Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Algebra 319 (2008) 1285–1304
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Quasi-Lie structure of σ-derivations of $\mathbb{C}[t^{\pm 1}]$

Lionel Richard [a], Sergei D. Silvestrov [b,*]

[a] *University of Edinburgh, School of Mathematics, JCMB-King's Buildings, EH9 3JZ Edinburgh, United Kingdom*
[b] *Centre for Mathematical Sciences, Lund University, Box 118, SE-221 00 Lund, Sweden*

Received 6 February 2007
Available online 26 November 2007
Communicated by Michel Van den Bergh

Abstract

Hartwig, Larsson and Silvestrov in [J.T. Hartwig, D. Larsson, S.D. Silvestrov, Deformations of Lie algebras using σ-derivations, J. Algebra 295 (2) (2006) 314–361] defined a bracket on σ-derivations of a commutative algebra. We show that this bracket preserves inner derivations, and based on this obtain structural results providing new insights into σ-derivations on Laurent polynomials in one variable.

Keywords: Quasi-Lie algebras; σ-derivations; Twisted bracket; q-deformed Witt algebras

1. Introduction

In [16,23–25] a new class of algebras called quasi-Lie algebras and its subclasses, quasi-hom-Lie algebras and hom-Lie algebras, have been introduced. An important characteristic feature of those algebras is that they obey some deformed or twisted versions of skew-symmetry and Jacobi identity with respect to some possibly deformed or twisted bilinear bracket multiplication. Quasi-Lie algebras include color Lie algebras, and in particular Lie algebras and Lie superalgebras, as well as various interesting quantum deformations of Lie algebras. Let us mention here as significant examples deformations of the Heisenberg Lie algebra, oscillator algebras, sl_2 and of other finite-dimensional Lie algebras, of infinite-dimensional Lie algebras of Witt and Virasoro type

* Corresponding author.
E-mail addresses: lionel.richard@ed.ac.uk (L. Richard), sergei.silvestrov@math.lth.se (S.D. Silvestrov).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.09.029

applied in physics within the string theory, vertex operator models, quantum scattering, lattice models and other contexts, as well as various algebras arising in connection to non-commutative geometry (see [3–14,16–21,26–29,34] and references therein). Many of these quantum deformations of Lie algebras can be shown to play the role of underlying algebraic objects for calculi of twisted, discretized or deformed derivations and difference type operators and thus in corresponding general non-commutative differential calculi.

Considering these deformed differential calculi, vector fields are replaced by twisted vector fields when derivations are replaced by twisted derivations. In [16, Theorem 5], it was proved that under some general assumptions these twisted vector fields are closed under a natural twisted non-associative skew-symmetric multiplication satisfying a twisted 6-term Jacobi identity. This identity generalizes the usual Lie algebras 3-term Jacobi identity that is recovered when no twisting is present (see Theorem 2.2.3). This result is shown to be instrumental to the construction of various examples and classes of quasi-Lie algebras. Both known and new one-parameter and multi-parameter deformations of Witt and Virasoro algebras and other Lie and color Lie algebras have been constructed within this framework in [16,23–25].

