Strong Resonance for Some Quasilinear Elliptic Equations

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We study the existence of the weak solutions of the nonlinear boundary value problem

\[- \Delta_p u = \lambda |u|^{p-2} u + g(u) - h(x) \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega,\]

where the nonlinearity \(g\) is of the Landesman–Lazer type. Our sufficient conditions generalize all previously published results about the strong resonance at the first eigenvalue.

Key Words: the \(p\)-Laplacian; resonance at the first eigenvalue; saddle point theorem; Landesman–Lazer-type conditions.

1. INTRODUCTION AND THE STATEMENT OF THE RESULT

In this paper we study the existence of the weak solutions of the boundary value problem

\[- \Delta_p u = \lambda |u|^{p-2} u + g(u) - h(x) \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega,\] \hspace{1cm} (1.1)

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where $\Omega$ is a bounded open set in $\mathbb{R}^N$, $N \geq 1$, $p > 1$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $h \in L^p(\Omega)$ ($p' = \frac{p}{p-1}$). Here, $\Delta_p$ is the $p$-Laplacian, i.e., $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$, and $\lambda_1 = \min \{ \int_{\Omega} |\nabla u|^p \, dx : u \in W^{1,p}_0(\Omega), \int_{\Omega} |u|^p \, dx = 1 \}$ is the first eigenvalue of $-\Delta_p$ on $\Omega$ subject to zero Dirichlet boundary conditions. Recall that $\lambda_1$ is simple, positive, and isolated. Moreover, there exists a unique positive eigenfunction $\varphi_1$ whose norm in $W^{1,p}_0(\Omega)$ is 1 (see [1, 4]). Our paper was motivated by the result in [2] and the generalized form of the Landesman–Lazer conditions considered in [8, 9]. While the semilinear problem is studied in [8, 9], it turns out that a different technique allows us to use these conditions also for the quasilinear problem (1.1) and to generalize the result of [2].

We suppose that
\[
\lim_{|t| \to \infty} \frac{g(t)}{|t|^{p-1}} = 0. \tag{1.2}
\]

Let us define
\[
F(t) = \begin{cases} \frac{p}{p-1} \int_0^t g(s) \, ds - g(t), & t \neq 0, \\ (p-1)g(0), & t = 0, \end{cases}
\]
and set
\[
\underline{F}(-\infty) = \limsup_{t \to -\infty} F(t), \quad \overline{F}(+\infty) = \liminf_{t \to +\infty} F(t),
\]
\[
\underline{F}(+\infty) = \limsup_{t \to +\infty} F(t), \quad \overline{F}(-\infty) = \liminf_{t \to -\infty} F(t).
\]

We suppose also
\[
\underline{F}(-\infty) \int_{\Omega} \varphi_1(x) \, dx < (p-1) \int_{\Omega} h(x) \varphi_1(x) \, dx < \overline{F}(+\infty) \int_{\Omega} \varphi_1(x) \, dx \tag{1.3}
\]
or
\[
\overline{F}(+\infty) \int_{\Omega} \varphi_1(x) \, dx < (p-1) \int_{\Omega} h(x) \varphi_1(x) \, dx < \underline{F}(-\infty) \int_{\Omega} \varphi_1(x) \, dx. \tag{1.3^*}
\]

Our main result is the following.

**Theorem 1.1.** Let $\Omega$ be a smooth bounded domain. Assume (1.2) and (1.3) (or (1.3*)) hold. Then the boundary value problem (1.1) has at least one weak solution $u \in W^{1,p}_0(\Omega)$.
Remark 1.1. There are functions $g$ and $h$ satisfying the assumptions of our theorem and not satisfying the corresponding assumptions in [2]. For example (cf. [9]), let $h = 0$ and let

$$g(t) = \begin{cases} 1 - \exp(-t^4\sin t)\ln(1 + t^2), & t \geq 0, \\ 2\exp(t) - 1, & t \leq 0. \end{cases}$$

It is not difficult to check that $F(-\infty) = -(p - 1)$ and $F(\infty) = (p - 1)$, which implies that $g$ and $h$ satisfy the assumptions of our theorem. On the other hand, $g$ and $h$ do not satisfy the assumptions of [2].

