On the convective Cahn–Hilliard equation with degenerate mobility

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Abstract
In this paper, we study the existence of weak solutions for the convective Cahn–Hilliard equation with degenerate mobility. Based on the Schauder type estimates, we establish the global existence of classical solutions for regularized problems. After establishing some necessary uniform estimates on the approximate solutions, we prove the existence of weak solutions. The nonnegativity and the finite speed of propagation of perturbations of solutions are also discussed.
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1. Introduction
In this paper, we investigate the convective Cahn–Hilliard equation

\[
\frac{\partial u}{\partial t} + D\left[m(u)\left(kD^3 u - DA(u)\right)\right] - \gamma DB(u) = 0, \quad \text{in } Q_T = I \times (0, T),
\]

where \( I = (0, 1) \), \( D = \frac{\partial}{\partial x} \), \( m(u) = |u|^n \), \( 0 < n < 1 \), \( B(u) = u^2 \), and \( k > 0 \), \( \gamma > 0 \) are constants. From the physical consideration, we prefer to consider a typical case of the potential \( H(u) \), that is \( H'(u) = A(u) \), in the following form

\[
(H1) \quad H(u) = \frac{1}{4}(u^2 - 1)^2,
\]

namely, the well-known double well potential.

Eq. (1.1) is supplemented by the boundary value conditions

\[
u(0, t) = u(1, t) = D^2 u(0, t) = D^2 u(1, t) = 0, \quad t > 0.
\]

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The boundary value conditions (1.2) are reasonable for the thin film equation or the Cahn–Hilliard equation (see [1,2,7]), and initial value condition
\[ u(x, 0) = u_0(x). \] (1.3)

Eq. (1.1) arises naturally as a continuous model for the formation of facets and corners in crystal growth (see [9,13,14]). Here \( u(x, t) \) denotes the slope of the interface. The convective term \( u \frac{\partial u}{\partial x} \) (see [13,14]) stems from the effect of kinetics (the finite rate of atoms or molecules attachment to the crystal surface) that provides an independent flux of the order parameter, similar to the effect of an external field in spinodal decomposition of a driven system.

During the past years, many authors have paid much attention to the convective Cahn–Hilliard equation for \( m(u) \) is a constant. It was K.H. Kwek [11] who first studied Eq. (1.1) for a special case with constant mobility and a special convection, namely, \( m(u) = 1, B(u) = u \). Based on the discontinuous Galerkin finite element method, he proved the existence of classical solutions. Liu [12] proved the existence, asymptotic behavior of classical solutions for \( m(u) \) is a constant.

Eden and Kalantarov [6], Zaks et al. [15] also considered the problem (1.1)–(1.3) with constant mobility, i.e. \( m(u) = 1 \). However, only a few works have been devoted to the equation with concentration dependent mobility. Bertozzi and Shearer [5] considered the convective–diffusive equation of the form
\[ \frac{\partial u}{\partial t} + Df(u) + D(u^3 D^3 u) - \alpha D(u^3 Du) = 0, \]
where \( f(u) = u^2 - u^3 \) is the flux function and \( \alpha \geq 0 \) is a dimensionless parameter. The main interest is the existence and nonexistence of undercompressive traveling waves. Basing on the parameter \( \alpha \geq 0 \), the authors proved the existence of an undercompressive traveling wave solution for sufficiently small \( \alpha \) and the nonexistence when \( \alpha \) is sufficiently large. We also refer the following relevant equation
\[ \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} \left( \mu^n \frac{\partial^3 u}{\partial x^3} \right), \]
which has been extensively studied. F. Bernis and A. Friedman [4] have studied the initial boundary value problems to the thin film equation \( n \geq 0 \) and proved existence of weak solutions preserving nonnegativity (see also [2,10]). They proved that if \( n \geq 2 \) the support of the solutions \( u(\cdot, t) \) is nondecreasing with respect to \( t \).

In this paper, we study the problem (1.1)–(1.3). Because of the degeneracy, the problem does not admit classical solutions in general. So, we introduce the weak solutions in the following sense.

**Definition.** A function \( u \) is said to be a weak solution of (1.1)–(1.3), if the following conditions are satisfied:

1. \( u \in C^a(\bar{Q}_T), \alpha \in (0, 1), Du \) is locally Hölder continuous in \( P, u \in L^\infty(0, T; H^1(0, 1)), |u|^{n/2} D^3 u \in L^2(P) \).
2. For \( \varphi \in C^1(\bar{Q}_T) \) and \( Q_T = (0, 1) \times (0, T) \),
\[ -\int_0^1 u(x, T)\varphi(x, T) \, dx + \int_0^1 u_0(x)\varphi(x, 0) \, dx + \int \int_{Q_T} u \frac{\partial \varphi}{\partial t} \, dx \, dt \]
\[ + \int \int_P |u|^n (k D^3 u - DA(u)) D\varphi \, dx \, dt - \gamma \int \int_{Q_T} B(u) D\varphi \, dx \, dt = 0, \]
where \( P = \bar{Q}_T \setminus \{u(x, t) = 0\} \cup \{t = 0\} \).

We investigate the existence of weak solutions. Because of the degeneracy, we will first consider the regularized problem. To prove the existence of classical solutions for the regularized problem, the basic a priori estimates are the \( L^2 \) norm estimates on \( u \) and \( Du \). For the usual Cahn–Hilliard equation (i.e. \( B(u) = 0 \)) with following boundary conditions
\[ Du(0, t) = Du(1, t) = D^3 u(0, t) = D^3 u(1, t) = 0, \quad t > 0, \] (1.4)
the two estimates can be easily obtained, since the problem (1.1), (1.4), (1.3) with $B(u) = 0$ has two important properties:

1. the conservation of mass, namely
   \[
   \int_0^1 u(x, t) \, dx = \int_0^1 u_0(x) \, dx;
   \]
2. there exists a Lyapunov functional
   \[
   F[u] = \int_0^1 \left( k \frac{1}{2} |Du|^2 + H(u) \right) \, dx,
   \]
   which is decreasing in time.

However, for the problem (1.1)–(1.3) above two properties might not be existent. This means that we should find a new approach to establish the required estimates on $\|u\|_{L^2(\Omega)}$ and $\|Du\|_{L^2(\Omega)}$. Nevertheless, because of the nonlinearity of both the diffusive and the convective factors, the method used in [11] seems not applicable to the present situation. Our method is based on uniform Schauder type estimates for local in time solutions via the framework of Campanato spaces. To this purpose, we require some delicate local integral estimates rather than the global energy estimates used in the discussion for the Cahn–Hilliard equation with constant mobility. Based on the uniform estimates for the approximate solutions, we obtain the existence. Owing to the background, we are much interested in the nonnegativity of the weak solutions and the solutions with the property of finite speed of propagation of perturbations. Using weighted Nirenberg’s inequality and Hardy’s inequality, we proved these properties.

This paper is arranged as follows. We first study the regularized problem in Section 2, and then establish the existence in Section 3. Subsequently, we discuss the nonnegativity of weak solutions in Section 4 and the finite speed of propagation in Section 5.

2. Regularized problems

To discuss the existence, we adopt the method of parabolic regularization, namely, the desired solution will be obtained as the limit of some subsequence of solutions of the following regularized problem

\[
\frac{\partial u_\varepsilon}{\partial t} + D[m_\varepsilon(u_\varepsilon)(kD^3u_\varepsilon - DA(u_\varepsilon))] - \gamma DB(u_\varepsilon) = 0 \quad \text{in } Q_T, \tag{2.1}
\]

\[
u_\varepsilon(0, t) = u_\varepsilon(1, t) = D^2u_\varepsilon(0, t) = D^2u_\varepsilon(1, t) = 0, \quad t > 0, \tag{2.2}
\]

\[
u_\varepsilon(x, 0) = u_{0\varepsilon}(x), \tag{2.3}
\]

where $m_\varepsilon(u_\varepsilon) = (|u_\varepsilon|^2 + \varepsilon)^{1/2}$.

**Theorem 2.1.** For each fixed $\varepsilon > 0$ and under assumption (H1),

\[
u_{0\varepsilon} \in C^{4+\alpha}(\bar{T}), \quad D^i u_{0\varepsilon}(0) = D^i u_{0\varepsilon}(1) = 0 \quad (i = 0, 2),
\]

then the problem (2.1)–(2.3) admits a unique classical solution $u_\varepsilon \in C^{4+\alpha, 1+\alpha/4}(\bar{Q}_T), \text{ for some } \alpha \in (0, 1)$.

From the classical approach, it is not difficult to conclude that the problem (2.1)–(2.3) admits a unique classical solution local in time. So, it is sufficient to make a priori estimates. As an important step, we give the Hölder norm estimate on the local in time solutions.

