) \) and \( \| I_\varepsilon w - w \| = o(1) \) for \( \varepsilon \to 0 \). If \( w \in H^1(a, b)^m \), \( \| I_\varepsilon w - w \| = O(\varepsilon^{1/2}) \). Moreover, there is a constant \( c \in \mathbb{R} \) such that \( \| I_\varepsilon \| \leq c \) for all \( \varepsilon > 0 \).
Lemma 4. There is an $\varepsilon_*>0$ such that, for all $\varepsilon\in (0, \varepsilon_*)$, $f, g \in L^2(a, b)^m$, $y \in M$, and for all $\beta \in L$, the system (3.8) with the right-hand sides $PQ_1A_2^{-1}q$ and $PP_1A_2^{-1}q$ replaced by $f$ and $g$, respectively, has exactly one solution $(y_\varepsilon, z_\varepsilon)$ subject to the boundary conditions $D_\alpha z(a) + D_\beta z(b) = \gamma$, $(I-PP_1)z(a) = 0$, $y(a) = \beta$. Moreover, there exists a $c \in \mathbb{R}$ such that

\[
\| y \| \leq c(\varepsilon |\gamma| + \varepsilon^{1/2} |\beta| + \| f \| + \varepsilon \| g \|),
\]

\[
\| z_\varepsilon \| \leq c(\varepsilon |\gamma| + \varepsilon^{1/2} |\beta| + \| f \| + \| g \|).
\]

The proof of Lemma 4 can be given using standard arguments from the singular perturbation theory (cf., e.g., [17, 19]).

Proof of Theorem 2. Lemmas 2 and 4 imply the bijectivity of $T_\varepsilon$. It remains to show that $(A + \varepsilon B_0 P)(P x)' + B_0 x = q$ is a transferable DAE. According to Definition 1 we consider the pair $\{A_*, B_*\}$ with $A_* = A + \varepsilon B_0 P$, $B_* = B_0$. We show that, for $\varepsilon > 0$ sufficiently small and $t \in [a, b]$,

(i) $N(A(t)) = N(A_*(t))$ and

(ii) $A_*(t) + B_*(t) Q(t)$ is nonsingular.

ad (i): Trivially, $N(A(t)) \subseteq N(A_*(t))$. Let now $z \in N(A_*(t))$ for $t \in [a, b]$ fixed. Then, $0 = A_* z = (A_2 + \varepsilon B_0 P - B_0 Q - B_0 PQ_1) z$. Multiplying this equation by $Q_1 A_2^{-1}$ and regarding $Q_1 = Q_{1, z} = Q_1 A_2^{-1} B_0 P$, $A_2^{-1} B_0 Q = Q$ leads to $\varepsilon Q_1 z = 0$. On the other hand, by multiplying with $PA_2^{-1}$ we obtain $Pz + \varepsilon PA_2^{-1} B_0 P z = 0$. Hence, for sufficiently small $\varepsilon > 0$, $Pz = 0$. But this is equivalent to $z \in N(A(t))$. Therefore, $N(A(t)) = N(A_*(t))$ for every $t \in [a, b]$.

as (ii): Let $t \in [a, b]$ fixed and $\varepsilon > 0$. We omit the argument $t$. Let $z \in \mathbb{R}^m$ with $(A_* + B_*Q) z = 0$. This is equivalent to $(A_2 + \varepsilon B_0 P - B_0 PQ_1) z = 0$. Multiplication by $Q_1 A_2^{-1}$ yields $\varepsilon Q_1 z = 0$. On the other hand, by multiplying with $PA_2^{-1}$ we obtain $(I + \varepsilon PA_2^{-1}) Pz = 0$. This implies $Pz = 0$ for $\varepsilon$ sufficiently small. Hence, $A_2 z = 0$. Since $A_2$ is nonsingular, $z = 0$ follows.

Proof of Theorem 3. Set $f = PQ_1A_2^{-1}q$, $g = PP_1A_2^{-1}q$, $\beta = PQ_1A_2^{-1}q(a)$. Define $y = PQ_1x$, $z = PP_1x$, $v = Qx$, $y_\varepsilon = PQ_1x_\varepsilon$, $z_\varepsilon = PP_1x_\varepsilon$, $v_\varepsilon = Px_\varepsilon$. By (3.8) and (3.9),

\[
y_\varepsilon(t) - y(t) = \exp \left( -\frac{1}{\varepsilon} (t - a) \right) \beta + \int_a^t \exp \left( -\frac{1}{\varepsilon} (t - s) \right) \frac{1}{\varepsilon} f(s) \, ds
\]

\[
+ \varepsilon I_\varepsilon V_{11} y_\varepsilon(t) + \varepsilon I_\varepsilon V_{12} z_\varepsilon(t) - f(t)
\]

\[
= \varepsilon I_\varepsilon (V_{11} y_\varepsilon + V_{12} z_\varepsilon - f')(t).
\]