In this article, we gain further insight in the particular class of quasi-Lie algebra deformations of the Witt algebra. These were introduced in [16] via the general twisted bracket construction, and associated with twisted discretization of derivations generalizing the Jackson q-derivatives to the case of twistings by general endomorphisms of Laurent polynomials. In Section 2 we present necessary definitions, facts and constructions on σ-derivations that are central for this article. In Proposition 2.3.1 we observe that inner derivations are stable under the bracket defined in [16]. In the last part of this section, we present a characterization of the set of inner derivations for UFD (Proposition 2.4.1), and also general inclusions concerning sets of inner derivations and image and pre-image subsets with respect to the twisted bracket (Proposition 2.4.6). In Section 3 we develop the preceding framework for a particular important UFD, the algebra $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ of Laurent polynomials in one variable. For this specialization more deep and precise results can be obtained. We shall focus here on the fact that in the present paper we deal with an endomorphism σ of $\mathcal{A}$ which is NOT assumed to be an automorphism. The space of σ-derivations $\mathcal{D}_\sigma(\mathcal{A})$ endowed with the twisted bracket mentioned above is the deformation of the Witt algebra within the class of quasi-hom-Lie algebras in the sense of [16]. In Theorem 3.2.1 we show that the space of σ-derivations can be decomposed into a direct sum of the space of inner σ-derivations and a finite number of one-dimensional subspaces. In Theorem 3.3.4, we show that for arbitrary σ the $\mathbb{Z}$-gradation of this non-linearly q-deformed Witt algebra with coefficients in $\mathbb{C}$ becomes a $\mathbb{Z}/d\mathbb{Z}$-gradation with coefficients in $\mathbb{C}[T^{\pm 1}]$ for some element T of $\mathcal{A}$. The usual q-deformed Witt algebra associated to ordinary Jackson q-derivative corresponds to the automorphism $\sigma : t \mapsto qt$, and appears as a “limit case” where $d = 0$ and all σ-derivations are inner. In Subsection 3.4, we provide a more detailed description of what relations for the bracket in the non-linearly q-deformed Witt algebra become modulo inner σ-derivations. Finally, in Subsection 3.5, we describe normalizer-like subsets in detail for the non-linearly q-deformed Witt algebra.

Throughout this article, $\mathcal{A}$ will denote an associative, commutative and unital algebra over the field $\mathbb{C}$ of complex numbers. We will sometimes mention more general results concerning non-commutative algebras, and we will precise our assumptions on $\mathcal{A}$ in these cases. In the last section the algebra $\mathcal{A}$ will be the algebra of Laurent polynomials $\mathbb{C}[t^{\pm 1}]$.

2. Some general facts on σ-derivations

2.1. Definitions

In this section we do not assume that $\mathcal{A}$ is commutative, and as in the rest of the paper the endomorphism σ is not assumed to be an automorphism. We recall here some basic definitions and facts concerning σ-derivations. On this subject and more generally on Ore extensions one may see the reference books [30] or [22, Section 1.7].

Definition 2.1.1. Let $\mathcal{A}$ be an algebra, and let σ be an endomorphism of $\mathcal{A}$. A σ-derivation is a linear map D satisfying $D(ab) = \sigma(a)D(b) + D(a)b$ for all $a, b \in \mathcal{A}$. We denote the set of all σ-derivations by $\mathcal{D}_\sigma(\mathcal{A})$.

Example. It is easy to check that for any $p \in \mathcal{A}$, the $\mathbb{C}$-linear map Δ_p defined by $\Delta_p(a) = pa - \sigma(a)p$ for all $a \in \mathcal{A}$ is a σ-derivation of $\mathcal{A}$. Note that if $\mathcal{A}$ is commutative, then we have $\Delta_p = p(\mathrm{id} - \sigma)$.

Definition 2.1.2. The map Δ_p defined above is called the *inner* σ-derivation associated to p. The set of all inner derivations of $\mathcal{A}$ will be denoted $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

For any map $\tau : \mathcal{A} \to \mathcal{A}$ denote $\mathrm{Ann}(\tau) = \{a \in \mathcal{A} \mid a\tau(b) = 0\ \forall b \in \mathcal{A}\}$, the left annihilator ideal of τ. In particular if $\mathcal{A}$ is commutative then this is a two-sided ideal, and also $\Delta_p = \Delta_q \Leftrightarrow (p - q) \in \mathrm{Ann}(\mathrm{id} - \sigma)$.

The σ-derivations play a crucial role in the definition of Ore extensions.

Definition 2.1.3. Let $\mathcal{A}$ be an algebra, σ an endomorphism of $\mathcal{A}$, and Δ a σ-derivation. Then the Ore extension $R = \mathcal{A}[X; \sigma, \Delta]$ is the algebra such that:

- $\mathcal{A}$ is a subalgebra of R;
- R is a free $\mathcal{A}$-module with basis $\{X^n,\ n \in \mathbb{N}\}$;
- the multiplication is defined in R by the rule $Xa = \sigma(a)X + \Delta(a)$ for all $a \in \mathcal{A}$.