**Proposition 2.1.** Assume that (H1) holds, and $u_\varepsilon$ is a smooth solution of the problem (2.1)–(2.3). Then there exists a constant $C$ depending only on the known quantities, such that for any $(x_1, t_1), (x_2, t_2) \in Q_T$ and some $0 < \alpha < 1$,

\[
|u_\varepsilon(x_1, t_1) - u_\varepsilon(x_2, t_2)| \leq C(|t_1 - t_2|^\alpha + |x_1 - x_2|^\alpha). \tag{2.4}
\]
Proof. Let \( z = kD^2u_\varepsilon - A(u_\varepsilon) \). Multiplying both sides of Eq. (2.1) by \( z \) and then integrating the resulting relation with respect to \( x \) over \( \Omega \), we have

\[
\int_0^1 \frac{\partial u_\varepsilon}{\partial t} (kD^2u_\varepsilon - A(u_\varepsilon)) \, dx + \int_\Omega D(m_\varepsilon(u_\varepsilon))z \, dx - \int_0^1 \gamma DB(u_\varepsilon)z \, dx = 0.
\]

After integrating by parts, and using the boundary value conditions,

\[
\frac{d}{dt} \left( \int_0^1 kD^2u_\varepsilon \, dx \right) + \int_0^1 m_\varepsilon(u_\varepsilon) |Dz|^2 \, dx - \int_0^1 \gamma B(u_\varepsilon)Dz \, dx = 0.
\]

Using Hölder’s inequality, we have

\[
\frac{d}{dt} \left( \int_0^1 kD^2u_\varepsilon \, dx \right) + \int_0^1 m_\varepsilon(u_\varepsilon) |Dz|^2 \, dx \leq C \int_0^1 (k(Du_\varepsilon)^2 + H(u_\varepsilon)) \, dx.
\]

From the assumption (H1) and \( m_\varepsilon(u_\varepsilon) = (|u_\varepsilon|^2 + \varepsilon)^n \), \( 0 < n < 1 \), we obtain

\[
\frac{d}{dt} \left( \int_0^1 k(Du_\varepsilon)^2 \, dx \right) + \tilde{\int}_0^1 m_\varepsilon(u_\varepsilon) |Dz|^2 \, dx \leq C \int_0^1 (k(Du_\varepsilon)^2 + H(u_\varepsilon)) \, dx.
\]

The Gronwall inequality implies that

\[
\int_0^1 m_\varepsilon(u_\varepsilon)(D^3u_\varepsilon)^2 \, dx \, dt \leq C, \quad (2.5)
\]

\[
\int_0^1 |Du_\varepsilon|^2 \, dx \leq C, \quad 0 \leq t \leq T, \quad (2.6)
\]

\[
\int_0^1 u_\varepsilon^4 \, dx \leq C, \quad 0 \leq t \leq T. \quad (2.7)
\]

By (2.6), (2.7) we have

\[
|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \leq C|x_1 - x_2|^\alpha, \quad 0 < \alpha < \frac{1}{2}. \quad (2.8)
\]

Integrating Eq. (2.1) with respect to \( (x,t) \) over \( (y, y + (\Delta t)^{1/4}) \times (t_1, t_2) \), where \( 0 < t_1 < t_2 < T \), \( \Delta t = t_2 - t_1 \), we see that

\[
\int_y^{y+(\Delta t)^{1/4}} \left[ u_\varepsilon(z,t_2) - u_\varepsilon(z,t_1) \right] \, dz
\]

\[
= - \int_{t_1}^{t_2} \left[ m_\varepsilon(u_\varepsilon(y',s)) (kD^3u_\varepsilon(y',s) - A'(u_\varepsilon)Du_\varepsilon(y',s)) - \gamma B(u_\varepsilon(y',s)) \right. \\
\left. - m_\varepsilon(u_\varepsilon(y,s)) (kD^3u_\varepsilon(y,s) - A'(u_\varepsilon)Du_\varepsilon(y,s)) + \gamma B(u_\varepsilon(y,s)) \right] \, ds.
\]

Set

\[
N(s,y) = m_\varepsilon(u_\varepsilon(y',s)) (kD^3u_\varepsilon(y',s) - A'(u_\varepsilon)Du_\varepsilon(y',s)) - \gamma B(u_\varepsilon(y',s))
\]

\[
- m_\varepsilon(u_\varepsilon(y,s)) (kD^3u_\varepsilon(y,s) - A'(u_\varepsilon)Du_\varepsilon(y,s)) + \gamma B(u_\varepsilon(y,s)),
\]

where \( y' = y + (\Delta t)^{1/4} \).
Then (2.9) is converted into
\[
(\Delta t)^{1/4} \int_0^1 (u_\varepsilon(y + \theta(\Delta t)^{1/4}, t_2) - u_\varepsilon(y + \theta(\Delta t)^{1/4}, t_1)) \, d\theta = - \int_{t_1}^{t_2} N(s, y) \, ds.
\]
Integrating the above equality with respect to \( y \) over \((x, x + (\Delta t)^{1/4})\), we get
\[
(\Delta t)^{1/2} \int_0^{(\Delta t)^{1/4}} u_\varepsilon(x^*, t_2) - u_\varepsilon(x^*, t_1) = - \int_{t_1}^{t_2} \int_{x} N(s, y) \, dy \, ds.
\]
Here, we have used the mean value theorem, where \( x^* = y^* + \theta^*(\Delta t)^{1/4}, y^* \in (x, x + (\Delta t)^{1/4}), \theta^* \in (0, 1) \). Hence by Hölder’s inequality and (2.6)–(2.8), we get
\[
|u_\varepsilon(x^*, t_2) - u_\varepsilon(x^*, t_1)| \leq C(\Delta t)^{\alpha/4}, \quad 0 < \alpha < 1.
\]
The proof is complete. \( \square \)

To prove Theorem 2.1, the key estimate is the Hölder estimate for \( Du_\varepsilon \). We consider the following linear problem
\[
\frac{\partial u}{\partial t} + D(a(x, t)D^3 u) = Df, \quad (2.10)
\]
\[
u|_{x=0,1} = D^2 u|_{x=0,1} = 0, \quad (2.11)
\]
\[
u(x, 0) = 0, \quad (2.12)
\]
where
\[
a(x, t) = km(u(x, t)), \quad f = m(u(x, t)) DA(u(x, t)) + \gamma B(u).
\]
Here we do not restrict the smoothness of the given functions \( a(x, t) \) and \( f(x, t) \), but simply assume that they are sufficiently smooth. Our main purpose is to find the relation between the Hölder norm of the solution \( u \) and \( a(x, t) \), \( f(x, t) \).

The crucial step is to establish the estimates on the Hölder norm of \( u \). Let \((x_0, t_0) \in (0, 1) \times (0, T)\) be fixed and define
\[
\psi(\rho) = \int_{S_\rho} \left( |D u - (Du)_\rho|^2 + \rho^4 |D^3 u|^2 \right) \, dx \, dt \quad (\rho > 0),
\]
where
\[
S_\rho = B_\rho(x_0) \times (t_0 - \rho^4, t_0 + \rho^4), \quad u_\rho = \frac{1}{|S_\rho|} \int_{S_\rho} u \, dx \, dt
\]
and \( B_\rho(x_0) = (x_0 - \rho, x_0 + \rho) \).

Let \( u \) be the solution of the problem (2.10)–(2.12). We split \( u \) on \( S_\rho \) into \( u = u_1 + u_2 \), where \( u_1 \) is the solution of the problem
\[
\frac{\partial u_1}{\partial t} + a(x_0, t_0)D^4 u_1 = 0, \quad (x, t) \in S_\rho, \quad (2.13)
\]
\[
u_1 = u, \quad D^2 u_1 = D^2 u, \quad (x, t) \in \partial B_R(x_0) \times (t_0 - R^4, t_0 + R^4), \quad (2.14)
\]
\[
u_1 = u, \quad t = t_0 - R^4, \quad x \in B_R(x_0), \quad (2.15)
\]
and \( u_2 \) solves the problem
\[
\frac{\partial u_2}{\partial t} + a(x_0, t_0)D^4 u_2 = D[\left(a(x_0, t_0) - a(x, t)\right)D^3 u] + Df, \quad (x, t) \in S_\rho, \quad (2.16)
\]
\[
u_2 = 0, \quad D^2 u_2 = 0, \quad (x, t) \in \partial B_R(x_0) \times (t_0 - R^4, t_0 + R^4), \quad (2.17)
\]
\[
u_2 = 0, \quad t = t_0 - R^4, \quad x \in B_R(x_0). \quad (2.18)
\]
By classical linear theory, the above decomposition is uniquely determined by $u$.

We need several lemmas on $u_1$ and $u_2$.