Remark 1.2. Our proof is variational. If (1.3) is satisfied, we show that the energy functional associated with (1.1) has a saddle point geometry while if (1.3*) holds this functional attains its global minimum. That is why we can call the problem (1.1) at resonance above or below the first eigenvalue if (1.3) or (1.3*) holds, respectively.

2. THE PROOF

We prove Theorem 1.1 separately under assumption (1.3) and under assumption (1.3*), respectively. Let us assume first that (1.3) holds.

The main tool is the saddle point theorem (see [7]) in this case.

Let us introduce the action functional $J: W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with (1.1)

$$J(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda_1}{p} \int_\Omega |u|^p \, dx - \int_\Omega G(u) \, dx + \int_\Omega hu \, dx,$$

where

$$G(t) = \int_0^t g(s) \, ds.$$ 

The functional $J$ is continuously Fréchet differentiable at every point $u \in W_0^{1,p}(\Omega)$ and we have

$$\langle J'(u), v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx - \lambda_1 \int_\Omega |u|^{p-2} uv \, dx$$

$$- \int_\Omega g(u)v \, dx + \int_\Omega hv \, dx$$

for every $u, v \in W_0^{1,p}(\Omega)$ (here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$). It is clear that the critical points of $J$ correspond to the weak solutions of (1.1).
Lemma 2.1. Let us assume (1.2) and (1.3). Then the functional $J$ satisfies the Palais–Smale condition; that is, if $(u_n)$ is a sequence of functions in $W_0^{1,p}(\Omega)$ such that $J(u_n)$ is bounded in $\mathbb{R}$ and $J'(u_n) \to 0$ in $W^{-1,p'}(\Omega)$, then $(u_n)$ has a subsequence that is strongly convergent in $W_0^{1,p}(\Omega)$.

Proof. The proof will be performed in two steps. We prove first that $(u_n)$ is bounded in $W_0^{1,p}(\Omega)$. Suppose, by contradiction, that $\|u_n\| \to \infty$ and define $v_n := u_n/\|u_n\|$ (here $\|u\| = (\int_\Omega |\nabla u|^p \, dx)^{1/p}$). Due to the reflexivity of $W_0^{1,p}(\Omega)$ and the compact embedding $W_0^{1,p}(\Omega) \subset L^p(\Omega)$, there exists $v_0 \in W_0^{1,p}(\Omega)$ such that (up to subsequences)

\[
v_n \rightharpoonup v_0 \quad \text{(i.e., weakly) in } W_0^{1,p}(\Omega),
\]

\[
v_n \rightarrow v_0 \quad \text{(i.e., strongly) in } L^p(\Omega).
\]

The boundedness of $J(u_n)$ and $\|u_n\| \to \infty$ imply

\[
J(u_n) = \frac{1}{p} \int_\Omega |\nabla v_n|^p \, dx - \frac{\lambda_1}{p} \int_\Omega |v_n|^p \, dx
\]

\[-\int_\Omega \frac{G(u_n)}{\|u_n\|^p} \, dx + \int_\Omega h \frac{u_n}{\|u_n\|^p} \, dx \to 0.
\]

(2.1)

Clearly,

\[
\int_\Omega h \frac{u_n}{\|u_n\|^p} \, dx \to 0,
\]

(2.2)

and because of (1.2) also

\[
\int_\Omega \frac{G(u_n)}{\|u_n\|^p} \, dx \to 0.
\]

(2.3)