The following facts can be easily checked [15, Lemmas 1.5, 2.4].

Lemma 2.1.4. *Let σ be an endomorphism of an algebra $\mathcal{A}$.*

(1) *Let $p \in \mathcal{A}$, and $\Delta_p \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. Then the identity map on $\mathcal{A}$ extends to an isomorphism τ between the Ore extensions $\mathcal{A}[X; \sigma, \Delta_p]$ and $\mathcal{A}[Y; \sigma]$, such that $\tau(X) = Y + p$.*
(2) *Assume $\mathcal{A}$ is commutative. Then for all $\Delta \in \mathcal{D}_\sigma(\mathcal{A})$, and for all $a, b \in \mathcal{A}$ one has*

$$\big(a - \sigma(a)\big)\Delta(b) = \big(b - \sigma(b)\big)\Delta(a).$$

The first statement of this lemma provides one of the reasons why we get interested in σ-derivations up to inner in the following sections.

2.2. *A bracket on σ-derivations*

Henceforth, $\mathcal{A}$ is supposed to be a *commutative* algebra. Then $\mathcal{D}_\sigma(\mathcal{A})$ becomes a left $\mathcal{A}$-module by $(a\Delta)(r) = a\Delta(r)$ for all $a, r \in \mathcal{A}$. Note that in the non-commutative case this operation makes $\mathcal{D}_\sigma(\mathcal{A})$ a left module only over the center of $\mathcal{A}$.

Now we fix a σ-derivation Δ, and consider the cyclic left $\mathcal{A}$-submodule of $\mathcal{D}_\sigma(\mathcal{A})$ generated by Δ. The interest of considering cyclic submodules is reinforced by the following result proved by Hartwig, Larsson and Silvestrov in [16, Theorem 4], in the case where $\mathcal{A}$ is a unique factorization domain.

Theorem 2.2.1. *(See Hartwig, Larsson, Silvestrov [16].) Let σ be an endomorphism of a unique factorization domain $\mathcal{A}$, and $\sigma \neq \mathrm{id}$. Then $\mathcal{D}_\sigma(\mathcal{A})$ is a free $\mathcal{A}$-module of rank one with generator*

$$\Delta_0 = \frac{\mathrm{id} - \sigma}{g}, \quad \textit{with } g = \gcd\big((\mathrm{id} - \sigma)(\mathcal{A})\big).$$

Remark 2.2.2. We do not know of any example of a commutative algebra $\mathcal{A}$ such that $\mathcal{D}_\sigma(\mathcal{A})$ is an $\mathcal{A}$-module of rank greater than 1.

The authors also define in [16, Theorem 5] a bracket (i.e. a non-associative antisymmetric multiplication, but not of Lie type in general) on this cyclic submodule and prove the following results.

Theorem 2.2.3. *(See Hartwig, Larsson, Silvestrov [16].) Let σ be an endomorphism of a commutative algebra $\mathcal{A}$, and $\sigma \neq \mathrm{id}$. Let $\Delta \in \mathcal{D}_\sigma(\mathcal{A})$ be a σ-derivation such that*:

- $\sigma(\mathrm{Ann}(\Delta)) \subseteq \mathrm{Ann}(\Delta)$;
- $\exists \delta \in \mathcal{A}$ *such that* $\Delta \circ \sigma = \delta\sigma \circ \Delta$.

