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Total colorings of equibipartite graphs

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Abstract

The total chromatic number $\chi_T(G)$ of a graph G is the least number of colors needed to color the vertices and the edges of G such that no adjacent or incident pair of elements receive the same color.
A simple graph G is called *type 1* if $\chi_T(G) = \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. In this paper we prove the following conjecture of Chen et al.: An $(n-2)$-regular equibipartite graph $K_{n,n} - E(J)$ is type 1 if and only if J contains a 4-cycle.

Keywords: Total coloring; Latin square

1. Introduction

In this paper, all graphs are finite, simple and undirected. We denote the vertex set, edge set, maximum degree, and order of a graph G by $V(G), E(G)$, $\Delta(G)$, and $|G|$, respectively. The degree of $v \in V(G)$ is denoted by $d_G(v)$. A vertex of degree $\Delta(G)$ is called a *major vertex*, otherwise a *minor vertex*. The *deficiency* $\mathrm{def}(G)$ of a graph G is defined as $\sum_{v \in V(G)} (\Delta(G) - d_G(v))$. The complete equibipartite graph of order $2n$, the cycle of order n and the star of order n are denoted by $K_{n,n}$, C_n and S_n, respectively.

A *total coloring* of a graph G is a mapping π: $V(G) \cup E(G) \to C$ such that no incident or adjacent pair of elements of $V(G) \cup E(G)$ receive the same color, where C is a color set. The *total chromatic number* $\chi_T(G)$ is the least cardinality of C for which G has a total coloring. From the definition of total chromatic number, it is clear that $\chi_T(G) \geqslant \Delta(G) + 1$. In 1965, Behzad [1] and Vizing [8] independently made the following conjecture.

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Total coloring conjecture (TCC). For any graph G, $\chi_T(G) \leqslant \Delta(G)+2$.

This conjecture has been proved for complete graphs, for graphs G having $\Delta(G) \leqslant 5$, for complete r-partite graphs, for graphs G having $\Delta(G) \geqslant \frac{3}{4}|G|$ and for graphs G having $\Delta(G) \geqslant |G|-5$. For details, see [9].

A graph G is called *type i*, if $\chi_T(G)=\Delta(G)+i$, $i \geqslant 1$. Hence, if the TCC holds for certain classes of graphs G, then either G is type 1 or type 2. Many people, including J.C. Bermond, B.L. Chen, A.G. Chetwynd, J.K. Dugdale, H.L. Fu, A.J.W. Hilton, C.A. Rodger and H.P. Yap, have studied the problem of classifying the graphs according to their total chromatic number. For details, also see [9].

The notion of biconformability was introduced by Chetwynd and Hilton [4] for determining the exact total chromatic number of equibipartite graphs. A bipartite graph G with bipartition (A, B) is called *biconformable* if $|A| = |B|$ and G has a $(\Delta(G)+1)$-vertex-coloring $\phi : V(G) \to \{c_1, c_2, \dots, c_{\Delta(G)+1}\}$ such that the following conditions hold:

(1) $\mathrm{def}(G) \geqslant \sum_{i=1}^{\Delta(G)+1} |a_i - b_i|$;

(2) $|V_{<\Delta(G)}(A \backslash A_j)| \geqslant b_j - a_j$ and $|V_{<\Delta(G)}(B \backslash B_j)| \geqslant a_j - b_j$,

where $|V_{<\Delta(G)}(S)|$ is the number of minor vertices in $S \subseteq V(G)$, $A_j = \phi^{-1}(c_j) \cap A$, $B_j = \phi^{-1}(c_j) \cap B$, $a_j = |A_j|$ and $b_j = |B_j|$.

The following lemma is proved in [4].

Lemma 1.1 (Chetwynd and Hilton [4]). *Let G be an equibipartite graph. If G is type* 1, *then G is biconformable.*

The following conjecture is also made in [4].

Conjecture 1. Let G be a bipartite graph with $\Delta(G) \geqslant \frac{3}{14}(|G|+1)$. Then G is type 2 if and only if G contains an induced equibipartite subgraph H with $\Delta(H)=\Delta(G)$ which is not biconformable.

However, counterexamples to this conjecture have been found for equibipartite graphs of order $2n$ with $\Delta(G)=n-1$ by Chen et al. [2]. In [2], the authors also studied the problem of classifying equibipartite graphs with $\Delta(G)=n-2$. In this paper, we prove the following theorem which is a conjecture made in [3].

