Asymptotic behavior of solutions of reaction-diffusion equations with nonlocal boundary conditions

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Abstract

This paper is concerned with some dynamical property of a reaction–diffusion equation with nonlocal boundary condition. Under some conditions on the kernel in the boundary condition and suitable conditions on the reaction function, the asymptotic behavior of the time-dependent solution is characterized in relation to a finite or an infinite set of constant steady-state solutions. This characterization is determined solely by the initial function and it leads to the stability and instability of the various steady-state solutions. In the case of finite constant steady-state solutions, the time-dependent solution blows up in finite time when the initial function is greater than the largest constant solution. Also discussed is the decay property of the solution when the kernel function in the boundary condition possesses alternating sign in its domain. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we investigate the dynamics of the following parabolic boundary-value problem with nonlocal boundary condition:

\[
\begin{align*}
  u_t - Lu &= f(x, u) \quad (t > 0, \ x \in \Omega), \\
  Bu &= \int_{\Omega} K(x, y) u(t, y) \, dy \quad (t > 0, \ x \in \partial\Omega), \\
  u(0, x) &= u_0(x) \quad (x \in \Omega),
\end{align*}
\]

(1.1)
where $\Omega$ is a bounded domain in $\mathbb{R}^n \ (n = 1, 2, \ldots)$, $\partial \Omega$ is the boundary of $\Omega$, and $L$ and $B$ are the differential and boundary operators given in the form

$$
Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{j=1}^{n} b_j(x)u_j,
$$

$$
Bu = \alpha_0 \frac{\partial u}{\partial v} + u.
$$

The operator $L$ is uniformly elliptic with smooth coefficients in $\Omega$, $\partial / \partial v$ denotes the outward normal derivative on $\partial \Omega$, and either $u_0 = 0$ (Dirichlet condition) or $u_0 > 0$ (Robin condition). It is assumed that $\partial \Omega$ is smooth and the functions $f(x, \cdot)$, $K(x,y)$ and $u_0(x)$ are Hölder continuous in their respective domains, and $u_0(x)$ satisfies the boundary condition at $t = 0$. It is also assumed that $f(\cdot, u)$ is a $C^1$-function of $u$ for $u$ in bounded subsets of $\Omega$.

The dynamics of problem (1.1) has been treated in [9] where the large time behavior of the solution in relation to the solutions of the corresponding steady-state problem

$$
-Lu = f(x, u) \quad (x \in \Omega),
$$

$$
Bu = \int_{\partial \Omega} K(x,y)u(y) \, dy \quad (x \in \partial \Omega) \tag{1.2}
$$

is investigated. It has been shown in [9] that if $K(x,y)$ satisfies either the condition

$$
K(x,y) \geq 0, \quad \hat{K}(x) < 1 \quad (x \in \partial \Omega, \; y \in \Omega) \tag{1.3}
$$

or the condition

$$
K(x,y) > 0, \quad \hat{K}(x) \leq 1, \quad \hat{K}(x) \neq 1 \quad (x \in \partial \Omega, \; y \in \Omega), \tag{1.4}
$$

where

$$
\hat{K}(x) \equiv \int_{\partial \Omega} K(x,y) \, dy \quad (x \in \partial \Omega), \tag{1.5}
$$

then for a certain class of nonlinear functions $f(x,u)$ the solution $u(t,x)$ of (1.1) converges to a solution of (1.2) as $t \to \infty$.

The purpose of this paper is to investigate the dynamics of (1.1) for the case $\hat{K}(x) = 1$ and $\hat{K}(x) \geq 1$ as well as the case where $K(x,y)$ has both positive and negative values on $\partial \Omega \times \Omega$, including the case $K(x,y) \leq 0$. It turns out that for the case $\hat{K}(x) \equiv 1$ the asymptotic behavior of the solution may be quite different from the case under condition (1.3) or (1.4). For example, if $K(x,y) \geq 0$ and $\hat{K}(x) \equiv 1$, then for a class of functions $f(x,u)$ problem (1.2) has an infinite number of constant solutions and the time-dependent solutions of (1.1) converge to one set of these constant solutions and move away from the other set of constant solutions, depending solely on the initial function $u_0(x)$ (see Theorem 2.2 and Example 2.1). For another class of functions $f(x,u)$, the time-dependent solution may converge to a constant solution or blows up at some finite time, depending again on the initial function (see Theorem 3.2 and Example 3.1). This dynamic behavior characterizes the stability and instability of various constant steady-state solutions.

The nonlocal parabolic boundary-value problem (1.1) in the one-dimensional domain $\Omega = (-\ell, \ell)$ was initiated in Day [1, 2] and has been extended to multi-dimensional domains and more general
equations by a number of investigators (cf. [3, 5, 6, 9, 13]). Most of the discussions in these papers
are in relation to the asymptotic behavior of the time-dependent solution. Recently, this problem
It has also been extended to a parabolic equation with both nonlocal boundary and nonlocal initial
conditions (cf. [4, 10]). A common requirement in the above works is that the kernel \( K(x, y) \)
satisfies condition (1.3) (or \(|\hat{K}(x)| < 1\)). The present work is devoted to the case \( \hat{K}(x) \geq 1 \)
with emphasis on the special case \( \hat{K}(x) = 1 \).

In Section 2 we discuss the asymptotic behavior of the solution of (1.1) in relation to a finite or
an infinite set of constant steady-state solutions when \( K(x, y) \geq 0 \) and \( \hat{K}(x) \equiv 1 \). Section 3 is devoted
tо the blow-up property of the solution when \( K(x, y) \geq 0 \) and \( \hat{K}(x) \geq 1 \). The case where \( K(x, y) \)
has alternating sign is considered in Section 4.