**Lemma 2.1.** Assume that
\[ |a(x, t) - a(x_0, t_0)| \leq a_0(|t - t_0|^{\alpha/4} + |x - x_0|^{\alpha}), \quad x \in B_R(x_0), \ t \in (t_0 - R^4, t_0 + R^4). \]

Then
\[ \sup_{(t_0 - R^4, t_0 + R^4)} \int_{B_R(x_0)} (Du_2)^2(x, t) \, dx + \int_{S_R} \int (D^3u_2)^2 \, dx \, dt \leq CR^{2\alpha} \int_{S_R} (D^3u)^2 \, dx \, dt + C \sup_{S_R} |f|^2 R^5. \]

**Proof.** Multiply Eq. (2.16) by $D^2u_2$ and integrate the resulting relation over $(t_0 - R^4, t) \times B_R(x_0)$. Integrating by parts, we have
\[
\frac{1}{2} \int_{B_R} (Du_2)^2(x, t) \, dx + a(x_0, t_0) \int_{t_0 - R^4}^t ds \int_{B_R} (D^3u_2)^2 \, dx \\
= \int_{t_0 - R^4}^t ds \int_{B_R} [a(x_0, t_0) - a(x, t)] D^3u D^3u_2 \, dx + \int_{t_0 - R^4}^t ds \int_{B_R} f D^3u_2 \, dx.
\]

Noticing that
\[
\left| \int_{t_0 - R^4}^t ds \int_{B_R} [a(x_0, t_0) - a(x, t)] D^3u D^3u_2 \, dx \right| \leq a_0 \sup_{S_R} (Du_2)^3 \, dx \, ds + CR^{2\alpha/4} \int_{S_R} (D^3u)^2 \, dx \, ds
\]
and
\[
\left| \int_{t_0 - R^4}^t ds \int_{B_R} f D^3u_2 \, dx \right| \leq C \, \sup_{S_R} |f|^2,
\]

hence we obtain the estimate and the proof is complete. \(\square\)

**Lemma 2.2.** For any $(x_1, t_1), (x_2, t_2) \in S_{\rho}$,
\[
\frac{|Du_1(x_1, t_1) - Du_1(x_2, t_2)|^2}{|t_1 - t_2|^{1/4} + |x_1 - x_2|} \leq C \sup_{(t_0 - \rho^4, t_0 + \rho^4)} \int_{B_\rho(x_0)} (D^2u_1(x, t))^2 \, dx + C \int_{S_\rho} (D^4u_1)^2 \, dx \, dt.
\]

**Proof.** From the Sobolev embedding theorem, we have for any $(x_1, t_1), (x_2, t_2) \in S_{\rho}$,
\[
\frac{|u_1(x_1, t_1) - u_1(x_2, t_2)|^2}{|x_1 - x_2|} \leq C \sup_{(t_0 - \rho^4, t_0 + \rho^4)} \int_{B_\rho(x_0)} (D^2u_1(x, t))^2 \, dx.
\]  

(2.19)

Differentiating Eq. (2.13) gives
\[
\frac{\partial Du_1}{\partial t} + a(x_0, t_0) D^5u_1 = 0, \quad (x, t) \in S_R.
\]

Integrating the above equation with respect to $(x, t) \in (y, y + (\Delta t)^{1/4}) \times (t_1, t_2)$, where $0 < t_1 < t_2 < T$, $\Delta t = t_2 - t_1$,
we see that
\[
\int_y (Du_1(z, t_2) - Du_1(z, t_1)) \, dz + a(x_0, t_0) \int_{t_1}^{t_2} [D^4u_1(y', s) - D^4u_1(y, s)] \, ds = 0,
\]

where $y' = y + (\Delta t)^{1/4}$. 
That is
\[ (\Delta t)^{1/4} \int_0^1 [Du_1(y + \theta(\Delta t)^{1/4}, t_2) - Du_1(y + \theta(\Delta t)^{1/4}, t_1)] d\theta \]
\[ + a(x_0, t_0) \int_{t_1}^{t_2} \int_{x} D^4 u_1(y + (\Delta t)^{1/4}, s) - D^4 u_1(y, s) ds = 0. \]

Integrating the above equality with respect to \( y \) over \( (x, x + (\Delta t)^{1/4}) \), we get
\[ (\Delta t)^{1/2}(Du_1(x^*, t_2) - Du_1(x^*, t_1)) = a(x_0, t_0) \int_{t_1}^{t_2} \int_{x} [D^4 u_1(y + (\Delta t)^{1/4}, s) - D^4 u_1(y, s)] dy \, ds. \]

Hence,
\[ |Du_1(x^*, t_2) - Du_1(x^*, t_1)| \leq C|t_1 - t_2|^{1/4} \iint_{S_{t_2}} (D^4 u_1)^2 \, dx \, dt, \]

where \( x^* = y^* + \theta^*(\Delta t)^{1/4}, y^* \in (x, x + (\Delta t)^{1/4}), \theta \in (0, 1) \). This and (2.19) yield the desired conclusion and the proof is complete. \( \square \)

**Lemma 2.3 (Caccioppoli type inequality).**

\[ \sup_{(t_0 - (R/2)^4, t_0 + (R/2)^4)} \int_{B_{R/2}(x_0)} |Du_1(x, t) - (Du_1)_R|^2 \, dx + \iint_{S_{t_2}} (D^3 u_1)^2 \, dx \, dt \]
\[ \leq C \int_{S_R} |Du_1(t, x) - (Du_1)_R|^2 \, dx \, dt, \]
\[ \sup_{(t_0 - (R/2)^4, t_0 + (R/2)^4)} \int_{B_{R/2}(x_0)} |D^2 u_1|^2 \, dx + \iint_{S_{t_2}} (D^4 u_1)^2 \, dx \, dt \]
\[ \leq C \int_{S_R} |D^2 u_1|^2 \, dx \, dt \leq \frac{C}{R^6} \iint_{S_{2R}} |Du_1(t, x) - (Du_1)_R|^2 \, dx \, dt. \]

**Proof.** For simplicity, we only prove the first inequality, since the other can be shown similarly. Choose a cut-off function \( \chi(x) \) defined on \((x_0 - R, x_0 + R)\) such that \( \chi(x) = 1 \) in \((x_0 - \frac{R}{2}, x_0 + \frac{R}{2})\) and
\[ |D\chi| \leq \frac{C}{R}, \quad |D^2 \chi| \leq \frac{C}{R^2}, \]
\[ |D^3 \chi| \leq \frac{C}{R^3}, \quad |D^4 \chi| \leq \frac{C}{R^4}. \]

Let \( g(t) \in C_{\infty}^2((t_0, +\infty) \) with \( 0 \leq g(t) \leq 1, 0 \leq g'(t) \leq \frac{C}{R} \) and \( g(t) = 1 \) for \( t \geq t_0 - \left(\frac{R}{2}\right)^4 \). Multiplying Eq. (2.13) by \( g(t)D[\chi^4(Du_1 - (Du_1)_R)] \) and then integrating the resulting relation over \((t_0 - R^4, t) \times (x_0 - R, x_0 + R)\), we have
\[ \int_{t_0 - R^4}^{t} g(s) \, ds \int_{B_R(x_0)} \frac{\partial u_1}{\partial t} D[\chi^4(Du_1 - (Du_1)_R)] \, dx \]
\[ + a(x_0, t_0) \int_{t_0 - R^4}^{t} g(s) \, ds \int_{B_R(x_0)} D^4 u_1 D[\chi^4(Du_1 - (Du_1)_R)] \, dx = 0. \]
It follows from integrating by parts and using the boundary value condition (2.14),

\[
\frac{1}{2} \int_{B_R(x_0)} g(s) \chi_4 |Du_1(t, x) - (Du_1)_R|^2 dx + a(x_0, t_0) \int_{t_0-R^4}^t g(s) ds \int_{B_R(x_0)} D^3 u_1 D[\chi_4 D^2 u_1 + D\chi_4 (Du_1 - (Du_1)_R)] dx \\
= \frac{1}{2} \int_{t_0-R^4}^t g(s) ds \int_{B_R(x_0)} D^3 u_1 D[\chi_4 D^2 u_1 + D\chi_4 (Du_1 - (Du_1)_R)] dx.
\]

Thus

\[
\frac{1}{2} \int_{B_R(x_0)} g(s) \chi_4 |Du_1(t, x) - (Du_1)_R|^2 dx + a(x_0, t_0) \int_{t_0-R^4}^t g(s) ds \int_{B_R(x_0)} 8\chi^3 \chi' D^3 u_1 D^2 u_1 dx\\
+ a(x_0, t_0) \int_{t_0-R^4}^t g(s) ds \int_{B_R(x_0)} (4\chi^3 \chi'' + 12\chi^2 (\chi')^2) [Du_1 - (Du_1)_R] D^3 u_1 dx \\
= \frac{1}{2} \int_{t_0-R^4}^t g(s) ds \int_{B_R(x_0)} \chi_4 |Du_1 - (Du_1)_R|^2 dx.
\]

