Chords of Longest Cycles in Cubic Graphs

Carsten Thomassen

Department of Mathematics, Technical University of Denmark,
Building 303, DK-2800 Lyngby, Denmark

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We describe a general sufficient condition for a Hamiltonian graph to contain another Hamiltonian cycle. We apply it to prove that every longest cycle in a 3-connected cubic graph has a chord. We also verify special cases of an old conjecture of Sheehan on Hamiltonian cycles in 4-regular graphs and a recent conjecture on a second Hamiltonian cycle by Triesch, Nolles, and Vygen.

1. INTRODUCTION

In 1976 I conjectured that every longest cycle in every 3-connected graph has a chord [3, Conjecture 6.11], see also [8, Conjecture 6; 2, Conjecture 8.1; and 11, Conjecture 8.3.14]. In this paper the conjecture is verified for cubic graphs.

The proof follows from a general sufficient condition for a second Hamiltonian cycle (using Thomason’s lollipop method) combined with the result of Fleischner and Stiebitz [5] (proved with the aid of the Alon and Tarsi result [1]) that every “cycle plus triangles graph” is 3-colorable. We prove that every “cycle plus triangles graph” has a second Hamiltonian cycle. This verifies a special case of the conjecture made in 1975 by Sheehan [6] that every 4-regular Hamiltonian graph has another Hamiltonian cycle. Motivated by a problem on optimum tours in the Travelling Salesman Problem, Triesch et al. [10] made the following conjecture: Let $C$ be a cycle with $n$ vertices. Let $c$ be a coloring of $V(C)$ in $m$ colors where $2m + 1 \leq n$. Consecutive vertices are allowed to have the same color. Add all edges between vertices of the same color. Then the resulting graph $G$ has a Hamiltonian cycle distinct from $C$. We verify this when $c$ is proper (i.e., no two consecutive vertices of $C$ have the same color) and all color classes have at least 3 vertices. The graph of Fleischner [4, Fig. 6] shows that the conjecture fails in general even if $c$ is proper and all color classes have at least 2 vertices.

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2. A SECOND HAMILTONIAN CYCLE

**Lemma 2.1.** Let $G$ be a graph with a Hamiltonian cycle $C$. Suppose that for some set $A$ of vertices, the subgraph $G - A$ has $|A|$ components each of which is a path whose ends are of odd degree in $G$. Then

1. for every Hamiltonian cycle $C'$ of $G$, $C' - A = C - A$, and
2. each edge of $G$ incident to a vertex of $A$ is included in an even number of Hamiltonian cycles of $G$.

**Proof.** (1) is obvious. Let $e = xy$ where $x \in A$. Every Hamiltonian path in $G$ which starts with $x$ and $e$ (if any) must end at an end of one of the paths in $G - A$. That vertex has odd degree in $G$. It follows by Thomason’s lollipop argument [7] (see also [9, Theorem 2.1]) that $G$ has an even number of Hamiltonian cycles containing $e$. □

**Theorem 2.2.** Let $G$ be a graph with a Hamiltonian cycle $C$. Let $A$ be a vertex set in $G$ such that

(i) $A$ is independent in $C$ (i.e., $A$ contains no two consecutive vertices of $C$), and
(ii) $A$ is dominating in $G - E(C)$ (i.e., every vertex of $G - A$ is joined to a vertex in $A$ by some chord of $C$).

Then $G$ has a Hamiltonian cycle $C'$ distinct from $C$. Moreover, $C'$ can be chosen such that

(iii) $C' - A = C - A$, and
(iv) there is a vertex $v$ in $A$ such that one of the two edges of $C'$ incident with $v$ is in $C$ and the other is not in $C$.

**Proof.** We consider the following subgraph $G'$ of $G$: $G'$ contains $C$ and, for each vertex $x$ which is adjacent on $C$ to a vertex of $A$, we let $G'$ contain precisely one chord of $C$ from $x$ to a vertex of $A$. So in $G'$, $x$ has degree 3. Every vertex of $C$ which is not in $A$ and not consecutive to a vertex of $A$ has degree 2 in $G'$. Note that $G' - A$ has precisely $|A|$ components, each of which is a path. By Lemma 2.1, $G'$ has a Hamiltonian cycle $C'$ distinct from $C$. Clearly, $C'$ satisfies (iii).