It follows from (2.1)–(2.3) that

\[
\int_\Omega |\nabla v_n|^p \, dx - \lambda_1 \int_\Omega |v_n|^p \, dx \to 0,
\]

which together with the variational characterization of $\lambda_1$ and the lower semicontinuity of the norm yield

\[
\lambda_1 \int_\Omega |v_0|^p \, dx \leq \int_\Omega |\nabla v_0|^p \, dx \leq \liminf_{n \to \infty} \int_\Omega |\nabla v_n|^p \, dx
\]

\[= \lim_{n \to \infty} \int_\Omega |\nabla v_n|^p \, dx = \lambda_1 \int_\Omega |v_0|^p \, dx.
\]
Hence \( \|v_n\| \to \|v_0\| \) and \( v_0 \) is the first eigenfunction. Moreover, \( v_n \to v_0 \) in \( W_0^{1,p}(\Omega) \). Since \( \|v_0\| = 1 \), there is either \( v_0 = \varphi_1 \) or \( v_0 = -\varphi_1 \). Let us assume that \( v_0 = \varphi_1 > 0 \) in \( \Omega \) (the other case is treated similarly). The boundedness of \( J(u_n), J'(u_n) \to 0 \) in \( W^{-1,p}(\Omega) \), and \( \|u_n\| \to \infty \) imply

\[
0 = \lim_{n \to \infty} \frac{\langle J'(u_n), u_n \rangle - pJ(u_n)\rangle}{\|u_n\|} = \lim_{n \to \infty} \left[ \int_\Omega \frac{pG(u_n) - g(u_n)u_n}{\|u_n\|} \, dx - (p - 1) \int_\Omega h \frac{u_n}{\|u_n\|} \, dx \right].
\]

Hence

\[
\lim_{n \to \infty} \int_\Omega \frac{pG(u_n) - g(u_n)u_n}{\|u_n\|} \, dx = \lim_{n \to \infty} \int_\Omega F(u_n) \frac{u_n}{\|u_n\|} \, dx = (p - 1) \int_\Omega h \varphi_1 \, dx. \tag{2.4}
\]

It follows from \( \beta \) that \( F(\infty) > -\infty \) and \( F(-\infty) < +\infty \). For arbitrary \( \varepsilon > 0 \) set

\[
\begin{align*}
e = \begin{cases} F(\infty) - \varepsilon & \text{if } F(\infty) \in \mathbb{R}, \\ 1 & \text{if } F(\infty) = +\infty, \end{cases} \\
\end{align*}
\]

\[
\begin{align*}
d = \begin{cases} F(-\infty) + \varepsilon & \text{if } F(-\infty) \in \mathbb{R}, \\ 1 & \text{if } F(-\infty) = -\infty. \end{cases}
\end{align*}
\]

Then for any \( \varepsilon > 0 \) there exists \( K > 0 \) such that

\[
F(t) \geq c_\varepsilon \quad \text{for any } t > K, \quad F(t) \leq d_\varepsilon \quad \text{for any } t < -K. \tag{2.5}
\]

On the other hand, the continuity of \( F \) on \( \mathbb{R} \) implies that for any \( K > 0 \) there exists \( c(K) > 0 \) such that

\[
|F(t)| \leq c(K) \quad \text{for any } t \in [-K, K]. \tag{2.6}
\]

Let us choose \( \varepsilon > 0 \) and consider the corresponding \( K > 0 \) and \( c(K) > 0 \) given by \( \beta \) and \( \gamma \), respectively. Set

\[
\int_\Omega F(u_n) v_n \, dx = A_{K,n} + B_{K,n} + C_{K,n},
\]
where

\[
A_{K,n} = \int_{|u_n(x)| \leq K} F(u_n) v_n \, dx,
\]

\[
B_{K,n} = \int_{u_n(x) > K} F(u_n) v_n \, dx,
\]

\[
C_{K,n} = \int_{u_n(x) < -K} F(u_n) v_n \, dx.
\]