Then the map

$$[\cdot,\cdot]_\sigma : \mathcal{A}\Delta \times \mathcal{A}\Delta \to \mathcal{A}\Delta$$

defined by setting

$$[a\Delta, b\Delta]_\sigma = \big(\sigma(a)\Delta\big) \circ (b\Delta) - \big(\sigma(b)\Delta\big) \circ (a\Delta), \quad \textit{for } a, b \in \mathcal{A}, \tag{1}$$

where $\circ$ denotes composition of functions, is a well-defined $\mathbb{C}$-algebra product on the $\mathbb{C}$-linear space $\mathcal{A}\Delta$, satisfying the following identities for $a, b, c \in \mathcal{A}$:

$$[a\Delta, b\Delta]_\sigma = \big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big)\Delta, \tag{2}$$

$$[a\Delta, b\Delta]_\sigma = -[b\Delta, a\Delta]_\sigma. \tag{3}$$

In addition,

$$\begin{aligned}&\big[\sigma(a)\Delta,[b\Delta,c\Delta]_\sigma\big]_\sigma+\delta\big[a\Delta,[b\Delta,c\Delta]_\sigma\big]_\sigma\\&\quad+\big[\sigma(b)\Delta,[c\Delta,a\Delta]_\sigma\big]_\sigma+\delta\big[b\Delta,[c\Delta,a\Delta]_\sigma\big]_\sigma\\&\quad+\big[\sigma(c)\Delta,[a\Delta,b\Delta]_\sigma\big]_\sigma+\delta\big[c\Delta,[a\Delta,b\Delta]_\sigma\big]_\sigma=0.\end{aligned}\tag{4}$$

Remark 2.2.4.

(1) The condition $\sigma(\mathrm{Ann}(\Delta))\subseteq\mathrm{Ann}(\Delta)$ is clearly satisfied if $\mathrm{Ann}(\Delta)=\{0\}$, which is the case for non-zero Δ for instance if $\mathcal{A}$ is a domain.
(2) The existence of δ such that $\Delta\circ\sigma=\delta\sigma\circ\Delta$, with the extra assumption that $\delta=q\in\mathbb{C}^*$, is the definition of q-skew derivations given in [15]. These particular σ-derivations play an important role in quantum groups, see for instance [2,31] or the book [1] and references therein. Note that in this case (4) can be written as a 3-term Jacobi-like identity, showing that $(\mathcal{A}\Delta,[\cdot,\cdot]_\sigma)$ is a hom-Lie algebra and thus in particular also a quasi-Lie algebra [16,24].
(3) The identity (2) is just a formula expressing the product defined in (1) as an element of $\mathcal{A}\Delta$. Identities (3) and (4) are expressing, respectively, skew-symmetry and a generalized ((σ,δ)-twisted) Jacobi identity for the product defined by (1). Theorem 2.2.3 shows, that under stated conditions, $(\mathcal{A}\Delta,[\cdot,\cdot]_\sigma)$ is a quasi-hom-Lie algebra and thus also a quasi-Lie algebra [16,23]. For special choices of $\mathcal{A},\sigma$ and Δ the algebra $(\mathcal{A}\Delta,[\cdot,\cdot]_\sigma)$ may belong to the class of hom-Lie algebras, a subclass of quasi-hom-Lie algebras with 3-term twisted Jacobi identity.

If $\mathcal{A}$ is a unique factorization domain then the space of σ-derivations is a free $\mathcal{A}$-module of rank one generated by a σ-derivation which satisfies the hypotheses of Theorem 2.2.3, with $\delta=\frac{\sigma(g)}{g}$ (see Theorems 2.2.1, 2.4.1 and [16, Theorem 4]). Nevertheless note that even if a σ-derivation Δ may satisfy these hypotheses, that may not be the case for any σ-derivation in the cyclic module $\mathcal{A}\Delta$. More precisely, for $a\in\mathcal{A}$ one has $a\Delta\circ\sigma=a\delta\sigma\circ\Delta$ which may not be written in the form $\delta'\sigma\circ(a\Delta)$, that is $\delta'\sigma(a)\sigma\circ(\Delta)$, if a does not divide $\sigma(a)$. Let us have a look at an example of a non-integral ring.