To prove (iv) we consider a Hamiltonian cycle $C'$ of $G'$ such that $C' \neq C$ and $C'$ contains as many edges of $C$ as possible. We claim that $C'$ satisfies (iv). Suppose (reductio ad absurdum) that $C'$ does not satisfy (iv). Let $H$ be the graph $C' \cup C$. Let $A'$ be those vertices of $A$ which contain an edge (and hence two edges) in $E(C') \setminus E(C)$. Then each vertex of $A'$ has degree 4 in $H$, each vertex in $V(G) \setminus A'$ which is adjacent (on $C$) to a vertex in $A'$ has degree 3 in $H$, and all other vertices have degree 2 in $H$. Also, $H - A'$
has precisely $|A'|$ components, each of which is a path whose ends are of odd degree in $H$. Now let $u$ be a vertex of $A'$ and let $uy$ be an edge of $C'$. By Lemma 2.1, $H$ has a Hamiltonian cycle $C''$ distinct from $C'$ containing $uy$. There must be some vertex $z$ in $A'$ such that $C''$ does not contain the two edges of $C'$ incident with $z$. (This follows because $C'' - A' = C' - A'$.) But then $C''$ contains an edge of $C$ incident with $z$. We claim that $C''$ also contains each edge $f$ in $C \cap A'$. This follows from Lemma 2.1 if $f$ is not incident with a vertex of $A$. On the other hand, if $f$ is incident with a vertex $w$ of $A$, then $w$ is not in $A'$ (by the definition of $A'$) and hence $w$ has degree 2 in $H$. Therefore $f$ is in every Hamiltonian cycle of $H$. This contradiction to the maximality property of $C'$ proves that $C''$ satisfies (iv).

3. APPLICATIONS

The following was proved in [9].

**Theorem 3.1.** If $C$ is a Hamiltonian cycle in a bipartite graph with bipartition $A, B$ such that every vertex of $B$ has degree at least 3 in $G$, then $G$ has a Hamiltonian cycle distinct from $C$.

Theorem 3.1 follows from Theorem 2.2 since the set $A$ in the former can play the role of $A'$ in the latter. Conversely, Theorem 2.2 except (iv) follows easily from Theorem 3.1 by subdividing each edge of $C - A$ once and adding the new vertices of degree 2 to $A$.

The next result is related to the conjecture of Triesch et al. mentioned in the Introduction.

**Theorem 3.2.** Let $G$ be a graph with a Hamiltonian cycle $C$ such that $G - E(C)$ is the disjoint union of complete graphs each of order at least 3. Then $G$ has a Hamiltonian cycle distinct from $C$.

**Proof.** Select a triangle in each component of $G - E(C)$. Let $G'$ denote the union of $C$ and these triangles. Then $G'$ is a subdivision of a “cycle plus triangles graph” which, by a result of Fleischner and Stiebitz [5] is 3-colorable. Since a subdivision of a 3-colorable graph is 3-colorable, $G'$ is 3-colorable. Any color class of $G'$ satisfies (in $G$) the assumption of Theorem 2.2. Hence $G$ has a Hamiltonian cycle distinct from $C$.

We can now prove the main result.

**Theorem 3.3.** Let $C$ be any longest cycle in a 3-connected cubic graph $G$. Then $C$ has a chord.
Proof. Suppose (reductio ad absurdum) that $C$ has no chord. For each component $H$ of $G - V(C)$ we select three vertices $x(H), y(H), z(H)$ of $C$ which are joined to vertices of $H$. This is possible because $G$ is 3-connected. As $G$ is cubic, $\{x(H), y(H), z(H)\} \cap \{x(H'), y(H'), z(H')\} = \emptyset$ when $H \neq H'$. We form a new graph $G'$ which consists of $C$ and all triangles of the form $x(H) \cup y(H) \cup z(H)$. By [5], $G'$ is 3-colorable. Let $A$ be a color class. Assume without loss of generality that $A$ contains all vertices of the form $x(H)$ where $H$ is a component of $G - V(C)$. We form a new graph $G'$ by contracting each component $H$ of $G - V(C)$ into $x(H)$. Clearly, $G'$ and $A$ satisfy (i) and (ii) in Theorem 2.2 (with $G'$ instead of $G$). Let $C'$ be a Hamiltonian cycle of $G'$ satisfying (iii) and (iv). $C'$ can easily be modified into a cycle $C''$ of $G$. The condition (iv) implies that $C''$ is longer than $C$, a contradiction which completes the proof.

REFERENCES