(3.11)
$I_\varepsilon$ is given by (3.10). Because of Lemma 4 and $f' \in L^2(a, b)^m$ we obtain

$$\| y_\varepsilon - y \| = O(\varepsilon). \quad (3.12)$$

By (3.12), the equation $(I + \varepsilon C) z_\varepsilon' = U_{21}(\varepsilon) y_\varepsilon + U_{22} z_\varepsilon + PP_1 A_2^{-1} q$ is a regular $O(\varepsilon)$-perturbation of $z' = U_{21}(0) y + U_{22} z + PP_1 A_2^{-1} q$. A usual argument gives now the estimate

$$\| z_\varepsilon' - z' \|^2 + \| z_\varepsilon - z \|^2 = O(\varepsilon^2). \quad (3.13)$$

Differentiating (3.11) yields

$$y_\varepsilon' - y' = I_\varepsilon f' - f' + V_{11} y_\varepsilon - I_\varepsilon V_{11} y_\varepsilon + V_{12} z_\varepsilon - I_\varepsilon V_{12} z_\varepsilon. \quad (3.14)$$

Again using Lemma 3 we may estimate

$$\| V_{11} y_\varepsilon - I_\varepsilon V_{11} y_\varepsilon \|$$

$$\leq \| V_{11} (y_\varepsilon - y) \| + \| V_{11} y - I_\varepsilon V_{11} y \| + \| I_\varepsilon V_{11} (y - y_\varepsilon) \| = o(1).$$

Analogously, $\| V_{12} z_\varepsilon - I_\varepsilon V_{12} z \| = o(1)$. This gives finally

$$\| y_\varepsilon' - y' \| = o(1). \quad (3.15)$$

If $PP_1, PQ_1,$ and $f$ are twice differentiable, then we may partially integrate (3.14) to obtain

$$y_\varepsilon' - y' = \varepsilon I_\varepsilon (-f'' + (V_{11} y_\varepsilon)' + (V_{12} z_\varepsilon)')$$

$$+ \exp \left( -\frac{1}{\varepsilon} (\cdot - a) \right) (-f'(a) + V_{11} y_\varepsilon(a) + V_{12} z_\varepsilon(a)). \quad (3.16)$$

Consequently,

$$\| y_\varepsilon' - y' \| = O(\varepsilon^{1/2}). \quad (3.17)$$

Now, (3.1), (3.2), (3.13), and (3.15) imply

$$\| v_\varepsilon - v \| = \| (QP_1 A_2^{-1} B_0 P - (QP_1 P)')(z_\varepsilon - z) + QP_1 P(y_\varepsilon' - y')$$

$$+ \varepsilon QP_1 A_2^{-1} B_0 P(z_\varepsilon' + y_\varepsilon') \|$$

$$= O(\| y_\varepsilon' - y' \| + \varepsilon). \quad (3.18)$$

Since $\| x \|^2_\mathcal{A} = \| y + z + v \|^2 + \| y' + z' \|^2$, the assertions follows from (3.12), (3.13), (3.15), (3.17), and (3.18).

Equation (3.16) shows that the convergence rate $O(\varepsilon)$ can be obtained if the boundary term $V_{11} y(a) + V_{12} z(a) - f'(a)$ vanishes.
Proof of Theorem 4. Let us assume that there is a sequence \((\varepsilon_n)\) such that \(\varepsilon_n \to 0\) and \(\|x_{e_n}\|_A \leq c\). Then there is a subsequence (again denoted by \((\varepsilon_n)\)) and an \(x \in H^1_0(a, b)\) such that \(x_{e_n} \to x\) (in \(H^1_0(a, b)\)). \((-\to\) denotes the weak convergence.) Moreover, we have \(Tx_{e_n} \to Tx\). On the other hand,

\[
Tx_{e_n} = \left( D_a x_{e_n}(a) + D_b x_{e_n}(b) \right) = \left( q - \varepsilon_n BP (P x_{e_n})' \right) \to \left( q \right).
\]

Hence, \((q, \gamma) \in R(T)\) in contradiction to the assumption.

Now we consider the case of noisy data. The next lemma is an immediate consequence of Lemma 4.