Before estimating these integrals we claim that

\[
\lim_{n \to \infty} \text{meas}\{x \in \Omega; u_n(x) \leq K\} = 0. \tag{2.7}
\]

Since \(\Omega\) is smooth, it follows from the strong maximum principle (see, e.g., [6]) that for any \(\delta > 0\) there exists \(\eta(\delta) > 0\) such that

\[
\text{meas}\{x \in \Omega; \varphi_1(x) \leq \eta(\delta)\} < \delta. \tag{2.8}
\]

It follows from \(v_n \to \varphi_1\) in \(L^p(\Omega)\) and [5, Theorems 12.3–12.5] that (passing to a subsequence if necessary) for any \(\delta > 0\) and for any \(\eta > 0\) there exist \(M \subset \Omega\) and \(n_0 \in \mathbb{N}\) such that for any \(x \in M\) and any \(n \geq n_0\) we have

\[
\text{meas}(\Omega \setminus M) < \delta \quad \text{and} \quad |\psi_n(x) - \varphi_1(x)| < \eta. \tag{2.9}
\]

Let \(\delta > 0\) be arbitrary, let \(\eta = \eta(\delta) > 0\) be given by (2.8), and let \(M \subset \Omega, n_0 \in \mathbb{N}\) correspond to \(\delta > 0, \eta > 0\) via (2.9). Then

\[
\text{meas}\{x \in \Omega; u_n(x) \leq K\}
\]

\[
= \text{meas}\{x \in \Omega \setminus M; u_n(x) \leq K\}
\]

\[
+ \text{meas}\{x \in M; \varphi_1(x) \leq \eta(\delta) \text{ and } u_n(x) \leq K\}
\]

\[
+ \text{meas}\{x \in M; \varphi_1(x) > \eta(\delta) \text{ and } u_n(x) \leq K\}
\]

\[
\leq \delta + \delta + \text{meas}\{x \in M; \varphi_1(x) > \eta(\delta) \text{ and } u_n(x) \leq K\}.
\]

Since, for \(x \in M, \varphi_1(x) > \eta(\delta)\) we have (due to (2.9))

\[
0 < \varphi_1(x) - \eta(\delta) < \frac{u_n(x)}{\|u_n\|} \quad \text{for any } n \geq n_0,
\]

and \(\|u_n\| \to \infty, \{x \in M; \varphi_1(x) > \eta(\delta) \text{ and } u_n(x) \leq K\} = \emptyset \) for \(n\) large enough (i.e., for \(n \geq n_0\) and \(K/\|u_n\| < \varphi_1(x) - \eta(\delta))\). This proves (2.7).
We are now ready to estimate $A_{K,n}$, $B_{K,n}$, and $C_{K,n}$

$$|A_{K,n}| \leq \int_{|u_n(x)| \leq K} |F(u_n)| \cdot \frac{|u_n|}{||u_n||} \, dx \leq \frac{c(K) \cdot K \cdot \text{meas}(\Omega)}{||u_n||} \to 0,$$

$$B_{K,n} \geq c_\varepsilon \int_{u_n(x) > K} v_n \, dx = c_\varepsilon \left( \int_{\Omega} v_n \, dx - \int_{u_n(x) \leq K} v_n \, dx \right) \to c_\varepsilon \int_{\Omega} \varphi_1 \, dx,$$

due to (2.7), and

$$C_{K,n} \geq d_\varepsilon \int_{u_n(x) < -K} v_n \, dx \to 0,$$

using again (2.7). Hence

$$\liminf_{n \to \infty} \int_{\Omega} F(u_n) v_n \, dx = \liminf_{n \to \infty} (A_{K,n} + B_{K,n} + C_{K,n}) \geq c_\varepsilon \int_{\Omega} \varphi_1 \, dx,$$

which together with (2.4) implies that

$$(p - 1) \int_{\Omega} h \varphi_1 \, dx \geq c_\varepsilon \int_{\Omega} \varphi_1 \, dx$$

for any $\varepsilon \searrow 0$, i.e.,

$$(p - 1) \int_{\Omega} h \varphi_1 \, dx \geq \frac{F(+\infty)}{\lambda_1} \int_{\Omega} \varphi_1 \, dx,$$

contradicting (1.3). Thus $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

The rest of the proof follows the proof of Lemma 2.1 in [2]. We include it for completeness. Passing to subsequences, we can assume that there is $u \in W_0^{1,p}(\Omega)$ such that