By Cauchy’s inequality, we have

\[
\left| 8 \int_{t_0-R^4}^t g(s) ds a(x_0, t_0) \chi_4 \chi' D^3 u_1 D^2 u_1 dx ds \right| \leq \frac{1}{4} a(x_0, t_0) \int_{t_0-R^4}^t g(s) ds \int_{B_R(x_0)} \chi_4 (D^3 u_1)^2 dx ds \\
+ C \int_{t_0-R^4}^t g(s) ds \chi_2 (\chi')^2 (D^3 u_1)^2 dx ds.
\]

Noticing that

\[
\int_{t_0-R^4}^t g(s) ds \chi_2 (\chi')^2 (D^3 u_1)^2 dx ds = - \int_{t_0-R^4}^t g(s) ds (Du_1 - (Du_1)_R) D(\chi^2 (\chi')^2 D^2 u_1) dx ds \\
= - \int_{t_0-R^4}^t g(s) ds (Du_1 - (Du_1)_R) \chi^2 (\chi')^2 D^3 u_1 dx ds
\]
Lemma 2.4. Assume that

\[ |a(x, t) - a(x_0, t_0)| \leq a_\sigma \{ |t - t_0|^{\sigma/4} + |x - x_0|^{\sigma} \}, \quad x \in B_R(x_0), \ t \in (t_0 - R^4, t_0 + R^4). \]

Then for any \( \rho \in (0, R) \),

\[ \frac{1}{\rho^6} \iint_{S_{\rho}} \left( |Du_1 - (Du_1)_R|^2 + \rho^4 |D^3u_1|^2 \right) \, dx \, dt \leq \frac{C}{R^6} \iint_{S_R} \left( |Du_1 - (Du_1)_R|^2 + R^4 |D^3u_1|^2 \right) \, dx \, dt. \]

**Proof.** One needs only to check the inequality for \( \rho \leq \frac{R}{2} \). From Lemmas 2.2 and 2.3, we have

\[
\frac{1}{\rho^6} \iint_{S_{\rho}} |Du_1 - (Du_1)_R|^2 \, dx \, dt \leq C \sup_{(t_0 - (\xi)^2, t_0 + (\xi)^2)_{B_{\frac{R}{2}}(x_0)}} |D^2u_1|^2 \, dx + C \iint_{S_R} |D^4u_1|^2 \, dx \, dt
\]

\[ \leq \frac{C}{R^6} \iint_{S_R} |Du_1 - (Du_1)_R|^2 \, dx \, dt. \]

On the other hand,

\[
\iint_{S_{\rho}} \rho^4 |D^3u_1|^2 \, dx \, dt \leq C_1 \iint_{S_{\rho}} \rho^6 (D^4u_1)^2 \, dx \, dt + C_2 \iint_{S_{\rho}} \rho^2 (D^2u_1)^2 \, dx \, dt
\]

\[ \leq C_1 \rho^6 \iint_{S_{R/2}} (D^4u_1)^2 \, dx \, dt + C_2 \rho^6 \sup_{(t_0 - (\xi)^2, t_0 + (\xi)^2)_{B_{R/2}(x_0)}} (D^2u_1)^2 \, dx
\]

\[ \leq C \left( \frac{\rho}{R} \right)^6 \iint_{S_{R/2}} R^2 (D^2u_1)^2 \, dx \, dt
\]

\[ \leq C \left( \frac{\rho}{R} \right)^6 \left[ \iint_{S_R} R^4 (D^3u_1)^2 \, dx \, dt + \iint_{S_R} (Du_1 - (Du_1)_R)^2 \, dx \, dt \right]. \]

The conclusion of the lemma follows at once. \( \square \)
Lemma 2.5. For \( \lambda \in (5, 6) \),
\[
\varphi(\rho) \leq C_\lambda \left( \varphi(R_0) + \sup_{S_{R_0}} |f| \right) \rho^\lambda, \quad \rho \leq R_0 = \min(\text{dist}(x_0, \partial \Omega), t_0^{1/4}),
\]
where \( C_\lambda \) depends on \( \lambda \), \( R_0 \) and the known quantities.

By Lemma 2.4,
\[
\varphi(\rho) = \int \int \left( |D\rho - (D\rho)_\rho|^2 + \rho^4|D^3u|^2 \right) dx \, dt
\]
\[
\leq 2 \int \int \left( |D\rho_1 - (D\rho_1)_\rho|^2 + \rho^4|D^3u_1|^2 \right) dx \, dt + 2 \int \int \left( |D\rho_2 - (D\rho_2)_\rho|^2 + \rho^4|D^3u_2|^2 \right) dx \, dt
\]
\[
\leq C \left( \frac{\rho}{R} \right)^6 \int \int \left( |D\rho - (D\rho)_\rho|^2 + R^4|D^3u|^2 \right) dx \, dt + C \int \int \left( |D\rho_1|^2 + R^4|D^3u_1|^2 \right) dx \, dt
\]
\[
\leq C \left[ \left( \frac{\rho}{R} \right)^6 + R^{2\sigma} \right] \varphi(R) + C \sup_{S_{R_0}} |f|^2 R^9.
\]
The conclusion follows immediately from [8].

Proof of Theorem 2.1. Similar to the discussion about the Campanato spaces in [8], we first conclude from Lemma 2.5 that
\[
|Du_\varepsilon(x_1, t_1) - Du_\varepsilon(x_2, t_2)| \leq C \left( |x_1 - x_2|^{\alpha/2} + |t_1 - t_2|^{\alpha/8} \right).
\]
(2.20)
The conclusion follows immediately from the classical theory, since we can transform Eq. (2.1) into the form
\[
\frac{\partial u_\varepsilon}{\partial t} + a_1(x, t)D^4u_\varepsilon + b_1(x, t)D^3u_\varepsilon + a_2(x, t)D^2u_\varepsilon + b_2(x, t)Du_\varepsilon = 0,
\]
where the Hölder norms on
\[
a_1(x, t) = km_\varepsilon (u_\varepsilon(x, t)), \quad b_1(x, t) = km_\varepsilon (u_\varepsilon(x, t)) Du_\varepsilon(x, t),
\]
\[
a_2(x, t) = -m_\varepsilon(u_\varepsilon) A'(u_\varepsilon(x, t)), \quad b_2(x, t) = -m_\varepsilon(u_\varepsilon) A'(u_\varepsilon) Du_\varepsilon - \gamma B'(u_\varepsilon(x, t))
\]
have been estimated in the above discussion. The proof is complete. \( \square \)

3. Existence

After the discussion of the regularized problem, we can now turn to the investigation of the existence of weak solutions of the problem (1.1)–(1.3). The main existence result is the following

Theorem 3.1. Assume that \( u_0 \in H^1_0(I) \cap H^3(I) \). Then the problem (1.1)–(1.3) admits at least one weak solution.

Proof. Let \( u_\varepsilon \) be the approximate solution of the problem (2.1)–(2.3) constructed in the previous section. Using the estimates (2.4), (2.6), (2.7), (2.20), we can extract a subsequence from \( \{u_\varepsilon\} \), denoted also by \( \{u_\varepsilon\} \), such that
\[
u_\varepsilon(x, t) \to u(x, t) \quad \text{uniformly in } \mathring{Q}_T,
\]
\[
Du_\varepsilon(x, t) \to Du(x, t) \quad \text{uniformly in } P,
\]
and the limiting function \( u \in C^{1/4, 1/16} (\mathring{Q}_T), Du \in C^{1/4, 1/16} (P) \). By (2.6), we also have \( u \in L^\infty(0, T; H^1(I)) \).