2. Asymptotic behavior of solutions

In this section we investigate the asymptotic behavior of the solution of (1.1) in relation to constant
steady-state solutions of (1.2) under the following condition on \( K(x, y) \):

\[(H_1) \quad K(x, y) \geq 0, \quad \hat{K}(x) \equiv 1 \quad \text{and either} \quad x_0 > 0 \quad \text{or} \quad K(x, y) > 0 \quad (x \in \partial \Omega, \ y \in \Omega).\]

Let \( D_T = (0, T] \times \Omega, \ S_T = (0, T] \times \partial \Omega \) and \( \bar{D}_T = [0, T] \times \bar{\Omega} \), where \( T > 0 \) is an arbitrary finite number
and \( \bar{\Omega} = \Omega \cup \partial \Omega \). Denote by \( C(D_T) \) the set of continuous functions in \( \bar{D}_T \) and by \( C^{1,2}(D_T) \) the set of
functions which are once continuously differentiable in \( t \) and twice continuously differentiable in \( x \)
for \((t, x) \in D_T \). Similar notations are used for other function spaces and other domains.

Consider problem (1.1) in \( \bar{D}_T \). We say that \( u \in C(D_T) \cap C^{1,2}(D_T) \) is a lower solution of (1.1) if

\[
\begin{aligned}
   &\dot{u} - Lu < f(x, \dot{u}) \quad \text{in} \quad D_T, \\
   &B\dot{u} \leq \int_0^T K(x, y)\dot{u}(t, y) \, dy \quad \text{on} \quad S_T, \\
   &\dot{u}(0, x) \leq u_0(x) \quad \text{in} \quad \Omega
\end{aligned}
\]

and \( \bar{u} \in C(\bar{D}_T) \cap C^{1,2}(D_T) \) is an upper solution if it satisfies (2.1) in reversed order. The pair \( \dot{u} \) and
\( \bar{u} \) are said to be ordered if \( \dot{u} \leq \bar{u} \) on \( \bar{D}_T \). Similarly, a pair of functions \( \bar{u}_i, \bar{u}_e \) in \( C(\bar{\Omega}) \cap C^2(\Omega) \) are
called ordered upper and lower solutions of (1.2) if \( \bar{u}_i \leq \bar{u}_e \) on \( \bar{\Omega} \) and they satisfy the corresponding
inequalities (and reversed inequalities) in (2.1) without the time derivative term and the initial
condition. It is known that if \( K(x, y) \geq 0 \) and problem (1.1) has a pair of ordered upper and lower
solutions \( \bar{u} \) and \( \dot{u} \), then there exists a unique solution \( u \) to (1.1) and \( \bar{u} \leq u \leq \dot{u} \) on \( \bar{D}_T \) (cf. [9]). To
investigate the asymptotic behavior of the solution we extend a lemma in [9] to the case \( \hat{K}(x) \equiv 1 \).

Lemma 2.1. Let Hypothesis (H_1) hold, and let \( w \in C(\bar{D}_T) \cap C^{1,2}(D_T) \) and satisfy the relation

\[
\begin{aligned}
   &w_t - Lw + cw \leq 0 \quad \text{in} \quad D_T, \\
   &Bw \leq \int_\Omega K(x, y)w(t, y) \, dy \quad \text{on} \quad S_T, \\
   &w(0, x) \leq 0 \quad \text{in} \quad \Omega
\end{aligned}
\]

where \( c \equiv c(t, x) \) is a bounded function on \( \bar{D}_T \). Then \( w(t, x) \leq 0 \) on \( \bar{D}_T \).
Proof. The proof is a modification of that given in [9] and we sketch it as follows: If the lemma was not true then there would exist a point \((t_0, x_0) \in \overline{D_T}\) such that \(w(t_0, x_0)\) is a positive maximum. By (2.2) and the maximum principle, \(t_0 > 0\) and \(x_0 \in \partial \Omega\) (cf. [12]). This implies that

\[
\partial w \over\partial v (t_0, x_0) + w(t_0, x_0) \leq \int_{\Omega} K(x_0, y)w(t_0, y) \, dy \leq \hat{K}(x_0)w(t_0, x_0).
\]

Since \(\hat{K}(x_0) = 1\) and \((\partial w/\partial v)(t_0, x_0) > 0\) the above relation is impossible if \(x_0 > 0\), and when \(x_0 = 0\) and \(K(x, y) > 0\) it is possible only when \(w(t_0, y) = w(t_0, x_0)\) for all \(y \in \Omega\). In the latter case, \(w(t_0, x)\) is an interior maximum of \(w\) for every \(x \in \Omega\) which contradicts the maximum principle since \(w(t, x)\) cannot be a positive constant. This contradiction leads to the result \(w(t, x) \leq 0\) on \(\overline{D_T}\).

Following the reasoning in the proof of the above lemma we have an analogous result for the elliptic problem (1.2).

Lemma 2.2. Let Hypothesis \((H_1)\) hold, and let \(w_s \in C(\overline{\Omega}) \cap C^2(\Omega)\) and satisfy the relation

\[
-Lw_s + cw_s \leq 0 \quad \text{in} \quad \Omega,
\]

\[
Bw_s \leq \int_{\Omega} K(x, y)w_s(y) \, dy \quad \text{on} \quad \partial \Omega,
\]

where \(c \equiv c(x) \geq 0\). Then \(w_s(x) \leq 0\) on \(\overline{\Omega}\) unless \(c(x) \equiv 0\) and \(w_s(x)\) is a constant.

It is known that if problem \((1.2)\) has a pair of ordered upper and lower solutions \(\hat{u}_s, \hat{u}_s\) then it has a maximal solution \(\bar{u}_s\) and a minimal solution \(u_s\) such that \(\hat{u}_s \leq u_s \leq \bar{u}_s \leq \bar{u}_s\) in \(\overline{\Omega}\) (cf. [9]). The following theorem gives the asymptotic behavior of the time-dependent solution in relation to \(\bar{u}_s\) and \(u_s\).

Theorem 2.3. Let \(\hat{u}_s, \hat{u}_s\) be a pair of ordered upper and lower solutions of (1.2), and let hypothesis \((H_1)\) hold. Denote the solution of (1.1) by \(\bar{u}(t, x)\) when \(u_0 = \hat{u}_s\) and by \(u(t, x)\) when \(u_0 = \hat{u}_s\). Then

\[
\lim_{t \to \infty} \bar{u}(t, x) = \bar{u}_s(x), \quad \lim_{t \to \infty} u(t, x) = u_s(x) \quad (x \in \overline{\Omega}).
\]

For arbitrary \(\hat{u}_s \leq u_0 \leq \hat{u}_s\) the corresponding solution \(u(t, x)\) of (1.1) satisfies the relation

\[
u(t, x) \leq u(t, x) \leq \bar{u}(t, x) \quad (t > 0, x \in \overline{\Omega})
\]

and if \(\bar{u}_s(x) = u_s(x)\) then \(u(t, x) \to \bar{u}_s(x)\) as \(t \to \infty\).