Main Theorem. *Suppose J is a subgraph of $K_{n,n}$ such that $G=K_{n,n}-E(J)$ is $(n-2)$-regular, where $n \geqslant 5$. Then G is type* 1 *if and only if J contains a* 4*-cycle.*

2. Some preliminary results

In this section, we shall introduce some results in completing partial latin squares and extending partial edge colorings, which will be used in our proof of the main theorem. The part of embedding partial latin squares is abstracted from [4].

A *partial latin square* of side n defined on n distinct symbols is an $n \times n$ array in which some cells may be empty and each of the non-empty cells contains exactly one symbol such that no symbol occurs more than once in any row or in any column. It is a (complete) *latin square* if there is no empty cell. A partial latin square is called *symmetric* if whenever a cell (i,j) contains a symbol σ, the cell (j,i) also contains σ. A partial latin square L is called *unipotent* if all the cells of the main diagonal of L are filled with the same symbol.

A set Q of $n-1$ cells of a (partial) latin square P of side n on symbols $1,2,\ldots,n$ is called a *near transversal excluding* (*row* i, *column* j, *symbol* k) if it satisfies:

(1) each row, except row i, has exactly one cell of Q, and each column, except column j, has exactly one cell of Q;

(2) no two cells of Q contain the same symbol;

(3) symbol k is not filled in any cell of Q.

Note that if P is unipotent and Q contains no main diagonal cell, then Q does not contain the symbol in the main diagonal. Hence, for convenience, in this case, we say that Q excludes (i,j) instead of saying that Q excludes (row i, column j, symbol k). Two sets Q and Q' of cells are said to avoid one another if they have no common cell.

Let P be a unipotent partial latin square of side n consisting of $n-1$ occupied off-diagonal cells which form a near transversal Q. Suppose P has a sequence of occupied off-diagonal cells $(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),\ldots,(\alpha_k,\alpha_1)$. Since each row R_i (resp. column C_j) of P has at most one occupied cell other than cell (i,i) (resp. (j,j)), we know that $\alpha_1,\alpha_2,\ldots,\alpha_k$ are all distinct. We call such a sequence $(\alpha_1,\alpha_2),(\alpha_2,\alpha_3),\ldots,(\alpha_k,\alpha_1)$ of cells a *cycle* $(\alpha_1,\alpha_2,\ldots,\alpha_k)$.

Suppose that P has two cycles $C^1=(\alpha_1,\alpha_2,\ldots,\alpha_r)$ and $C^2=(\beta_1,\beta_2,\ldots,\beta_s)$. Let a and b be the symbols in (α_1,α_2) and (β_1,β_2) respectively. Let P' be a new partial latin square obtained from P by:

(1) transfering symbol a from (α_1,α_2) to (α_1,β_2), and transfering symbol b from (β_1,β_2) to (β_1,α_2);

(2) keeping the symbols in all the other cells of P unchanged.

Clearly, in P', the two cycles $(\alpha_1,\alpha_2,\ldots,\alpha_r)$ and $(\beta_1,\beta_2,\ldots,\beta_s)$ have been combined into one cycle $C^3=(\alpha_1,\beta_2,\beta_3,\ldots,\beta_s,\beta_1,\alpha_2,\alpha_3,\ldots,\alpha_r)$. We say that P' is obtained from P by *swopping* the two preassigned off-diagonal symbols of columns α_2 and β_2, and we call columns α_2 and β_2 *used columns* and all the other columns of C^1 and C^2 *unused columns*. Since $r,s\geqslant 2$, the number of unused columns in C^3 is at least 2.

In the proof of the main theorem, we will encounter the case that P' has been completed to a latin square $L'=(l'_{ij})$ of side n such that $l'_{\alpha_2,\beta_2}=l'_{\beta_2,\alpha_2}$. Since $\alpha_1,\alpha_2,\beta_1$ and β_2 are distinct, the cells (α_2,β_2) and (β_2,α_2) are not in C^1,C^2 or C^3. We can obtain a new latin square L of side n from L' by interchanging the symbols between cells (α_2,α_2) and (α_2,β_2), and between cells (β_2,β_2) and (β_2,α_2), and then interchanging column α_2 with column β_2. It is easy to see that L contains P. We call this a *resuming process*. The resuming process has the following properties:

P1. L is still an unipotent latin square;

P2. For any near transversal Q' of L' that contains no main diagonal cells, if $(\alpha_2,\beta_2),(\beta_2,\alpha_2)\notin S'$, then the cells of L corresponding to those of S' also form a near transversal.