Now, let \( \delta > 0 \) be fixed and set \( P_\delta = \{ (x, t); |u|^\alpha(x, t) > \delta \} \). We choose \( \varepsilon(\delta) > 0 \), such that
\[
\left( |u_\varepsilon|^2(x, t) + \varepsilon \right)^2 \geq \frac{\delta}{2}, \quad (x, t) \in P_\delta, \quad 0 < \varepsilon < \varepsilon_0(\delta).
\]
(3.1)
Then from (2.5)
\[\int\int_{P_\delta} (D^3 u_\varepsilon)^2 \, dx \, dt \leq \frac{C}{\delta}, \tag{3.2}\]
where the constant \(C\) is independent of \(\varepsilon\) and \(\delta\). By employing a diagonal selection, we obtain a subsequence from \(\{u_\varepsilon\}\), denoted also by \(\{u_\varepsilon\}\), such that
\[D^3 u_\varepsilon \to D^3 u, \quad \text{weakly in } L^2(P_\delta).\]

Noting that
\[\int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt \leq \int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt + \int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt + \frac{1}{2} \int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt\]
\[+ \frac{1}{2} \int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt,\]
hence
\[\int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt \leq 2 \int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt + \int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt.\]

This and the fact that
\[\lim_{\varepsilon \to 0} \int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt = 0,\]
\[\lim_{\varepsilon \to 0} \int\int_{P_\delta} |(|u_\varepsilon|^2 + \varepsilon)^2 - |u|^n|^2 (D^3 u_\varepsilon)^2 \, dx \, dt = 0,\]
yield
\[\int\int_{P_\delta} |u|^n (D^3 u)^2 \, dx \, dt \leq \lim_{\varepsilon \to 0} \int\int_{P_\delta} (|u_\varepsilon|^n + \varepsilon)(D^3 u_\varepsilon)^2 \, dx \, dt \leq C.\]

To prove the integral equality in the definition of solutions, it suffices to pass the limit as \(\varepsilon \to 0\) in
\[- \int_0^1 u_\varepsilon(x, T) \varphi(x, T) \, dx + \int_0^1 u_0 \varphi(x, 0) \, dx + \int\int_{Q_r} u_\varepsilon \frac{\partial \varphi}{\partial t} \, dx \, dt\]
\[+ \int\int_{Q_r} (|u_\varepsilon|^2 + \varepsilon)^2 (kD^3 u_\varepsilon - A'(u_\varepsilon)D\varphi) \, dx \, dt - \gamma \int\int_{Q_r} B(u_\varepsilon) D\varphi \, dx \, dt = 0.\]
The limits

\[
\lim_{\varepsilon \to 0} \int_0^1 u_{\varepsilon}(x, T) \phi(x, T) \, dx = \int_0^1 u(x, T) \phi(x, T) \, dx,
\]

\[
\lim_{\varepsilon \to 0} \int_0^1 u_0(x) \phi(x, 0) \, dx = \int_0 u_0(x) \phi(x, 0) \, dx,
\]

\[
\lim_{\varepsilon \to 0} \int_0^1 \frac{\partial \phi}{\partial t} \, dx \, dt = \int_0^1 \frac{\partial \phi}{\partial t} \, dx \, dt,
\]

\[
\lim_{\varepsilon \to 0} \int_0^1 B(u_{\varepsilon}) \phi \, dx \, dt = \int_0^1 B(u) \phi \, dx \, dt,
\]

are obvious. It remains to show

\[
\lim_{\varepsilon \to 0} \int_0^1 \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} D^3 u_{\varepsilon} \phi \, dx \, dt = \int_0^1 |u|^n D^3 u \phi \, dx \, dt,
\]

(3.3)

\[
\lim_{\varepsilon \to 0} \int_0^1 \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} A'(u_{\varepsilon}) Du_{\varepsilon} \phi \, dx \, dt = \int_0^1 |u|^n A'(u) Du \phi \, dx \, dt.
\]

(3.4)

In fact, for any fixed \(\delta > 0\),

\[
\left| \int_0^1 \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} D^3 u_{\varepsilon} \phi \, dx \, dt - \int_P |u|^n D^3 u \phi \, dx \, dt \right| 
\leq \left| \int_0^1 \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} D^3 u_{\varepsilon} \phi \, dx \, dt - \int_{P_\delta} |u|^n D^3 u \phi \, dx \, dt \right| 
+ \left| \int_{Q_T \setminus P_\delta} \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} D^3 u_{\varepsilon} \phi \, dx \, dt \right| + \left| \int_{P \setminus P_\delta} |u|^n D^3 u \phi \, dx \, dt \right|.
\]

From the estimates (2.5) and (3.1), we have

\[
\left| \int_{Q_T \setminus P_\delta} \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} D^3 u_{\varepsilon} \phi \, dx \, dt \right| \leq C(\delta + \varepsilon) \sup |D\phi|, \quad 0 < \varepsilon < \varepsilon_0(\delta),
\]

\[
\left| \int_{P \setminus P_\delta} |u|^n D^3 u \phi \, dx \, dt \right| \leq C \delta \sup |D\phi|,
\]

\[
\left| \int_{P_\delta} \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} D^3 u_{\varepsilon} \phi \, dx \, dt - \int_{P_\delta} |u|^n D^3 u \phi \, dx \, dt \right| 
\leq \int_{P_\delta} \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} |D^3 u_{\varepsilon}| |D\phi| \, dx \, dt + \left| \int_{P_\delta} |u|^n (D^3 u_{\varepsilon} - D^3 u) \phi \, dx \, dt \right| 
\leq \sup \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} - |u|^n |D\phi| \frac{C}{\sqrt{\delta}} + \left| \int_{P_\delta} |u|^n (D^3 u_{\varepsilon} - D^3 u) \phi \, dx \, dt \right|
\]

and hence

\[
\lim_{\varepsilon \to 0} \left| \int_0^1 \left( |u_{\varepsilon}|^2 + \varepsilon \right)^{\frac{n}{2}} D^3 u_{\varepsilon} \phi \, dx \, dt - \int_P |u|^n D^3 u \phi \, dx \, dt \right| \leq C \delta \sup |D\phi|.
\]
By the arbitrariness of $\delta$, we see that the limit (3.3) holds.

Finally, from the uniform convergence of $u_{\epsilon}$ to $u$, we immediately obtain

$$
\lim_{\epsilon \to 0} \int \int_{Q_T} \left( |u_{\epsilon}|^2 + \epsilon \right)^{\frac{3}{2}} DA(u_{\epsilon}) D\varphi \, dx \, dt
$$

$$
= \lim_{\epsilon \to 0} \int \int_{Q_T} DH(u_{\epsilon}) D\varphi \, dx \, dt
$$

$$
= \lim_{\epsilon \to 0} \int_0^T \int H(u_{\epsilon}(1, t)) D\varphi(1, t) \, dx - \lim_{\epsilon \to 0} \int_0^T \int H(u_{\epsilon}(0, t)) D\varphi(0, t) \, dx - \lim_{\epsilon \to 0} \int \int_{Q_T} H(u_{\epsilon}) D^2 \varphi \, dx \, dt
$$

$$
= \int_0^T \int \left( u(1, t) \right) D\varphi(1, t) \, dx - \int_0^T \int \left( u(0, t) \right) D\varphi(0, t) \, dx - \int \int_{Q_T} H(u) D^2 \varphi \, dx \, dt
$$

where $H(s) = \int_0^s (|s|^2 + \epsilon)^{\frac{3}{2}} A'(s) \, ds$. The proof is complete. $\square$

4. Nonnegativity

Just as mentioned by several authors, it is much interesting to discuss the nonnegativity of solutions.

**Theorem 4.1.** The weak solutions $u$ obtained in Section 3 satisfy $u(x, t) \geq 0$, if $u_0(x) \geq 0$.

**Proof.** Suppose the contrary, that is, the set

$$
E = \{(x, t) \in Q_T; \ u(x, t) < 0\}
$$

(4.1)

is nonempty.

For any fixed $\delta > 0$, choose a $C^\infty$ function $H_\delta(s)$ such that $H_\delta(s) = -\delta$ for $s \geq -\delta$, $H_\delta(s) = -1$, for $s \leq -2\delta$ and that $H_\delta(s)$ is nondecreasing for $-2\delta < s < -\delta$. Also, we extend the function $u(x, t)$ to be defined in the whole plane $\mathbb{R}^2$ such that the extension $\bar{u}(x, t) = 0$ for $t \geq T + 1$ and $t \leq -1$. Let $\alpha(s)$ be the kernel of mollifier in one dimension, that is, $\alpha(s) \in C^\infty(\mathbb{R})$, supp $\alpha = [-1, 1]$, $\alpha(s) > 0$ in $(-1, 1)$, and $\int_{-1}^1 \alpha(s) \, ds = 1$. For any fixed $k > 0$, $\delta > 0$, define

$$
uh(x, t) = \int_{\mathbb{R}} \bar{u}(s, x) \alpha_h(t - s) \, ds,$$

$$
\beta_h(t) = \int_t^{+\infty} \alpha \left( \frac{s - T}{T - \delta} \right) \frac{1}{T - \delta} \, ds,
$$

where $\alpha_h(s) = \frac{1}{h} \alpha(\frac{s}{h})$.