Proof. The proof follows from the same argument as that in [9] using the result of Lemma 2.1. Details are omitted.

To investigate the asymptotic behavior of the solution of (1.1) in relation to constant solutions of (1.2) we prepare the following lemma.

Lemma 2.4. Let hypothesis \((H_1)\) hold, and let \(c_1, c_2\) be constants such that \(c_1 < c_2\) and \(f(x, c_1) = f(x, c_2) = 0\). Then \(u_s = c_2\) is the unique solution of (1.2) in \((c_1, c_2)\) if \(f(x, u) > 0\) in \(\Omega \times (c_1, c_2)\), and \(u_s = c_1\) is the unique solution in \([c_1, c_2)\) if \(f(x, u) < 0\) in \(\Omega \times (c_1, c_2)\).
Proof. It is obvious from $\hat{K}(x) = 1$ on $\partial \Omega$ that $u_s = c_1$ and $u_s = c_2$ are both solutions of (1.2). Let $u_s(x)$ be any solution of (1.2) such that $c_1 \leq u_s(x) \leq c_2$. If $u_s(x) \neq c_1$ and $f(x, u) > 0$ for $u \in (c_1, c_2)$ then $u_s(x)$ cannot be a constant solution other than $c_2$. Let $w_s(x) = c_2 - u_s(x)$. Then $w_s(x) \geq 0$, and by (H1) and $f(x, u) > 0$ in $\Omega \times (c_1, c_2)$,

$$-Lw_s = f(x, c_2) - f(x, u_s) \leq 0 \quad \text{in } \Omega,$$

$$Bw_s = c_2 - \int_{\Omega} K(x, y)u_s(y) \, dy = \int_{\Omega} K(x, y)w_s(y) \, dy \quad \text{on } \partial \Omega.$$ 

Since $w_s(x)$ is not a constant (unless $u_s(x) = c_2$), Lemma 2.2 ensures $w_s(x) \leq 0$. This leads to $w_s(x) = 0$ which shows that $u_s = c_2$ is the unique solution in $(c_1, c_2]$. On the other hand, if $u_s(x) \neq c_2$ and $f(x, u) < 0$ for $u \in (c_1, c_2)$ then $u_s(x)$ cannot be a constant solution other than $c_1$. It follows from the same argument as that for $c_2$ that $u_s(x) = c_1$ is the unique solution in $(c_1, c_2)$. This proves the lemma. \(\square\)

As a consequence of Lemma 2.4 and Theorem 2.3 we have the following result.

**Lemma 2.5.** Let the conditions in Lemma 2.4 hold. Then for any $u_0$ with $c_1 \leq u_0 \leq c_2$ a unique solution $u(t, x)$ to (1.1) exists and possesses the asymptotic limit

$$\lim_{t \to \infty} u(t, x) = \begin{cases} 
    c_2 & \text{if } c_1 < u_0(x) \leq c_2 \quad \text{and } f(x, u) > 0 \quad \text{in } \Omega \times (c_1, c_2), \\
    c_1 & \text{if } c_1 \leq u_0(x) < c_2 \quad \text{and } f(x, u) < 0 \quad \text{in } \Omega \times (c_1, c_2). 
\end{cases} \quad (2.4)$$

Proof. Since $\tilde{u} = c_2$ and $\hat{u} = c_1$ are ordered upper and lower solutions of (1.1) when $c_1 \leq u_0 \leq c_2$, the existence of a unique solution $u(t, x)$ and the relation $c_1 \leq u(t, x) \leq c_2$ follow from Theorem 2.1 of [9]. Consider the case $c_1 < u_0 \leq c_2$ and $f(x, u) > 0$ in $\Omega \times (c_1, c_2)$. Then there exists a small constant $\delta > 0$ such that $u_0 \geq c_1 + \delta$ and $f(x, c_1 + \delta) \geq 0$. This implies that the constant functions $\tilde{u}_s = c_2$ and $\hat{u}_s = c_1 + \delta$ satisfy the respective relation

$$-L\tilde{u}_s = f(x, \tilde{u}_s) \quad \text{and } L\hat{u}_s \leq f(x, \hat{u}_s) \quad \text{in } \Omega$$

and boundary condition (1.2), and therefore, they are ordered upper and lower solutions of (1.2). Since by Lemma 2.4, $u_s = c_2$ is the unique solution of (1.2) in $[c_1 + \delta, c_2]$, Theorem 2.3 ensures that $u(t, x) \to c_2$ as $t \to \infty$. This proves the relation (2.4) for the case $f(x, u) > 0$ in $\Omega \times (c_1, c_2)$. The proof for the case $f(x, u) < 0$ in $\Omega \times (c_1, c_2)$ is similar. \(\square\)

The implication of Lemma 2.5 is that if there exists an increasing sequence of real constants $\eta_0, \eta_1, \eta_2, \ldots$ (finite or infinite) such that $f(x, \eta_k) = 0$ for $k = 0, 1, 2, \ldots$, and either

$$f(x, u) > 0 \quad \text{in } \Omega \times (\eta_{2m}, \eta_{2m+1}) \quad \text{and } f(x, u) < 0 \quad \text{in } \Omega \times (\eta_{2m+1}, \eta_{2m+2}), \quad (2.5)$$

or

$$f(x, u) > 0 \quad \text{in } \Omega \times (\eta_{2m+1}, \eta_{2m}) \quad \text{and } f(x, u) < 0 \quad \text{in } \Omega \times (\eta_{2m}, \eta_{2m+1}), \quad (2.6)$$

where $m = 0, 1, 2, \ldots$, then we have the following convergence property of the time-dependent solution in relation to the constant steady-state solutions $\eta_0, \eta_1, \eta_2, \ldots$.\(\square\)
**Theorem 2.6.** Let Hypothesis \((H_1)\) hold, and let there exist a set of real numbers \(\{\eta_0, \eta_1, \eta_2, \ldots\}\), finite or infinite, such that \(f(x, \eta_k) = 0\) for \(k = 0, 1, 2, \ldots\). Then for any fixed \(m = 1, 2, \ldots\), a unique solution \(u(t, x)\) to (1.1) exists and converges to \(\eta_{2m+1}\) if condition (2.5) holds and either \(\eta_{2m} < U_0 < \eta_{2m+1}\) or \(\eta_{2m+1} < U_0 < \eta_{2m+2}\); and it converges to \(\eta_{2m}\) if condition (2.6) holds and either \(\eta_{2m-1} < U_0 < \eta_{2m}\) or \(\eta_{2m} < U_0 < \eta_{2m+1}\).