We will use the following lemma to prove the main theorem. It is a special case of Theorem 3.1 in [6].

Lemma 2.1. *Let P be a unipotent partial latin square of side n such that all the main diagonal cells are filled with symbol n, where n is even and $n\geqslant 4$. Suppose that all the off-diagonal occupied cells form a near transversal Q excluding $(1,1)$. Then P can be completed to form a unipotent latin square L of side n such that L has a near transversal Q' excluding $(1,1)$ and avoiding Q.*

We also need the following lemma in the proof of the main theorem. This lemma derives from a special 1-factorization of K_{2n} which is called cyclic of type 2 by Korovina [5] or factor-1-rotational by Mendelsohn and Rosa [7].

Lemma 2.2. *Let $G=S_2\cup C_{2n-2}$ be a spanning subgraph of K_{2n}, where $n\geqslant 3$. Then there exists a $(2n-1)$-edge-coloring of K_{2n} such that all the edges of G receive distinct colors.*

3. Proof of Main Theorem

Let $G=(X,Y)$. Suppose G is a type 1 graph. Then by Lemma 1.1, G is biconformable. Thus there exists a biconformable $(\Delta(G)+1)$-vertex-coloring ϕ of G. Let $X_j\in X$ and $Y_j\in Y$ be the vertex sets colored with color j. Then $|X_j|=|Y_j|$, $j=1,2,\ldots,\Delta+1$. Since G is $(n-2)$-regular, then $|X_j|=|Y_j|\leqslant 2$. Since $|X|=|Y|=n$, there is at least one X_k (Y_k) having 2 vertices. Then the vertices in X_k and Y_k form a 4-cycle in J.

Now suppose that J contains a 4-cycle C. Clearly J consists of two 1-factors F_1 and F_2. Then we can color F_1 with color n, and color F_2 with colors $1,2,\ldots,n-1$ such that the two edges of F_2 in the 4-cycle C are colored with color 1 and all the other edges of F_2 receive distinct colors $2,3,\ldots,n-1$. An example is shown in Fig. 1.

Now we shall show that there is a 1–1 correspondence between n-edge-colorings of $K_{n,n}$ using colors $1,2,\ldots,n$ and latin squares of side n on symbols $1,2,\ldots,n$: each n-

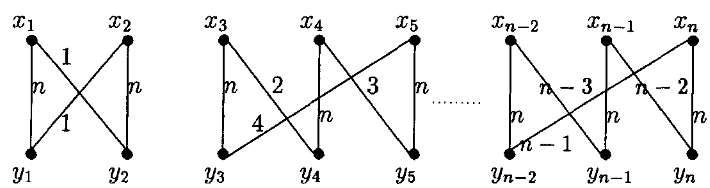

Fig. 1. An n-edge coloring ϕ of J.

	x_1	x_2	x_3	x_4	x_5	·······	x_{n-1}	x_n
y_1	n	1						
y_2	1	n						
y_3			n		4			
y_4			2	n				
y_5				3	n			
⋮						⋱		
y_{n-1}							n	
y_n							$n-2$	n

Fig. 2. The $n \times n$ partial latin square P corresponding to ϕ in Fig. 1.

edge-coloring ϕ of $K_{n,n}$ gives rise to a latin square $L=(\phi(y_ix_j))$ of side n; conversely, for any latin square $L=(l_{ij})$ of side n, we can construct an n-edge-coloring ϕ of $K_{n,n}$ by putting $\phi(y_ix_j)=l_{ij}$. Thus completing a partial latin square of side n is equivalent to extending a partial n-edge-coloring of $K_{n,n}$ to an n-edge-coloring of $K_{n,n}$. Similiarly, there also exists a 1–1 correspondence between $(2n-1)$-edge-colorings of K_{2n} using colors $1,2,\dots,2n-1$ and unipotent symmetric latin squares of side $2n$ on symbols $1,2,\dots,2n$, whose diagonal cells are all occupied by $2n$.

The n-edge-coloring ϕ of J is also a partial n-edge-coloring of $K_{n,n}$. By the 1–1 correspondence between n-edge-colorings of $K_{n,n}$ and latin squares of side n, ϕ is identified with a unipotent partial latin square P of side n. A unipotent partial latin square corresponding to the above example is shown in Fig. 2. Next we consider two cases:

Case 1: *n is odd.* We can obtain a unipotent partial latin square P' of side $n-1$ with symbols $2,3,\dots,n$ by deleting the first row and the first column of P (see Fig. 3).