The function

$$
\varphi^h_b(x, t) \equiv \left[ \beta_h(t) H_\delta(\nuh) \right]^h
$$

is clearly an admissible test function, that is the following integral equality holds
\[- \int_0^1 u(x, T) \varphi_h^h(T, x) \, dx + \int_0^1 u_0(x) \varphi_h^h(x, 0) \, dx + \int_{\mathcal{Q}_T} \frac{\partial \varphi_h^h}{\partial t} \, dx \, dt \]
\[+ \int_{\mathcal{Q}_T} m(u)(k \Delta^3 u - D A(u)) D \varphi_h^h \, dx \, dt - \gamma \int_{\mathcal{Q}_T} B(u) D \varphi_h^h \, dx \, dt = 0. \quad (4.2)\]

To proceed further, we analyze the properties of the test function $\varphi_h^h(x, t)$. The definition of $\beta_\delta(t)$ implies that
\[\varphi_h^h(x, t) = 0, \quad t \geq T - \frac{\delta}{2}, \quad h < \frac{\delta}{2}. \quad (4.3)\]

Since $\bar{u}(x, t)$ is continuous, for fixed $\delta$, there exists $\eta_1(\delta) > 0$, such that
\[u^h(x, t) \geq -\frac{\delta}{2}, \quad t \leq \eta_1(\delta), \quad 0 \leq u \leq 1, \quad h < \eta_1(\delta), \quad (4.4)\]

which together with the definition of $\beta_\delta(t), H_\delta(s)$ imply that
\[H_\delta(u^h(x, t)) = -\delta, \quad t \leq \eta_1(\delta), \quad 0 \leq x \leq 1, \quad h < \eta_1(\delta), \quad (4.5)\]

and hence
\[\varphi_\delta^h = -\delta, \quad t \leq \frac{1}{2} \eta_1(\delta), \quad 0 \leq x \leq 1, \quad h < \frac{1}{2} \eta_1(\delta). \quad (4.6)\]

We note also that for any functions $f(t), g(t) \in L^2(\mathbb{R})$,
\[\int_{\mathbb{R}} f(t)g^h(t) \, dt = \int_{\mathbb{R}} f(t) \, dt \int_{\mathbb{R}} g(s) \alpha_h(t - s) \, ds = \int_{\mathbb{R}} f(t) \, dt \int_{\mathbb{R}} g(s) \alpha_h(s - t) \, ds \]
\[= \int_{\mathbb{R}} g(s) \, ds \int_{\mathbb{R}} f(t) \alpha_h(s - t) \, dt = \int_{\mathbb{R}} f^h(t)g(t) \, dt.\]

Taking this into account and using (4.3), (4.5), (4.6), we have
\[\int_{\mathcal{Q}_T} u \frac{\partial}{\partial t} \varphi_\delta^h \, dx \, dt - \int_{-\infty}^{+\infty} dt \int_0^1 u \left[ \frac{\partial}{\partial t} \left( \beta_\delta(t) H_\delta(u^h) \right) \right]^h \, dx = \int_{\mathcal{Q}_T} \frac{\partial}{\partial t} \left( \beta_\delta(t) H_\delta(u^h) \right) \, dx \, dt \]

and hence by integrating by parts
\[\int_{\mathcal{Q}_T} u^h \frac{\partial}{\partial t} \left( \beta_\delta(t) H_\delta(u^h) \right) \, dx \, dt = \int_0^1 u^h(x, T) \beta_\delta(T) H_\delta(u^h(x, T)) \, dx - \int_0^1 u^h(x, 0) \beta_\delta(0) H_\delta(u^h(x, 0)) \, dx \]
\[+ \int_{\mathcal{Q}_T} \beta_\delta(t) H_\delta(u^h) \frac{\partial u^h}{\partial t} \, dx \, dt \]
\[= \delta \int_0^1 u^h(x, 0) \, dx - \int_{\mathcal{Q}_T} \beta_\delta(t) \frac{\partial}{\partial t} F_\delta(u^h) \, dx \, dt,\]

where $F_\delta(s) = \int_0^s H_\delta(\sigma) \, d\sigma$.

Again by (4.5)
\[F_\delta(u^h(x, 0)) = \int_0^1 H_\delta(\sigma) \, d\sigma = \int_0^1 H_\delta(\lambda u^h(x, 0)) \, d\lambda \cdot u^h(x, 0) = -\delta u^h(x, 0)\]
and hence
\[
\int_Q \int R \left( \frac{\partial}{\partial t} (\beta_\delta(t) H_\delta(u^h)) \right) dx dt = \delta \int_0^1 (u^h(x, 0)) dx + \int_0^1 \beta_\delta(0) F_\delta(u^h(x, 0)) dx + \int_Q F_\delta(u^h) \beta'_\delta(t) dx dt
\]
\[
= -\frac{1}{T - \delta} \int_Q F_\delta(u^h) \left( \frac{t - \frac{T}{2}}{T - \delta} \right) dx dt.
\] (4.7)

From (4.3), (4.6) it is clear that
\[
- \int_0^1 u(x, T) \phi^h_\delta(T, x) dx = 0, \quad 0 < h < \frac{1}{2} \eta_1(\delta),
\] (4.8)
\[
\int_0^1 u_0(x) \phi^h_\delta(x, 0) dx = -\delta \int_0^1 u_0(x) dx.
\] (4.9)

Substituting (4.7)–(4.9) into (4.2), we have
\[
- \frac{2}{T - 2\delta} \int_Q F_\delta(u^h) \left( \frac{t - \frac{T}{2}}{T - \delta} \right) dx dt - \delta \int_0^1 u_0(x) dx
\]
\[
+ \int_P m(u) (k D^3 u - DA(u)) D\phi^h_\delta dx dt - \gamma \int_Q B(u) D\phi^h_\delta dx dt = 0.
\] (4.10)

By the uniform continuity of \( u(x, t) \) in \( \overline{Q} \), there exists \( \eta_2(\delta) > 0 \), such that
\[ u(x, t) \geq -\frac{\delta}{2}, \quad \forall (x, t) \in P^\delta, \] (4.11)
where \( P^\delta = \{(x, t); \text{dist}((x, t), P) < \eta_2(\delta)\} \). Here we have used the fact that \( u(x, t) > 0 \) in \( P \). Thus
\[ H_\delta(u^h(x, t)) = -\delta, \quad \forall (x, t) \in P^{\delta/2}, \quad 0 < h < \frac{1}{2} \eta_2(\delta), \]
where \( P^{\delta/2} = \{(x, t); \text{dist}((x, t), P) < \frac{1}{2} \eta_2(\delta)\} \).

This and the definition of \( u^h, H_\delta(s) \) show that the function \( \phi^h_\delta(x, t) \) is only a function of \( t \) in \( P \), whenever \( h < \frac{1}{2} \eta_2(\delta) \). Therefore
\[ D\phi^h_\delta(x, t) = 0, \quad \forall (x, t) \in P, \quad 0 < h < \frac{1}{2} \eta_2(\delta), \] (4.12)
and so (4.10) becomes
\[
-\delta \int_0^1 u_0(x) dx - \frac{2}{T - 2\delta} \int_Q F_\delta(u^h) \left( \frac{2t - T}{T - 2\delta} \right) dx dt = 0,
\] (4.13)
where \( \eta(\delta) = \min(\eta_1(\delta), \eta_2(\delta)) \). Letting \( h \) tend to zero, we have
\[
-\delta \int_0^1 u_0(x) dx - \frac{2}{T - 2\delta} \int_Q F_\delta(u) \left( \frac{2t - T}{T - 2\delta} \right) dx dt = 0.
\] (4.14)

From the definition of \( F_\delta(s), H_\delta(s) \), it is easily seen that
\[ F_\delta(u(x, t)) \to -\chi_E(x, t) u(x, t) \quad (\delta \to 0) \]
and so by letting $\delta$ tend to zero in (4.14), we have
\[
\iint_E |u(x,t)| \alpha \left( \frac{2t-T}{T} \right) \, dx \, dt = 0,
\]
which contradicts the fact that $\alpha \left( \frac{2t-T}{T} \right) > 0$ for $0 < t < T$. We have thus proved the theorem. □

5. Finite speed of propagation of perturbations

As is well known, one of the important properties of solutions of the porous medium equation is the finite speed of propagation of perturbations. So from the point of view of physical background, it seems to be natural to investigate this property for thin film equation or Cahn–Hilliard equation. F. Bernis and A. Friedman [4], F. Bernis [3] considered this property for thin film equation. On the other hand, the mathematical description of this property is that if $\text{supp} \, u_0$ is bounded, then for any $t > 0$, $\text{supp} \, u(\cdot, t)$ is also bounded. So from the point of view of mathematics, this problem seems to be quite interesting. We adopt the weighted energy method and the main technical tools are weighted Nirenberg’s inequality and Hardy’s inequality.

**Theorem 5.1.** Assume $u_0 \in H^1(I) \cap H^3(I)$, $u_0 \geq 0$, $\text{supp} \, u_0 \subset [x_1, x_2]$, $0 < x_1 < x_2 < 1$, and $u$ is the weak solution of the problem (1.1)–(1.3), then for any fixed $t > 0$, we have
\[
\text{supp} \, u(x, \cdot) \subset [x_1(t), x_2(t)] \cap [0, 1],
\]
where $x_1(t) = x_1 - C_1 t^\gamma$, $x_2(t) = x_2 + C_2 t^\gamma$, $C_1, C_2, \gamma > 0$.