**Lemma 5.** Let \((q, \gamma) \in R(T)\) and \((q^\delta, \gamma^\delta, \beta^\delta) \in L^2(a, b)^m \times M \times L\). Denote by \(x_e\) and \(x_e^\delta\) solutions of \(T_e x = (q, \gamma, PQ_1 A_2^{-1} q(a))\) and \(T_e x = (q^\delta, \gamma^\delta, \beta^\delta)\), respectively. Then, the estimate

\[
\|x_e^\delta - x_e\|_A \leq c(|\gamma^\delta - \gamma| + \varepsilon^{-1/2} \|\beta^\delta - PQ_1 A_2^{-1} q(a)\| + \varepsilon \|q^\delta - q\|)
\]

holds for sufficiently small \(\varepsilon > 0\).

Theorems 5 and 6 follow now immediately from Theorem 3, Lemma 5, and the estimate \(\|x_e^\delta - x_e\|_A \leq \|x_e^\delta - x_e\|_A + \|x_e - x\|_A\). Now we turn to the pencil regularization (2.8). Again, a representation theorem is essential. For the time-independent nullspace \(N = N(A(t))\) we choose \(Q\) to be constant, too.

**Lemma 6.** Let \(N(A(t)) = \text{const}\). The boundary-value problem \(T_e x = (q, \gamma, PQ_1 A_2^{-1} q(a), v_a)\) is equivalent to the system

\[
e\gamma' + (I - \varepsilon(PQ_1)') y - \varepsilon(PQ_1)' PP_1 z = PQ_1 A_2^{-1} q,
\]

\[
(I + \varepsilon PP_1 A_2^{-1} BP) z' + (PP_1 A_2^{-1} BP - (PP_1)') z - ((PP_1)' + \varepsilon(PP_1 A_2^{-1} BP)') y = PP_1 A_2^{-1} q,
\]

\[
ev' + (I - \varepsilon(QP_1)') v = QP_1 A_2^{-1} q - QP_1 A_2^{-1} Bz + (QP_1 P)' z - \varepsilon QP_1 A_2^{-1} BP(z' + y').
\]

\[
D_a z(a) + D_b z(b) = \gamma, \quad (I - PP_1) z(a) = 0,
\]

\[
y(a) = PQ_1 A_2^{-1} q(a), \quad v(a) = v_a,
\]

where \(y = PQ_1 x, z = PP_1 x, v = Qx\).

Moreover, if \((\tilde{y}_e, \tilde{z}_e, \tilde{v}_e)\) and \((y_e, z_e, v_e)\) are solutions of (3.24) and (3.2), respectively, then \(\tilde{y}_e = y_e, \tilde{z}_e = z_e,\) and \(\varepsilon \tilde{v}_e' + (I - \varepsilon (QP_1)') \tilde{v}_e = v_e\).

The proof follows the lines of that of Lemma 2.
Proof of Theorem 7. Since (1.1) is tractable with index 2, the matrix pencil \((A(t), B(t))\) is regular for every \(t \in [a, b]\) (cf. [8, 15]). Moreover, one can compute the following representation of \((A + \varepsilon B)^{-1}\) for sufficiently small \(\varepsilon > 0\):
\[
(A + \varepsilon B) u = w \quad \text{iff} \quad u = u_1 + u_2 + u_3,
\]
\[
u_1 = \frac{1}{\varepsilon} A_2^{-1} w,
\]
\[
u_2 = (I + \varepsilon PA_2^{-1} B)^{-1} PA_2^{-1} (w - (A + \varepsilon B) u_1),
\]
\[
u_3 = \frac{1}{\varepsilon} A_2^{-1} (z - (A + \varepsilon B) (u_1 + u_2)).
\]
Hence, \((A + \varepsilon B)^{-1} \in C([a, b], B(\mathbb{R}^n))\). But then \(\tilde{T}_\varepsilon\) is a Fredholm operator. Applying Lemmas 6 and 4 completes the proof.

Proof of Theorem 8. Using the notations of Lemmas 3 and 6 we may write
\[
\bar{v}_\varepsilon(t) = \exp \left( -\frac{1}{\varepsilon} (\cdot - a) \right) \bar{v}_0 + I_\varepsilon \bar{v}_\varepsilon + \varepsilon I_\varepsilon (Q P_1)' \bar{v}_\varepsilon.
\] (3.19)
By Theorem 3, \(\|v_\varepsilon\| \leq c\) for \(\varepsilon > 0\) sufficiently small. Hence, (3.19) implies
\[
\|\bar{v}_\varepsilon\| \leq c(1 + \varepsilon \|\bar{v}_\varepsilon\|)
\]
for \(\varepsilon\) sufficiently small and some \(c\) independent of \(\varepsilon\). Therefore, \(\|v_\varepsilon\| = O(1)\) and (3.19) yield
\[
\|\bar{v}_\varepsilon - v_\varepsilon\| = O(\varepsilon^{1/2} \|v_\varepsilon\| + \|I_\varepsilon v_\varepsilon - v_\varepsilon\| + \varepsilon).
\]
Now, assertion (i) follows from Theorem 3 and Lemmas 3, 6. If the assumptions of part (ii) are fulfilled, \(\|I_\varepsilon v_\varepsilon - v_\varepsilon\| \leq \|I_\varepsilon v_\varepsilon - I_\varepsilon v\| + \|I_\varepsilon v - v\| + \|v - v_\varepsilon\| = O(\varepsilon^{1/2})\) by Theorem 3 and Lemma 3. Here, \(v = Q x\) where \(x\) is the solution of \(T x = (q, \gamma)\). Hence, (ii) holds.