**Proof.** Consider the case where condition (2.5) holds. By letting \(c_1 = \eta_{2m}\) and \(c_2 = \eta_{2m+1}\), when \(f(x, u) > 0\) in \(\Omega \times (\eta_{2m}, \eta_{2m+1})\) and \(\eta_{2m} < U_0 < \eta_{2m+1}\) Lemma 2.5 implies that \(u(t, x) \rightarrow \eta_{2m+1}\) as \(t \rightarrow \infty\). Similarly, by letting \(c_1 = \eta_{2m+1}\) and \(c_2 = \eta_{2m+2}\) when \(f(x, u) < 0\) in \(\Omega \times (\eta_{2m+1}, \eta_{2m+2})\) and \(\eta_{2m+1} < U_0 < \eta_{2m+2}\), the same lemma ensures that \(u(t, x) \rightarrow \eta_{2m+1}\) as \(t \rightarrow \infty\). This proves the convergence of \(u(t, x)\) to \(\eta_{2m+1}\). The proof for the convergence of \(u(t, x)\) to \(\eta_{2m}\) is similar. 

Theorem 2.6 implies that under condition (2.5) all the constant steady-state solutions \(\eta_{2m+1}\), \(m = 0, 1, 2, \ldots\), are asymptotically stable with a stability region \((\eta_{2m}, \eta_{2m+1})\) and \([\eta_{2m+1}, \eta_{2m+2})\) while all the constants \(\eta_{2m}, m = 0, 1, 2, \ldots\), are unstable. These stability and instability properties are reversed if condition (2.6) holds.

**Example 2.7.** As an application of Theorem 2.6 we consider problem (1.1) with the function

\[
f(x, u) = \sigma(x) \sin au \quad (a > 0),
\]

where \(\sigma(x) > 0\) on \(\Omega\). It is clear that \(f(x, \eta_k) = 0\) for all \(\eta_k = k\pi/a\), \(k = 0, \pm 1, \pm 2, \ldots\). Since

\[
f(x, u) \begin{cases} > 0 & \text{when } 2m\pi/a < u < (2m + 1)\pi/a \\ < 0 & \text{when } (2m + 1)\pi/a < u < (2m + 2)\pi/a,
\end{cases}
\]

we conclude from Theorem 2.6 that for any \(u_0(x)\) with either \(2m\pi/a < u_0(x) \leq (2m + 1)\pi/a\) or \((2m + 1)\pi/a \leq u_0(x) < (2m + 2)\pi/a\) where \(m = 0, \pm 1, \pm 2, \ldots\), the corresponding time-dependent solution \(u(t, x)\) converges to \((2m + 1)\pi/a\) as \(t \rightarrow \infty\). This implies that for each integer \(m\) the constant solution \(u_* = (2m + 1)\pi/a\) is asymptotically stable from above and below while the solutions \(u_* = 2m\pi/a\) and \(u_* = (2m + 2)\pi/a\) are unstable. On the other hand, if the function \(\sigma(x)\) in (2.7) is negative on \(\Omega\) then for each \(m\), \(u_* = 2m\pi/a\) is asymptotically stable from above and below while \(u_* = (2m - 1)\pi/a\) and \(u_* = (2m + 1)\pi/a\) are unstable. Similar conclusions can be drawn for the functions \(f(x, u) = \sigma(x) \cos au\) and

\[
f(x, u) = \sigma(x)(u - \eta_0)^{n_0}(u - \eta_1)^{n_1}\cdots(u - \eta_p)^{n_p},
\]

where \(n_0, n_1, \ldots, n_p\) are positive odd integers (see also Example 3.3).

3. Blow-up of solutions

It is seen from Theorem 2.3 that if \(\hat{K}(x) = 1\) and the initial function \(u_0(x)\) is between two consecutive zeros of \(f(\cdot, u)\), if any, then a global solution to (1.1) exists and converges to one of the zeros of \(f(\cdot, u)\) depending on the sign of \(f(\cdot, u)\). However, for a certain class of functions where \(f(\cdot, u)\) either has a finite number of zeros or has no zero, the corresponding solution \(u(t, x)\) of (1.1) may
To investigate the blow-up problem we make the following more general hypothesis on $K(x,y)$:

\[(H_2) \quad K(x,y) \geq 0 \text{ and } \frac{\partial K}{\partial y}(x) \geq 1 \quad (x \in \partial \Omega, \ y \in \Omega).\]

Our first goal is to establish an existence-comparison result for (1.1) with respect to a lower solution $\tilde{u}(t,x)$. In the absence of a negativity lemma as that in Lemma 2.1 we show a comparison result by using some modified functions of $f(x,u)$ and $K(x,y)u$.