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega), \quad u_n \to u \quad \text{in } L^p(\Omega).$$

Since

$$0 \leftarrow \langle J'(u_n), u_n - u \rangle = \int |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u) \, dx$$

$$- \lambda_1 \int \frac{|u_n|^{p-2}u_n(u_n - u)}{u_n} \, dx$$

$$- \int g(u_n)(u_n - u) \, dx + \int h(u_n - u) \, dx$$
and the last three terms approach 0, we have

\[ \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) \, dx \to 0, \]
i.e.,

\[ \int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) \nabla (u_n - u) \, dx \to 0. \]

It follows from Hölder’s inequality that

\[ 0 \leftarrow \int_{\Omega} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) (\nabla u_n - \nabla u) \, dx \]
\[ \geq \left( \|u_n\|^{p-1} - \|u\|^{p-1} \right) (\|u_n\| - \|u\|) \geq 0, \]

which implies \( \|u_n\| \to \|u\| \). The uniform convexity of \( W^{1,p}_0(\Omega) \) then yields \( u_n \to u \). The proof is complete.

**Lemma 2.2.** Let us assume (1.2) and (1.3). Then the functional \( J \) has a saddle point geometry; i.e., if we split \( W^{1,p}_0(\Omega) \) as the direct sum of \( E_1 = \langle \varphi_1 \rangle \) and \( E_2 = \{ u \in W^{1,p}_0(\Omega) \colon 0 = \int_\Omega u \varphi_1^{p-1} \, dx \} \), then

i) \( J(a \varphi_1) \to -\infty \) if \( |a| \to +\infty \),

ii) \( J \) is bounded from below on \( E_2 \).

**Proof.** Assume, via contradiction, that there is a sequence \( \{a_n\} \subset \mathbb{R} \), \( |a_n| \to +\infty \), and \( J(a_n \varphi_1) \geq c \) for any \( n \in \mathbb{N} \) and some \( c \in \mathbb{R} \). We can assume that \( a_n \to +\infty \) without loss of generality. Then

\[ \liminf_{n \to +\infty} \frac{J(a_n \varphi_1)}{a_n} \geq 0 \quad (2.10) \]

and

\[ \frac{J(a_n \varphi_1)}{a_n} = -\int_{\Omega} \frac{G(a_n \varphi_1)}{a_n} \, dx + \int_{\Omega} h \varphi_1 \, dx. \quad (2.11) \]

Let \( \varepsilon > 0 \) be arbitrary and let \( K > 0 \) be given by (2.5). Since