Proof of Theorem 9. Let us assume that there is a sequence \((\varepsilon_n)\) such that \(\varepsilon_n \to 0\) and \(\|\bar{x}_{\varepsilon_n}\|_A \leq c\). Denote by \(x_{\varepsilon_n}\) a solution of \(T_{\varepsilon_n} x = (q, \gamma, \beta)\). With the notations of Lemma 6, \(y_{\varepsilon_n} = \bar{y}_{\varepsilon_n}\) and \(z_{\varepsilon_n} = \bar{z}_{\varepsilon_n}\) are bounded in \(H^1(a, b)^m\). Hence, by (3.2), \(v_{\varepsilon_n}\) is also bounded. But then \(\|x_{\varepsilon_n}\|_A \leq c\) in contradiction to Theorem 4.

In order to prove Theorems 10 and 11 we need an estimate of \(\|\bar{x}_{\delta} - \bar{x}_\varepsilon\|_A\). Indeed, one can easily prove
Lemma 7. Let $N(A(t)) = \text{const.}$ Let $(q, \gamma) \in R(T)$, $v_\omega \in N$, and $(q^\delta, \gamma^\delta, \beta^\delta) \in L^2(a, b)^m \times M \times L$. Denote by $\tilde{x}_\omega$ and $\tilde{x}_\omega^\delta$ solutions of $\bar{T} x = (q, \gamma, P Q^1 A_2^{-1} q(a), v_\omega)$ and $T^e x = (q^\delta, \gamma^\delta, \beta^\delta, v_\omega)$, respectively. Then the estimate
\[
\| x^\delta_e - x_\omega \|_A \leq c (| \gamma^\delta - \gamma | + \varepsilon^{-1/2} | \beta^\delta - P Q^1 A_2^{-1} q(a) | + \varepsilon^{-1} \| q^\delta - q \|)
\]
holds for sufficiently small $\varepsilon > 0$.

Proof. Regarding Lemma 6 we must only estimate the difference $e_\varepsilon := \tilde{v}^\delta - \tilde{v}_\varepsilon$, where $\tilde{v}_\varepsilon^\delta = Q \tilde{x}_\omega^\delta$, $\tilde{v}_\varepsilon = Q \tilde{x}_\omega$. Again using Lemma 6 we obtain
\[
e\varepsilon' + (I - \varepsilon (PQ^1)' ) e_\varepsilon = r_\varepsilon, \quad e_\varepsilon(a) = 0
\]
with
\[
r_\varepsilon = O(\| q^\delta - q \| + \| z^\delta - z_\varepsilon \| + (\| y_\varepsilon^\delta \| - y_\varepsilon \| + \varepsilon (z_\varepsilon^\delta)' - z_\varepsilon' \|). \quad (3.20)
\]
This implies
\[
e_\varepsilon(t) = \int_a^t \exp \left( - \frac{1}{\varepsilon} (t - s) \right) \left( \frac{1}{\varepsilon} r_\varepsilon(s) + (QP^1)' e_\varepsilon(s) \right) ds.
\]
Consequently,
\[
\| e_\varepsilon \| = O(\| r_\varepsilon \|).
\]
Equation (3.20) gives the assertion.

4. Concluding Remark

In this paper we investigated the convergence properties of the parametrization (1.5) and (1.6) in the spirit of regularization methods. Therefore, we preferred to use the Hilbert space $H^1_{\delta}(a, b)$ as the underlying function space. The analysis showed that we really met singular perturbation problems. In this context, the convergence behaviour of the regularized solutions can be further characterized. Indeed, the results of [2, 4] concerning the pencil regularization for some special DAEs indicate clearly the initial layer behaviour. The theorems above show that this initial layer is always present in the nonstate components $Qx$. Moreover, it is often very hard to determine the right initial values (2.3) resp. (2.7) numerically. In order to design numerical methods for solving (1.1)–(1.2) using either parametrization, a rigorous asymptotic analysis should be done.
REFERENCES