For each large constant $N > u_0$ we define modified functions $f_N$ and $G_N$ by

\[
f_N(x,u) \equiv \begin{cases} f(x,u) & \text{when } u \leq N \\ f(x,N) & \text{when } u > N \end{cases} \quad (x \in \Omega),
\]

\[
G_N(x,y,u) \equiv \begin{cases} K(x,y)u & \text{when } u \leq N \\ K(x,y)N & \text{when } u > N \end{cases} \quad ((x, y) \in \partial \Omega \times \Omega),
\]

and consider the modified problem

\[
\begin{align*}
&u_t - Lu = f_N(x,u) \quad \text{in } D_T, \\
&Bu = \int_{\Omega} G_N(x,y,u(t,y)) \, dy \quad \text{on } S_T, \\
&u(0,x) = u_0(x) \quad \text{in } \Omega.
\end{align*}
\]

It is obvious that if $\tilde{u}(t,x)$ is a lower solution of (1.1) then it is also a lower solution of (3.1) whenever $\tilde{u}(t,x) \leq N$. We choose $T = T(N)$ in (3.2) such that $\tilde{u} \leq N$ in $\overline{D}_T$. Define

\[c_N(t,x) \equiv \max \left\{ -\frac{\partial f}{\partial u}(x,u); \ \tilde{u} \leq u \leq N \right\}.\]

By using $u^{(0)} = \tilde{u}$ as the initial iteration we construct a sequence $\{u^{(k)}\}$ from the linear iteration process

\[
\begin{align*}
u^{(k)} - Lu^{(k)} + c_N u^{(k)} &= c_N u^{(k-1)} + f_N(x,u^{(k-1)}) \quad \text{in } D_T, \\
Bu^{(k)} &= \int_{\Omega} G_N(x,y,u^{(k-1)}(t,y)) \, dy \quad \text{on } S_T, \\
u^{(k)}(0,x) &= u_0(x) \quad \text{in } \Omega.
\end{align*}
\]

We show that this sequence converges monotonically to a unique (local) solution of (1.1).

**Theorem 3.1.** Let $\tilde{u}(t,x)$ be a lower solution of (1.1), and let hypothesis $(H_2)$ hold. Then problem (1.1) has a unique solution $u(t,x)$ such that $u(t,x) \geq \tilde{u}(t,x)$ whenever it exists. Moreover, if
\[ f(x, u) \geq \varepsilon u \text{ for } u \geq 0, \text{ where } \varepsilon > 0 \text{ is a constant, then} \]
\[
\begin{align*}
    u(t, x) &\geq \delta e^{rt} \quad \text{when } u_0(x) \geq \delta. 
\end{align*}
\] (3.3)

**Proof.** It is easily seen from the argument in [9] that the sequence \( \{u^{(k)}\} \) given by (3.2) is monotone nondecreasing. Since \( f_N(\cdot, u) \) and \( G_N(\cdot, u) \) are uniformly bounded for \( u \geq \tilde{u} \), standard estimates for linear parabolic boundary-value problems ensure that the sequence \( \{u^{(k)}\} \) is bounded (e.g. see [8]). This implies that the pointwise limit \( u(t, x) \equiv \lim_{k \to \infty} u^{(k)}(t, x) \) as \( k \to \infty \) exists and \( u(t, x) \geq \tilde{u}(t, x) \). A regularity argument as that in [8] shows that \( u(t, x) \) is the unique solution of (3.1). By the definition of \( F_N(\cdot, u) \) and \( G_N(\cdot, u) \), \( u(t, x) \) is the unique solution of (1.1) when \( u(t, x) \leq N \). The arbitrariness of \( N \) ensures that \( u(t, x) \) is the solution of (1.1) and satisfies \( u(t, x) \geq \tilde{u}(t, x) \) for as long as it exists.

It is easy to verify that when \( f(x, u) \geq \varepsilon u \) for \( u \geq 0 \) the function \( \tilde{u} \equiv \delta e^{rt} \) satisfies all the inequalities in (2.1) when \( \hat{K}(x) \geq 1 \) and \( u_0(x) \geq \delta \). This shows that \( \delta e^{rt} \) is a lower solution of (1.1) which leads to the relation in (3.3). \( \square \)

It is seen from Theorem 3.1 that the blowing-up property of the solution of (1.1) can be determined by finding a lower solution which grows unbounded in a finite time. Consider the case \( f(x, u) \geq g(u) \) for \( x \in \Omega \) and \( u \geq \xi_0 \) for some \( \xi_0 \geq 0 \), where \( g(u) \) is a continuous nonnegative function on \([\xi_0, \infty)\). Then direct computation shows that the solution \( v(t) \) of the ordinary equation
\[
\begin{align*}
    \frac{dv}{dt} &= g(v), \quad v(0) = \xi_0 
\end{align*}
\] (3.4)
is a lower solution of (1.1) when \( \hat{K}(x) \geq 1 \) and \( u_0(x) \geq \xi_0 \). Since the solution \( v(t) \) of (3.4) grows unbounded at some finite \( T \) when
\[
\int_{\xi_0}^{\infty} \frac{dv}{g(v)} < \infty, 
\] (3.5)
we have the following conclusion:

**Theorem 3.2.** Let Hypothesis \((H_2)\) hold, and let \( f(x, u) \geq g(u) \geq 0 \) for \( x \in \Omega \) and \( u \geq \xi_0 \geq 0 \). If \( g(u) \) satisfies (3.5) then for any \( u_0 \geq \xi_0 \) the solution \( u(t, x) \) of (1.1) blows up at some finite \( T^* \).

**Example 3.3.** (a) Consider problem (1.1) with \( \hat{K}(x) \equiv 1 \) and
\[
\begin{align*}
    f(x, u) &\equiv q(x, u)(u - c_1)(u - c_2), 
\end{align*}
\] (3.6)
where \( q(x, u) \geq q_0 > 0 \) for some positive constant \( q_0 \) and \( c_2 > c_1 > 0 \). Since \( f(x, u) \geq q_0(u - c_1)(u - c_2) \) for \( u \geq c_2 \) and
\[
\int_{\xi_0}^{\infty} [(u - c_1)(u - c_2)]^{-1} du < \infty \quad \text{when } \xi_0 > c_2, 
\] we see from Theorem 3.2 that the solution \( u(t, x) \) of (1.1) blows up in finite time when \( u_0 > c_2 \). On the other hand, since \( f(u) > 0 \) when \( 0 < u < c_1 \), and \( f(u) < 0 \) when \( c_1 < u < c_2 \), Theorem 2.6 ensures that a unique global solution \( u(t, x) \) to (1.1) exists and converges to \( c_1 \) as \( t \to \infty \) if either \( 0 < u_0 \leq c_1 \) or \( c_1 \leq u_0 < c_2 \). This shows that the constant solution \( u_0 = c_1 \) is asymptotically stable while \( u_0 = c_2 \)
is unstable. The instability property of $c_2$ is “strong from above” in the sense that if the initial function is above $c_2$ the corresponding solution $u(t,x)$ not only moves away from $c_2$, but also grows unbounded in finite time.