We observe that the set of all the occupied off-diagonal cells in P' form a near transversal Q excluding $(1,1)$. By Lemma 2.1, P' can be completed to a unipotent latin square L' of side $n-1$ such that L' contains a near transversal Q' excluding $(1,1)$ and avoiding Q. Now we extend L' to a unipotent latin square L of side n, which contains P, as follows: add one row and one column to L' as the first row and first column, then fill in cell $(1,1)$ with symbol n, and fill in cells $(1,2)$ and $(2,1)$ with symbol 1. Furthermore, if cell (i,j) in L is a cell of Q' in L', then replace the symbol, say α, in (i,j) with symbol 1 and fill in cells $(i,1)$ and $(1,j)$ with symbol α. In other words, P can be completed to a unipotent latin square of side n. It follows that ϕ can be extended to an n-edge coloring ϕ' of $K_{n,n}$. Now we can obtain an

	x_2	x_3	x_4	x_5	⋯	x_{n-1}	x_n
y_2	n						
y_3		n		4			
y_4		2	n				
y_5			3	n			
⋮					⋱		
y_{n-1}						n	
y_n						$n-2$	n

Fig. 3. The $(n-1)\times(n-1)$ partial latin square P'.

$(n-1)$-total-coloring ψ of G from ϕ' using colors $1,2,\ldots,n-1$:

$$\psi(x_iy_j)=\phi'(x_iy_j) \quad \text{if } x_iy_j\in E(G),\ i=1,2,\ldots,n,$$

$$\psi(x_i)=\psi(y_j)=\phi'(x_iy_j) \quad \text{if } x_iy_j\in E(J) \quad \text{and} \quad \phi'(x_iy_j)\neq n.$$

Hence G is type 1.

Case 2: *n is even.* Suppose P has only two cycles $(1,2)$ and $(3,4,\ldots,n)$. Then for any occupied off-diagonal cell (i,j), except $(1,2)$ and $(2,1)$, cell (j,i) is not occupied. (Otherwise cells (j,i) and (i,j) form a cycle of length 2 which contradicts the fact that P has only two cycles.) We fill in cell (j,i) the same symbol as in cell (i,j). (We can do this because all the occupied off-diagonal cells except $(1,2)$ and $(2,1)$ have distinct symbols.) We thus obtain a unipotent symmetric partial latin square P' of side n. By the 1–1 correspondence between unipotent symmetric latin squares of side n and $(n-1)$-edge-colorings of K_n as mentioned before, P' can be identified with a partial $(n-1)$-edge-coloring π of K_n. The preassigned $n-1$ distinct symbols in P' correspond to $n-1$ distinct colors on the edges of $S_2\cup C_{n-2}$ in K_n. By Lemma 2.2, π can be extended to an $(n-1)$-edge-coloring of K_n. In other words, P' (so is P) can be completed to a symmetric latin square of side n.

Suppose P has $m>2$ cycles $C^1,C^2,\ldots,C^m$, where $C^1=(1,2)$. We first combine C^2 and C^3 to a new cycle C^* by swopping a column of C^2 with a column of C^3. There are at least two unused columns in C^*. Then we combine C^* and C^4 by swopping an unused column of C^* with a column of C^4. We continue this process until we finally obtain a unipotent partial latin square P' which has only two cycles (Since at each step there are at least two unused columns in the new cycle, we know that each column is used at most once throughout this process.) As shown above, P' can be completed to a symmetric unipotent latin square L of side n. Now for any pair of used columns α and β, since L is symmetric, we can restore back these two columns to the original two columns in P by the resuming process. (We can do this because in the combining process each column was used at most once, and thus the pairs of used columns are mutually independent.) Thus we obtain a latin square L' of side n from L. It is easy to see that L' contains P.

Hence, we conclude that P can be completed to a unipotent latin square. It follows that ϕ can be extended to an n-edge-coloring ϕ' of $K_{n,n}$. Now we can obtain an $(n-1)$-total-coloring ψ of G from ϕ' using colors $1,2,\dots,n-1$ as follows:

$$\psi(x_i y_j) = \phi'(x_i y_j) \quad \text{if } x_i y_j \in E(G) \ i=1,2,\dots,n, \ j=1,2,\dots,n,$$

$$\psi(x_i) = \psi(y_j) = \phi'(x_i y_j) \quad \text{if } x_i y_j \in E(J) \quad \text{and} \quad \phi'(x_i y_j) \neq n.$$

It follows that G is type 1. □

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