We need a series of uniform estimates on such approximate solutions $u_\varepsilon$.

**Lemma 5.1.** Let $u$ be the limit function of the approximate solutions, obtained above. Then the following integral inequality holds
\[
\int_0^1 u^{2-n} \, dx + \frac{k}{2} \iint_{Q_t} (D^2 u)^2 \, dx \, ds \leq \int_0^1 u_0^{2-n} \, dx.
\]

**Proof.** Let $u_\varepsilon$ be the solution of the problem (2.1)–(2.3). Denote
\[
g_\varepsilon(u) = \int_0^u \frac{dr}{(|r|^2 + \varepsilon)^{\gamma}}, \quad G_\varepsilon(u) = \int_0^u g_\varepsilon(r) \, dr.
\]
Multiplying both sides of Eq. (2.1) by $g_\varepsilon(u_\varepsilon)$, and then integrating over $Q_t$, we obtain
\[
\int_0^1 G_\varepsilon(u_\varepsilon(x, t)) \, dx + k \iint_{Q_t} (D^2 u_\varepsilon)^2 \, dx \, ds + \int_0^1 (3u_\varepsilon^2 - 1) (Du_\varepsilon)^2 \, dx \, ds
\]
\[
+ \iint_{Q_t} \frac{1}{(|u_\varepsilon|^2 + \varepsilon)^{\gamma}} u_\varepsilon^2 Du_\varepsilon \, dx \, ds = \int_0^1 G_\varepsilon(u_0(x)) \, dx. \tag{5.1}
\]
Using Hölder’s inequality and by (2.8), we have
\[
\int_0^1 G_\varepsilon(u_\varepsilon(x, t)) \, dx + k \iint_{Q_t} (D^2 u_\varepsilon)^2 \, dx \, ds \leq \int_0^1 G_\varepsilon(u_0(x)) \, dx + C \iint_{Q_t} (Du_\varepsilon)^2 \, dx \, ds.
\]
By Poincaré’s inequality, as \( k \) enough large, we have
\[
\int_0^1 G_\varepsilon(u_\varepsilon(x,t)) \, dx + \frac{k}{2} \int_0^1 (D^2 u_\varepsilon)^2 \, dx = \frac{1}{2} G_\varepsilon(u_{0\varepsilon}(x)) \, dx.
\]
Letting \( \varepsilon \to 0 \) and using the fact that \( G_\varepsilon(u_\varepsilon) \to u^2 - n/(1 - n)(2 - n) \) and \( u_\varepsilon \to u \) pointwise and the lower semi-continuity of the integrals, we immediately get the conclusion of the lemma. The proof is complete.

**Lemma 5.2.** Let \( u \) be the limit function of the approximate solutions, obtained above. Then for any \( y \in \mathbb{R}^+ \), the following integral inequality holds
\[
\int_0^1 (x - y)^\alpha u^{2-n} \, dx + \frac{k}{2} \int_0^1 (x - y)^\alpha (D^2 u)^2 \, dx \leq C \int_0^1 (x - y)^\alpha g(u) \, dx
\]
where \( C \) depends only on \( n, u_0 \) and \( \alpha \geq 2p - 1 \), where \( (x - y)^\alpha \) denotes the positive part of \( x - y \).

**Proof.** Let \( g_\varepsilon(u) \) and \( G_\varepsilon(u) \) be defined as in the proof of Lemma 5.1. Let \( u_\varepsilon \) be the approximate solutions derived from the problem (2.1)–(2.3). Then, using Eq. (2.1) and integrating by parts, we get
\[
\int_0^1 (x - y)^\alpha u_\varepsilon^{2-n} \, dx + \frac{k}{2} \int_0^1 (x - y)^\alpha (D^2 u_\varepsilon)^2 \, dx ds \leq C \int_0^1 (x - y)^\alpha g_\varepsilon(u_\varepsilon) \, dx + C \int_0^1 (x - y)^\alpha (Du_\varepsilon)^2 \, dx ds + C \int_0^1 (x - y)^\alpha |u_0|^{2-n} \, dx
\]
As for \( I_1 \), integrating by parts, we have
\[
I_1 = \int \left[ kD^3 u_\varepsilon - A'(u_\varepsilon)D u_\varepsilon \right] (x - y)^\alpha D u_\varepsilon \, dx ds
\]
\[
= -\int kD^2 u_\varepsilon D[(x - y)^\alpha D u_\varepsilon] \, dx ds - \int A'(u_\varepsilon)(x - y)^\alpha (D u_\varepsilon)^2 \, dx ds
\]
\[
= -\int kD^2 u_\varepsilon (x - y)^\alpha D^2 u_\varepsilon \, dx ds - \int D^2 u_\varepsilon \alpha (x - y)^\alpha D u_\varepsilon \, dx ds
\]
\[
= -\int D^2 u_\varepsilon \alpha (x - y)^\alpha D u_\varepsilon \, dx ds
\]
As for $I_2$, we have

\[
I_2 = \int_Q \left( |u_\varepsilon|^2 + \varepsilon \right)^{\alpha} \left[ kD^2 u_\varepsilon - A'(u_\varepsilon) Du_\varepsilon \right] \left[ \alpha(x-y)^\alpha_{+} g_\varepsilon(u_\varepsilon) \right] dx \, ds
\]

\[
= - \int_Q kD^2 u_\varepsilon \left[ (|u_\varepsilon|^2 + \varepsilon)^{\frac{\alpha}{2}} g_\varepsilon(u_\varepsilon) \alpha(x-y)^{\alpha_{+}-1} \right] dx \, ds
\]

\[
- \int_Q \alpha(x-y)^{\alpha_{+}-1} (|u_\varepsilon|^2 + \varepsilon)^{\frac{\alpha}{2}} A'(u_\varepsilon) g_\varepsilon(u_\varepsilon) Du_\varepsilon dx \, ds
\]

\[
- \int_Q \alpha(x-y)^{\alpha_{+}-1} (|u_\varepsilon|^2 + \varepsilon)^{\frac{\alpha}{2}} A'(u_\varepsilon) g_\varepsilon(u_\varepsilon) Du_\varepsilon dx \, ds.
\]

Therefore

\[
\int_0^1 (x-y)^{\alpha_{+}} G_\varepsilon(u_\varepsilon) dx - \int_0^1 (x-y)^{\alpha_{+}} G_\varepsilon(u_0) dx + k \int_0^1 \left( |D^2 u_\varepsilon|^2 \right) dx \, ds
\]

\[
= -2k \int_0^1 \alpha(x-y)^{\alpha_{+}} \left( D^2 u_\varepsilon ight)^2 dx \, ds - \int_0^1 A'(u_\varepsilon)(x-y)^{\alpha_{+}} \left( D^2 u_\varepsilon \right)^2 dx \, ds
\]

\[
- \int_0^1 k \left( |u_\varepsilon|^2 + \varepsilon \right)^{\frac{\alpha}{2}} D^2 u_\varepsilon g_\varepsilon(u_\varepsilon) \alpha(x-y)^{\alpha_{+}-1} dx \, ds
\]

\[
- \int_0^1 k \left( |u_\varepsilon|^2 + \varepsilon \right)^{\frac{\alpha}{2}} A'(u_\varepsilon) g_\varepsilon(u_\varepsilon) Du_\varepsilon dx \, ds
\]

\[
- \gamma \int_0^1 B(u_\varepsilon)(x-y)^{\alpha_{+}} \left( \frac{1}{(|u_\varepsilon|^2 + \varepsilon)^{\frac{\alpha}{2}}} D^2 u_\varepsilon \right)^2 dx \, ds
\]

\[
\equiv I_a + I_b + I_c + I_d + I_e + I_f + I_g.
\]

Hölder’s inequality yields

\[
|I_a| \leq \frac{k}{8} \int_Q (x-y)^{\alpha_{+}} \left( D^2 u_\varepsilon \right)^2 dx \, ds + C \int_Q (x-y)^{\alpha_{+}-2} \left( Du_\varepsilon \right)^2 dx \, ds.
\]