(b) As a second example we consider problem (1.1) with $\hat{K}(x) \equiv 1$ and
\[
f(x,u) = q(x,u)(u - c_1)(u - c_2)(u - c_3)^m,
\]
where $c_3 > c_2 > c_1 > 0$, $m$ is a positive integer and $q(x,u)$ is the same function as that in (3.6). Consider the case where $m$ is an odd integer. Then $f(x,u) > 0$ when $c_1 < u < c_2$ and $f(x,u) < 0$ when $c_2 < u < c_3$. By Theorem 2.6, a unique global solution $u(t,x)$ to (1.1) exists and converges to $c_2$ as $t \to \infty$ when either $c_1 < u_0 \leq c_2$ or $c_2 \leq u_0 < c_3$. However, since
\[
\int_{\xi_0}^{\infty} [ (u - c_1)(u - c_2)(u - c_3)^m]^{-1} \, du < \infty \quad \text{when } \xi_0 > c_3,
\]
Theorem 3.1 implies that the solution $u(t,x)$ blows up in finite time when $u_0 > c_3$. This shows that the constant $c_2$ is asymptotically stable, and $c_1$ and $c_3$ are unstable.

On the other hand, if $m$ is an even integer then Theorem 2.6 implies that the solution $u(t,x)$ of (1.1) converges to $c_1$ as $t \to \infty$ when $0 < u_0 \leq c_1$ or $c_1 \leq u_0 < c_2$ and it converges to $c_3$ as $t \to \infty$ when $c_2 < u_0 \leq c_3$. However, in view of (3.8) the solution $u(t,x)$ blows up at some finite $T^*$ when $u_0 > c_3$. This implies that the constant solution $u_s \equiv c_3$ is asymptotically stable from below and strongly unstable from above.

4. Kernels with alternating signs

When the kernel $K(x,y)$ possesses both positive and negative values on $\partial \Omega \times \Omega$, including the case $K(x,y) \leq 0$, the method of monotone iteration can still be used to obtain an existence-uniqueness theorem for (1.1). In this situation, however, upper and lower solutions are required to satisfy some boundary inequalities which are coupled. One approach to describe this requirement is to set $K(x,y) \equiv K^+(x,y) + K^-(x,y)$ and write the boundary condition in (1.1) as
\[
Bu = \int_{\Omega} K^+(x,y)u(t,y) \, dy + \int_{\Omega} K^-(x,y)u(t,y) \, dy,
\]
where
\[
K^+(x,y) = \begin{cases} K(x,y) & \text{if } K(x,y) \geq 0, \\ 0 & \text{if } K(x,y) < 0, \end{cases}
\]
\[
K^-(x,y) = \begin{cases} K(x,y) & \text{if } K(x,y) < 0, \\ 0 & \text{if } K(x,y) \geq 0. \end{cases}
\]

Then we have the following definition:
Definition 4.1. A pair of functions \( \tilde{u}, \tilde{u} \) in \( C^{1,2}(\Omega) \cap C(\overline{\Omega}) \) are called coupled upper and lower solutions of (1.1) if \( \tilde{u} \geq \tilde{u} \) and satisfy the relation

\[
\tilde{u} - L\tilde{u} \geq f(x, \tilde{u}), \quad \tilde{u} - L\tilde{u} \leq f(x, \tilde{u}),
\]

\[
B\tilde{u} \geq \int_{\Omega} K^{+}(x, y)\tilde{u}(t, y) \, dy + \int_{\Omega} K^{-}(x, y)\tilde{u}(t, y) \, dy,
\]

\[
B\tilde{u} \leq \int_{\Omega} K^{+}(x, y)\tilde{u}(t, y) \, dy + \int_{\Omega} K^{-}(x, y)\tilde{u}(t, y) \, dy,
\]

\[
\tilde{u}(0, x) \geq u_{0}(x) \geq \tilde{u}(0, x).
\]

It is clear from (4.3) that upper and lower solutions are in general coupled, and if \( K(x, y) \geq 0 \) then the above requirement is reduced to that in (2.1). We assume that a pair of coupled upper and lower solutions exist and set

\[
(\tilde{u}, \tilde{u}) = \{u \in C(\overline{\Omega}), \tilde{u} \leq u \leq \tilde{u}\}.
\]

By using \( \tilde{u}^{(0)} = \tilde{u} \) and \( u^{(0)} = \tilde{u} \) as the initial iterations we can construct two sequences \( \{\tilde{u}^{(k)}\}, \{u^{(k)}\} \) from the linear iteration process

\[
\tilde{u}^{(k)} = c\tilde{u}^{(k-1)} + \int_{\Omega} K^{+}(x, y)\tilde{u}^{(k-1)}(t, y) \, dy + \int_{\Omega} K^{-}(x, y)\tilde{u}^{(k-1)}(t, y) \, dy,
\]

\[
u^{(k)} = c\nu^{(k-1)} + \int_{\Omega} K^{+}(x, y)\nu^{(k-1)}(t, y) \, dy + \int_{\Omega} K^{-}(x, y)\nu^{(k-1)}(t, y) \, dy,
\]

\[
\tilde{u}^{(k)}(0, x) = u^{(k)}(0, x) = u_{0}(x), \quad k = 1, 2, \ldots,
\]

where

\[
\mathcal{L} u = u_{t} - Lu + cu
\]

and \( c \equiv c(t, x) \) is any smooth function satisfying

\[
c(t, x) \geq \max \left\{ -\frac{\partial f}{\partial u}(x, u); \tilde{u} \leq u \leq \tilde{u} \right\}.
\]

It is obvious that these sequences are well defined and are reduced to that given in [9] when \( K(x, y) \geq 0 \). The requirement of \( c(t, x) \) ensures that

\[
cu + f(x, u) \geq cv + f(x, v) \quad \text{when} \quad \tilde{u} \geq u \geq v \geq \tilde{u}.
\]

In the following lemma we show the monotone property of these sequences.