\[ \left( -\frac{G(\tau)}{\tau^p} \right)' = \frac{F(\tau)}{\tau^p} \geq \frac{c_\varepsilon}{\tau^p} = \left( -\frac{1}{p-1} \cdot \frac{c_\varepsilon}{\tau^{p-1}} \right)' \]
for any $\tau > K$, we have
\[
\int_{t}^{s} \left( - \frac{G(\tau)}{\tau^{p}} \right) \, d\tau \geq \int_{t}^{s} \left( - \frac{1}{p-1} \cdot \frac{c_{e}}{\tau^{p-1}} \right) \, d\tau,
\]
i.e.,
\[
\frac{G(t)}{t^{p}} - \frac{G(s)}{s^{p}} \geq \frac{c_{e}}{p-1} \left( \frac{1}{t^{p-1}} - \frac{1}{s^{p-1}} \right)
\]  \hspace{1cm} (2.12)
for any $s > t > K$. The assumption (1.2) implies that $G(s)/s^{p} \to 0$ as $s \to +\infty$. So, passing to the limit for $s \to +\infty$ in (2.12), we obtain
\[
\frac{G(t)}{t} \geq \frac{c_{e}}{p-1}
\]  \hspace{1cm} (2.13)
for any $t > K$. Since $e > 0$ is arbitrary, it follows from (2.13) that
\[
\liminf_{t \to +\infty} \frac{G(t)}{t} \geq \frac{1}{p-1} F(+\infty).
\]
Now, it follows from this, Fatou’s lemma, (2.10), and (2.11) that
\[
0 \leq \liminf_{n \to \infty} \frac{J(a_{n}\varphi_{1})}{a_{n}} \leq \limsup_{n \to \infty} \frac{J(a_{n}\varphi_{1})}{a_{n}} = \limsup_{n \to \infty} \left( - \int_{\Omega} \frac{G(a_{n}\varphi_{1})}{a_{n}} \, dx + \int_{\Omega} h\varphi_{1} \, dx \right)
\]
\[
= \limsup_{n \to \infty} \left( - \int_{\Omega} \frac{G(a_{n}\varphi_{1})}{a_{n}} \, dx \right) + \int_{\Omega} h\varphi_{1} \, dx
\]
\[
= - \liminf_{n \to \infty} \left( \int_{\Omega} \frac{G(a_{n}\varphi_{1})}{a_{n}} \, dx \right) + \int_{\Omega} h\varphi_{1} \, dx
\]
\[
\leq - \int_{\Omega} \liminf_{n \to \infty} \frac{G(a_{n}\varphi_{1})}{a_{n}\varphi_{1}} \varphi_{1} \, dx + \int_{\Omega} h\varphi_{1} \, dx
\]
\[
\leq - \frac{1}{p-1} F(+\infty) \int_{\Omega} \varphi_{1} \, dx + \int_{\Omega} h\varphi_{1} \, dx,
\]
which contradicts (1.3). Thus (i) is proved.
To prove (ii) we use the fact that the standard compactness argument yields the existence of \( \bar{\lambda} > \lambda_1 > 0 \) such that, for any \( u \in E_2 \),

\[
\int_{\Omega} |\nabla u|^p \, dx \geq \bar{\lambda} \int_{\Omega} |u|^p \, dx
\]

(cf. [2]). Let \( u \in E_2 \). Then (using the assumption (1.2)) for any \( \varepsilon > 0 \) there exists \( c > 0 \) such that

\[
J(u) = \frac{1}{p} \left[ \int_{\Omega} |\nabla u|^p \, dx - \lambda_1 \int_{\Omega} |u|^p \, dx \right] - \int_{\Omega} G(u) \, dx + \int_{\Omega} hu \, dx
\]

\[
\geq \frac{1}{p} \left[ \|u\|^p - \frac{\lambda_1}{\bar{\lambda}} \|u\|^p \right] - \int_{\Omega} G(u) \, dx + \int_{\Omega} hu \, dx
\]

\[
\geq \frac{1}{p} \|u\|^p \left( 1 - \frac{\lambda_1}{\bar{\lambda}} \right) - c \int_{\Omega} |u| \, dx - \frac{\varepsilon}{p} \int_{\Omega} |u|^p \, dx - \|h\|_{L^p} \|u\|_{L^q}
\]

\[
\geq \frac{1}{p} \|u\|^p \left( 1 - \frac{\lambda_1}{\bar{\lambda}} \right) - cl\|u\| - \frac{\varepsilon k^p}{p} \|u\|^p - \|h\|_{L^p} k\|u\|
\]

\[
= \frac{1}{p} \|u\|^p \left( 1 - \frac{\lambda_1}{\bar{\lambda}} - \varepsilon k^p \right) - (cl + k\|h\|_{L^p})\|u\|,
\]