Noticing that

\[
\left( |u_\varepsilon|^2 + \varepsilon \right)^{\frac{\alpha}{2}} |g_\varepsilon(u_\varepsilon)| \leq \frac{2}{1-n} |u_\varepsilon|,
\]

using (2.8), we have

\[
|I_b| \leq C \int_Q (x-y)^{\alpha_{+}} \left( Du_\varepsilon \right)^2 dx \, ds \leq \int_Q (x-y)^{\alpha_{+}-2} \left( Du_\varepsilon \right)^2 dx \, ds.
\]
and using Hölder’s inequality again, we get

\[ |I_c| \leq \frac{k}{8} \int_{Q_t} (x-y)^\alpha_{+}(D^2 u_\varepsilon)^2 \, dx \, ds + C_3 \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 4} \left[ \left| u_\varepsilon \right| + \varepsilon \right]^\frac{n}{2} g_\varepsilon(u_\varepsilon)^2 \, dx \, ds \]

\[ \leq \frac{k}{8} \int_{Q_t} (x-y)^\alpha_{+}(D^2 u_\varepsilon)^2 \, dx \, ds + C_3 \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 4} \left| u_\varepsilon \right|^2 \, dx \, ds, \]

and

\[ |I_d| \leq \frac{k}{4} \int_{Q_t} (x-y)^\alpha_{+}(D^2 u_\varepsilon)^2 \, dx \, ds + C_4 \int_{Q_t} (x-y)^{\frac{\alpha}{2} - 2} \left( \left| u_\varepsilon \right|^2 + \varepsilon \right)^{\frac{n}{2} - 1} g_\varepsilon(u_\varepsilon)^2 (Du_\varepsilon)^2 \, dx \, ds \]

\[ \leq \frac{k}{4} \int_{Q_t} (x-y)^\alpha_{+}(D^2 u_\varepsilon)^2 \, dx \, ds + C_4 \int_{Q_t} (x-y)^{\frac{\alpha}{2} - 2} (Du_\varepsilon)^2 \, dx \, ds. \]

Similarly, using (2.8), we have

\[ |I_e| \leq \int_{Q_t} |Du_\varepsilon|^2 (x-y)^{\frac{\alpha}{4} - 2} \, dx \, ds + C_3 \int_{Q_t} (x-y)^\alpha_{+} \left[ \left| u_\varepsilon \right|^2 + \varepsilon \right]^\frac{n}{2} A'(u_\varepsilon) g_\varepsilon(u_\varepsilon)^2 \, dx \, ds \]

\[ \leq \int_{Q_t} |Du_\varepsilon|^2 (x-y)^{\frac{\alpha}{4} - 2} \, dx \, ds + C \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 4} (Du_\varepsilon)^2 \, dx \, ds, \]

\[ |I_f| \leq C \int_{Q_t} |Du_\varepsilon|^2 (x-y)^{\frac{\alpha}{4} - 2} \, dx \, ds + C \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 4} (Du_\varepsilon)^2 \, dx \, ds, \]

and

\[ |I_g| \leq C \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 4} (Du_\varepsilon)^2 \, dx \, ds. \]

Summing up, we have

\[ \int_0^1 (x-y)^\alpha_{+} G_\varepsilon(u_\varepsilon) \, dx - \int_0^1 (x-y)^\alpha_{+} G_\varepsilon(u_0) \, dx + \frac{k}{2} \int_{Q_t} (x-y)^\alpha_{+} (D^2 u_\varepsilon)^2 \, dx \, ds \]

\[ \leq C \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 4} \left| u_\varepsilon \right|^2 \, dx \, ds + C \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 2} (Du_\varepsilon)^2 \, dx \, ds. \]

Letting \( \varepsilon \to 0 \), we immediately get the desired conclusion and complete the proof of the lemma. \( \square \)

**Proof of Theorem 5.1.** For any \( y \geq x_2 \), Lemma 5.2 and Hardy’s inequality imply that for any \( t \in [0, T] \),

\[ \int_{Q_t} (x-y)^\alpha_{+} u^{2-n} \, dx + \frac{k}{2} \int_{Q_t} (x-y)^\alpha_{+} |D^2 u|^2 \, dx \, ds \]

\[ \leq C \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 4} \left| u \right|^2 \, dx \, ds + C \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 2} |Du|^2 \, dx \, ds \]

\[ \leq C \int_{Q_t} (x-y)^{\frac{\alpha}{4} - 2} |Du|^2 \, dx \, ds. \] (5.2)
For any positive number \( m \), define
\[
    f_m(y) = \int_0^1 \int_0^1 (x - y)^m D^2 u(x, s)^2 \, dx \, ds,
\]
\[
    f_0(y) = \int_0^1 |D^2 u|^2 \, dx \, ds.
\]

Then, weighted Nirenberg’s inequality and estimate (5.2) imply that
\[
    f_{2p+1}(y) \leq C \int_Q (x - y)^{2p-1} |Du|^2 \, dx \, ds
\]
\[
    \leq C \int Q \left( \int_0^1 (x - y)^{2p-1} |D^2 u|^2 \, dx \right)^a \left( \int_0^1 (x - y)^{2p-1} |u|^q \, dx \right)^{2(1-a)/q} \, ds
\]
\[
    \leq C \sup_{0 < r < T} \left( \int_0^1 (x - y)^{2p-1} |u|^q \, dx \right)^{2(1-a)/q} r^{1-a} \left( \int Q (x - y)^{2p-1} |D^2 u|^2 \, dx \, ds \right)^a.
\]

Using (5.2) and Hardy’s inequality, we have
\[
    \sup_{0 < r < T} \int_0^1 (x - y)^{2p-1} |u|^q \, dx \leq C \int Q (x - y)^{2p-1} |D^2 u|^2 \, dx \, ds
\]
and hence
\[
    f_{2p-1}(y) \leq C t^{1-a} \left( \int Q (x - y)^{2p-1} |D^2 u|^2 \, dx \, ds \right)^{a+2(1-a)/q},
\]
where \( q = 2 - n \) and \( a = \frac{1 - \frac{2n}{p} - \frac{1}{q}}{\frac{2n}{p} - \frac{1}{q}} \).

Denote \( \lambda = 1 - a, \mu = a + 2(1 - a)/q \), then \( \lambda > 0, 1 < \mu < 3/2 \). Applying Hölder’s inequality, we have
\[
    f_{2p-1}(y) \leq C t^\lambda \left[ \int Q (x - y)^{2p-1} |D^2 u|^2 \, dx \, ds \right]^{\mu}
\]
\[
    \leq C t^\lambda \left[ \int Q (x - y)^{2p+1} |D^2 u|^2 \, dx \, ds \right]^{(2p-1)\mu/(2p+1)} \left[ \int_0^1 |D^2 u|^2 \, dx \, ds \right]^{2\mu/(2p+1)}
\]
\[
    \leq C t^\lambda \left[ f_{2p+1}(y) \right]^{(2p-1)\mu/(2p+1)} \left[ f_0(y) \right]^{2\mu/(2p+1)}.
\]

Therefore
\[
    f_{2p+1}(y) \leq C t^{\lambda/\sigma} \left[ f_0(y) \right]^{2\mu/(2p+1)\sigma}, \quad \sigma = 1 - \frac{2p - 1}{2p + 1} \mu > 0.
\]

Using Hölder’s inequality again, we get
\[
    f_1(y) \leq \left[ f_0(y) \right]^{2p/2p+1} \left[ f_{2p+1}(y) \right]^{1/2p+1} \leq C t^\gamma \left[ f_0(y) \right]^{1+\theta},
\]
where
\[
    \gamma = \frac{\lambda}{(2p + 1)\sigma}, \quad \theta = \frac{2\mu}{(2p + 1)^2\sigma} - \frac{1}{2p + 1} > 0.
\]
Noticing that \( f_1'(y) = -f_0(y) \), we obtain
\[
f_1'(y) \leq -C t^{-\gamma/(\theta+1)} \left[ f_1(y) \right]^{1/(\theta+1)}.
\]
If \( f_1(x_2) = 0 \), then \( \text{supp} u \subset [0, x_2] \). If \( f_1(x_2) > 0 \), then there exists a maximal interval \((x_2, x_2^*)\) in which \( f_1(y) > 0 \) and
\[
\left[ f_1(y)^{\theta/(\theta+1)} \right]' = \frac{\theta}{\theta + 1} f_1(y)^{1/(\theta+1)} \leq -C t^{-\gamma/(\theta+1)}.
\]
Integrating the above inequality over \((x_2, x_2^*)\), we have
\[
f_1(x_2^*)^{\theta/(\theta+1)} - f_1(x_2)^{\theta/(\theta+1)} \leq -C t^{-\gamma/(\theta+1)} (x_2^* - x_2),
\]
which implies that
\[
x_2^* \leq x_2 + C t^{\gamma} \left( f_0(x_2) \right)^{\theta}.
\]
Lemma 5.1 implies that \( f_0(y) \) can be controlled by a constant \( C \) independent of \( y \). Therefore
\[
\text{sup supp } u(\cdot, t) \leq x_2 + C t^{\gamma} \equiv x_2(t).
\]
We have thus completed the proof of Theorem 5.1. \( \Box \)

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References