Lemma 4.2. The sequences \( \{\tilde{u}^{(k)}\}, \{u^{(k)}\} \) given by (4.4) possess the monotone property

\[
\hat{u} \leq u^{(k)} \leq u^{(k+1)} \leq \tilde{u}^{(k+1)} \leq \tilde{u}^{(k)} \leq \hat{u} \quad \text{on} \ \overline{\Omega}.
\]
Proof. Let \( \tilde{\eta}(0) = \tilde{u}(0) - \tilde{u}^{(1)} \equiv \tilde{u} - \tilde{u}^{(1)} \) and \( \tilde{\omega}(0) = \tilde{u}^{(1)} - \tilde{u}^{(0)} \equiv \tilde{u}^{(1)} - \tilde{u} \). By (4.4) and (4.3),
\[
\mathcal{L}\tilde{\omega}(0) = (\tilde{u}_t - L\tilde{u} + c\tilde{u}) - [c\tilde{u}(0) + f(x, \tilde{u}(0))] \geq 0,
\]
\[
\mathcal{L}\tilde{\omega}(0) = [cu(0) + f(x, u(0))] - [\tilde{u}_t - L\tilde{u} + c\tilde{u}] \geq 0,
\]
\[
B\tilde{\omega}(0) = B\tilde{u} - \left[ \int_{\Omega} K^+(x, y)\tilde{u}(0)(t, y) \, dy + \int_{\Omega} K^-(x, y)u(0)(t, y) \, dy \right] \geq 0,
\]
\[
B\omega(0) = \left[ \int_{\Omega} K^+(x, y)u(0)(t, y) \, dy + \int_{\Omega} K^-(x, y)\tilde{u}(0)(t, y) \, dy \right] - B\tilde{u} \geq 0.
\]

Since \( \tilde{\omega}(0)(0, x) = \tilde{u}(0, x) - \tilde{u}(0, x) \geq 0 \) and \( \omega(0)(0, x) = u(0) - \tilde{u}(0, x) \geq 0 \), an application of the positivity lemma for parabolic boundary-value problems gives \( \tilde{\omega}(0) \geq 0 \) and \( \omega(0) \geq 0 \) (cf. [8]). This leads to \( \tilde{\omega}(0) \geq \tilde{\omega}(1) \) and \( \omega(1) \geq \omega(0) \). Similarly, by (4.4), (4.5) and \( \tilde{\omega}(0) \geq \omega(0) \), \( \omega(1) = \omega(0) \) satisfies the relation
\[
\mathcal{L}\omega(1) = [cu(0) + f(x, u(0))] - [cu(0) + f(x, u(0))] \geq 0,
\]
\[
B\omega(1) = \int_{\Omega} K^+(x, y)[\tilde{u}(0)(t, y) - u(0)(t, y)] \, dy + \int_{\Omega} K^-(x, y)[u(0)(t, y) - \tilde{u}(0)(t, y)] \, dy \geq 0
\]
and the initial condition \( \omega(1)(0, x) = u(0)(x) - \tilde{u}(0, x) = 0 \). It follows again from the positivity lemma that \( \omega(1) \geq 0 \). The above conclusions show that \( \tilde{\omega}(0) \leq \omega(0) \leq \tilde{\omega}(1) \leq \omega(1) \). The monotone property (4.6) follows by an induction argument similar to that in [9].

In view of (4.6) the pointwise limits
\[
limit_{k \to \infty} \tilde{\omega}(k)(t, x) \equiv \tilde{\omega}(t, x), \quad \limit_{k \to \infty} u^{(k)}(t, x) \equiv u(t, x)
\]
exist and satisfy the relation \( \tilde{\omega} \leq u \leq \tilde{\omega} \leq \tilde{\omega} \). Letting \( k \to \infty \) in (4.4) and using a regularity argument as that in [8] shows that both \( \tilde{\omega} \) and \( u \) satisfy the equations in (1.1) except that the boundary condition be replaced by
\[
B\tilde{\omega} = \int_{\Omega} K^+(x, y)\tilde{u}(t, y) \, dy + \int_{\Omega} K^-(x, y)u(t, y) \, dy,
\]
\[
Bu = \int_{\Omega} K^+(x, y)u(t, y) \, dy + \int_{\Omega} K^-(x, y)\tilde{u}(t, y) \, dy.
\]

In the following theorem we show that \( \tilde{\omega} = u \) and is the unique solution of (1.1) in \( \langle \tilde{\omega}, \tilde{\omega} \rangle \).

**Theorem 4.3.** Let \( \tilde{\omega}, \tilde{\omega} \) be a pair of coupled upper and lower solutions of (1.1) with \( K(x, y) = K^+(x, y) + K^-(x, y) \). Then problem (1.1) has a unique solution \( u(t, x) \). Moreover, the sequences \( \{\tilde{\omega}(k)\}, \{u^{(k)}\} \) given by (4.4) converge monotonically to \( u(t, x) \) and satisfy
\[
\tilde{\omega} \leq u^{(k)} \leq u^{(k+1)} \leq u \leq \tilde{u}^{(k+1)} \leq \tilde{u}^{(k)} \leq \tilde{\omega} \text{ on } \bar{D}_r.
\]
Proof. It suffices to show that \( \bar{u} = u \) and is the unique solution of (1.1). Let \( w = \bar{u} - u \). Then by (1.1), (4.8) and the mean-value theorem,
\[
\begin{align*}
wt - Lw &= f(x, \bar{u}) - f(x, u) = f_u(x, \xi)w, \\
Bw &= \int_{\Omega} K^+(x, y)w(t, y) \, dy - \int_{\Omega} K^-(x, y)w(t, y) \, dy, \\
&= \int_{\Omega} |K(x, y)|w(t, y) \, dy \\
w(0, x) &= 0,
\end{align*}
\]
where \( \xi \equiv \xi(t, x) \) is an intermediate value between \( \bar{u} \) and \( u \). For each \( t > 0 \), define the maximum norm
\[
||w||_t \equiv \max \{ |w(s, x)|; 0 \leq s \leq t, \ x \in \Omega \}.
\]
Then
\[
\int_{\Omega} |K(x, y)| |w(t, y)| \, dy \leq \hat{K}(x)||w||_t \leq \overline{K}||w||_t,
\]
where \( \overline{K} \) is an upper bound of \( \hat{K}(x) \) on \( \partial \Omega \). Using an integral representation for the solution \( w(t, x) \) of (4.10) (in terms of the fundamental solution of the operator \( \mathcal{L} \equiv \partial / \partial t - L - f_u(x, \xi) \)) and applying the estimate (4.11) a ladder argument as that in \([8, \text{pp. 143-144}]\) shows that \( w(t, x) = 0 \) on \( \overline{D}_T \). This proves \( \bar{u} = u \). Now, if \( u^* \) is any other solution in \((\bar{u}, \bar{u})\) then \( w^* = u^* - u \) satisfies (4.10) where the boundary condition is replaced by
\[
|w^*|_{\partial \Omega} \leq \int_{\Omega} |K(x, y)| |w^*(t, y)| \, dy.
\]
Since the estimate (4.11) holds for \( w^* \) the same reasoning as that for \( w \) yields \( w^* = 0 \) on \( \overline{D}_T \). This proves \( u^* = u \) and thus the uniqueness of the solution. \( \square \)