(2.14)

where \( k \) and \( l \) are the constants of Sobolev embeddings \( W_0^{1,p}(\Omega) \subset L^p(\Omega) \) and \( W_0^{1,k}(\Omega) \subset L^k(\Omega) \), respectively. Since \( \varepsilon > 0 \) can be taken so small that

\[
1 - \frac{\lambda_1}{\bar{\lambda}} - \varepsilon k^p > 0,
\]

it follows from (2.14) that \( J \) is bounded below on \( E_2 \). The proof of Lemma 2.2 is complete.

The proof of Theorem 1.1 now follows from Lemmas 2.1 and 2.2 and the saddle point theorem (see [7]).

Let us assume now that instead of (1.3) the condition (1.3*) holds. We prove that \( J \) attains its global minimum in this case.

**Lemma 2.3.** Let us assume (1.2) and (1.3*). Then \( J \) is coercive, i.e.,

\[
\lim_{\|u\| \to \infty} J(u) = \infty.
\]

**Proof.** Let us assume, via contradiction, that \( \|u_n\| \to \infty \) and \( J(u_n) \leq c \) for some \( c \in \mathbb{R} \) as \( n \to \infty \). We can assume without loss of generality that

\[
v'_n := \frac{u_n}{\|u_n\|} \to v_0
\]
for some $v_0 \in W_0^{1,p}(\Omega)$. It follows from (2.2), (2.3), and the weak lower semicontinuity of the norm that
\[
0 = \limsup \frac{c}{\|u_n\|^p} \geq \limsup \frac{J(u_n)}{\|u_n\|^p}
\]
\[
= \limsup \left[ \frac{1}{p} \int_{\Omega} |\nabla v_n|^p \, dx - \frac{\lambda_1}{p} \int_{\Omega} |v_n|^p \, dx - \frac{G(u_n)}{\|u_n\|^p} \right]
\]
\[
= \limsup \left[ \frac{1}{p} \int_{\Omega} |\nabla v_n|^p \, dx - \frac{\lambda_1}{p} \int_{\Omega} |v_n|^p \, dx \right]
\]
\[
\geq \liminf \left[ \frac{1}{p} \int_{\Omega} |\nabla v_n|^p \, dx - \frac{\lambda_1}{p} \int_{\Omega} |v_n|^p \, dx \right] \geq 0,
\]
i.e.,
\[
\int_{\Omega} |\nabla v_n|^p \, dx - \lambda_1 \int_{\Omega} |v_n|^p \, dx \to 0.
\]
Hence $v_0 = \varphi_1$ or $v_0 = -\varphi_1$ (cf. the proof of Lemma 2.1). Assume $v_0 = \varphi_1$ (the other case is treated similarly). It follows from (1.3*) that $F(-\infty) > -\infty$ and $F(+\infty) < +\infty$. For arbitrary $\varepsilon > 0$ set
\[
c_\varepsilon = \begin{cases} 
F(-\infty) - \varepsilon & \text{if } F(-\infty) \in \mathbb{R}, \\
\frac{1}{\varepsilon} & \text{if } F(-\infty) = +\infty,
\end{cases}
\]
\[
d_\varepsilon = \begin{cases} 
F(+\infty) + \varepsilon & \text{if } F(+\infty) \in \mathbb{R}, \\
\frac{1}{\varepsilon} & \text{if } F(+\infty) = -\infty.
\end{cases}
\]
Then for any $\varepsilon > 0$ there exists $K > 0$ such that
\[
F(t) \geq c_\varepsilon \quad \text{for any } t < -K, \quad F(t) \leq d_\varepsilon \quad \text{for any } t > K.
\]
Arguing exactly as in the proof of Lemma 2.2, we derive
\[
\frac{G(t)}{t} \geq \frac{c_\varepsilon}{p - 1} \quad \text{(2.15)}
\]
for any $t < -K$ and
\[
\frac{G(t)}{t} \leq \frac{d_\varepsilon}{p - 1} \quad \text{(2.16)}
\]
for any $t > K$. 
Since \( \|u_n\| \to \infty \), \( J(u_n) \leq c \), and
\[
\frac{J(u_n)}{\|u_n\|} \geq - \int_{\Omega} \frac{G(u_n)}{\|u_n\|} \, dx + \int_{\Omega} \frac{hu_n}{\|u_n\|} \, dx,
\]
we get
\[
\limsup \left( - \int_{\Omega} \frac{G(u_n)}{\|u_n\|} \, dx \right) + \int_{\Omega} h\varphi_1 \, dx
\leq \limsup \frac{J(u_n)}{\|u_n\|} \leq \limsup \frac{c}{\|u_n\|} = 0.
\]