It is seen from Theorem 4.3 that the existence of a solution to (1.1) is guaranteed if there exist a pair of coupled upper and lower solutions. To find such a pair which also ensure the decay property of the solution of (1.1) we impose the following conditions on \( K(x, y) \) and \( f(x, u) \).

\((H_3)\) \( |K(x, y)| \) satisfies either condition (1.3) or condition (1.4), and there exists a constant \( M > 0 \) such that \( f(x, 0) = 0, \ f_u(x, u) \leq 0 \) for \( |u| \leq M \) and \( f(x, -u) \geq -f(x, u) \) for \( 0 \leq u \leq M \).

Under the above condition we have the following result:

**Theorem 4.4.** Let Hypothesis \((H_3)\) hold. Then for any initial function \( u_0(x) \) with \( |u_0(x)| \leq M \), a unique solution \( u(t, x) \) to (1.1) exists and possesses the decay property \( u(t, x) \to 0 \) as \( t \to \infty \).

**Proof.** Consider the problem
\[
\begin{align*}
U_t - LU &= f(x, U) \quad (t > 0, \ x \in \Omega), \\
BU &= \int_{\Omega} |K(x, y)| U(t, y) \, dy \quad (t > 0, \ x \in \partial \Omega), \\
U(0, x) &= u_0(x) \quad (x \in \Omega).
\end{align*}
\]

\[\text{Proof.} \quad \text{Consider the problem} \]
\[\begin{align*}
U_t - LU &= f(x, U) \quad (t > 0, \ x \in \Omega), \\
BU &= \int_{\Omega} |K(x, y)| U(t, y) \, dy \quad (t > 0, \ x \in \partial \Omega), \\
U(0, x) &= u_0(x) \quad (x \in \Omega).
\end{align*}\]
It is easy to verify that under the conditions in \((H_3)\) the pair \(\tilde{U} = M\) and \(\tilde{U} = -M\) are ordered upper and lower solutions of \((4.12)\) as well as its corresponding steady-state problem. This implies that a unique solution \(U(t,x)\) to \((4.12)\) exists and \(|U(t,x)| \leq M\). Moreover, by Corollary 3.1 and Theorem 4.2 of [9], \(u_s = 0\) is the unique steady-state solution of \((4.12)\) in \([0,M]\) and \(U(t,x) \to 0\) as \(t \to \infty\). Hence, to show the existence of a solution \(u(t,x)\) to \((1.1)\) and the decay property \(u(t,x) \to 0\) as \(t \to \infty\) we only need to show that \(\tilde{u} = U\) and \(\tilde{u} = -U\) are coupled upper and lower solutions of \((1.1)\).

It is clear from \((4.12)\) that \(\tilde{u} = U\) satisfies the differential inequality in \((4.3)\) and by the hypothesis \(f(x,-u) \geq -f(x,u)\), \(\tilde{u} = -U\) satisfies

\[
\dot{\tilde{u}} - L\tilde{u} = -f(x,U) \leq f(x,-U) = f(x,\tilde{u}).
\]

Moreover, by the relation \(|K(x,y)| = K^+(x,y) - K^-(x,y)\),

\[
\tilde{B}u = BU = \int_\Omega K^+(x,y)U(t,y)\,dy + \int_\Omega K^-(x,y)(-U(t,y))\,dy,
\]

\[
\tilde{B}u = -BU = \int_\Omega K^+(x,y)(-U(t,y))\,dy + \int_\Omega K^-(x,y)(U(t,y))\,dy.
\]

This shows that \(U\) and \(-U\) are coupled upper and lower solutions of \((1.1)\). It follows from Theorem 4.1 that the solution \(u(t,x)\) satisfies \(|u(t,x)| \leq |U(t,x)|\) and therefore converges to 0 as \(t \to \infty\). 

When the function \(f(x,u)\) is given in the form

\[
f(x,u) = c(x)u^{2p+1}, \quad p = 0, 1, 2 \ldots
\]

for some function \(c(x) \leq 0\), including the case \(f(x,u) = c(x)u\), the requirement in Theorem 4.4 are fulfilled by any constant \(M > 0\) if \(|K(x,y)|\) satisfies either \((1.3)\) or \((1.4)\). In this situation a unique solution to \((1.1)\) exists for every initial function \(u_0(x)\) and converges to 0 as \(t \to \infty\).

**Remark 4.5.** The conclusion \(|u(t,x)| \leq U(t,x)|\) in the proof of Theorem 4.4 was given in Theorem 4.4 of [9] by a different argument. However, there is a gap in its proof and the above argument gives a correct proof for the conclusion (and under the slightly weakened condition \(f(x,-u) \geq -f(x,u)\) instead of \(f(x,-u) = -f(x,u)\)).

**References**