Hence
\[
\int_{\Omega} h\varphi_1 \, dx \leq \liminf \int_{\Omega} \frac{G(u_n)}{\|u_n\|} \, dx \leq \limsup \int_{\Omega} \frac{G(u_n)}{\|u_n\|} \, dx.
\]

Let us split the last integral as follows
\[
\int_{\Omega} \frac{G(u_n)}{\|u_n\|} \, dx = \int_{\|u_n\| \leq K} \frac{G(u_n)}{\|u_n\|} \, dx + \int_{\|u_n\| < -K} \frac{G(u_n)}{\|u_n\|} \, dx
\]
\[
+ \int_{\|u_n\| > K} \frac{G(u_n)}{\|u_n\|} \, dx.
\]
We obtain (taking into account (2.15))
\[
\left| \int_{\|u_n\| \leq K} \frac{G(u_n)}{\|u_n\|} \, dx \right| \leq \max_{s \in [-K, K]} |g(s)| \cdot K \cdot \operatorname{meas}(\Omega) \cdot \frac{1}{\|u_n\|} \to 0,
\]
\[
\int_{\|u_n\| < -K} \frac{G(u_n)}{\|u_n\|} \, dx \leq \frac{c_p}{p-1} \int_{\|u_n\| < -K} \frac{u_n(x)}{\|u_n\|} \, dx \to 0.
\]

So we have (taking into account (2.16))
\[
\int_{\Omega} h\varphi_1 \, dx \leq \limsup \int_{\|u_n\| \leq K} \frac{G(u_n)}{\|u_n\|} \, dx \leq \limsup \int_{\|u_n\| > K} \frac{G(u_n)}{\|u_n\|} \, dx
\]
\[
\leq \limsup \frac{d_p}{p-1} \int_{\|u_n\| > K} \frac{u_n(x)}{\|u_n\|} \, dx = \frac{d_p}{p-1} \int_{\Omega} \varphi_1(x) \, dx.
\]
We obtain from here
\[ \int_{\Omega} h(x) \varphi_i(x) \, dx \leq \frac{1}{p-1} F(q) \int_{\Omega} \varphi_i(x) \, dx, \]
a contradiction with (1.3*). The proof of the lemma is complete.

**Lemma 2.4.** Let us assume (1.2) and (1.3*). Then J satisfies the Palais–Smale condition.

**Proof.** The boundedness of the Palais–Smale sequence follows from Lemma 2.3. The rest is the same as in the proof of Lemma 2.1.

The action functional J is clearly bounded on bounded sets. It follows then from Lemmas 2.3 and 2.4 that J attains its global minimum on \( W_0^{1, p}(\Omega) \) (cf. [10, Corollary 2.5]). This completes the proof of Theorem 1.1.

**REFERENCES**

4. P. Lindqvist, On the equation \( \text{div}(|\nabla u|^{p-2} \nabla u) + \lambda |u|^{p-2} u = 0 \), *Proc. Amer. Math. Soc.* **109** (1990), 157–164.