Example 2.2.5. Let $\mathcal{A}=\mathbb{C}[X,Y]/\langle XY\rangle$ be the quotient of the commutative polynomial algebra in 2 variables by the ideal generated by the product XY. This commutative algebra is not an integral domain, and as a vector space is isomorphic to $\mathbb{C}\oplus X\mathbb{C}[X]\oplus Y\mathbb{C}[Y]$. Consider the endomorphism σ defined by $\sigma(X)=Y$ and $\sigma(Y)=X$. Then it can be easily seen that any σ-derivation Δ has to satisfy $\Delta(X)+\Delta(Y)=0$. Conversely, any linear map Δ on $\mathbb{C}X\oplus\mathbb{C}Y$ satisfying $\Delta(X)=-\Delta(Y)$ can be uniquely extended to a σ-derivation of $\mathcal{A}$. So $\mathcal{D}_\sigma(\mathcal{A})$ is generated as an $\mathcal{A}$-module by the σ-derivation Δ_0 which is defined by $\Delta_0(X)=-\Delta_0(Y)=1$. Consider the σ-derivation $\Delta=X\Delta_0$. Then $\Delta(X)=X$ and $\Delta(Y)=-X$. The conditions of Theorem 2.2.3 are not satisfied. Indeed, $\mathrm{Ann}(\Delta)=Y\mathbb{C}[Y]$ yields $\sigma(\mathrm{Ann}(\Delta))=X\mathbb{C}[X]\subsetneq\mathrm{Ann}(\Delta)$. The condition $\Delta\sigma=\delta\sigma\Delta$ is not satisfied either for any choice of $\delta\in\mathcal{A}$, since assuming existence of such δ and applying this relation to X one gets $-X=X\Delta_0(Y)=\Delta(Y)=\Delta\sigma(X)=\delta\sigma\Delta(X)=\delta\sigma(X\Delta_0(X))=\delta\sigma(X)=\delta Y$, which is impossible for $\delta\in\mathcal{A}$. For any element $a\in\mathcal{A}$ let

$$a=\mu_a+XP_a(X)+YQ_a(Y)=\mu_a+X\sum_{k=0}^{\deg P_a}P_{a,k}X^k+Y\sum_{k=0}^{\deg Q_a}Q_{a,k}Y^k$$

be the unique representation of a with respect to the direct sum decomposition $\mathcal{A} = \mathbb{C} \oplus X\mathbb{C}[X] \oplus Y\mathbb{C}[Y]$. In $\mathcal{A}$ the following equalities hold for any $a, b \in \mathcal{A}$:

$$\begin{aligned}
\Delta_0(X^k) &= X^{k-1} + Y^{k-1}, \qquad \Delta_0(Y^k) = -(X^{k-1} + Y^{k-1}) \quad \text{for } k > 1; \\
\Delta(X^k) &= X\Delta_0(X^k) = X^k, \qquad \Delta(Y^k) = X\Delta_0(Y^k) = -X^k \quad \text{for } k > 1; \\
\Delta(P_a(X)) &= \Delta\left(\sum_{k=0}^{\deg P_a} P_{a,k}X^k\right) = \sum_{k=1}^{\deg P_a} P_{a,k}X^k = P_a(X) - P_{a,0}; \\
\sigma(X)\Delta(P_a(X)) &= Y\Delta(P_a(X)) = 0; \\
\Delta(Q_a(Y)) &= \Delta\left(\sum_{k=0}^{\deg Q_a} Q_{a,k}Y^k\right) = -\sum_{k=1}^{\deg Q_a} Q_{a,k}X^k = -Q_a(X) + Q_{a,0} \\
&= -\Delta(Q_a(X)); \\
\Delta(a) &= \Delta(\mu_a + XP_a(X) + YQ_a(Y)) \\
&= \Delta(X)P_a(X) + \sigma(X)\Delta(P_a(X)) + \Delta(Y)Q_a(Y) + \sigma(Y)\Delta(Q_a(Y)) \\
&= XP_a(X) - XQ_a(X) = X(P_a - Q_a)(X); \\
b\Delta(a) &= (\mu_b + XP_b(X))X(P_a - Q_a)(X) \\
&= (\mu_b X + X^2 P_b(X))(P_a - Q_a)(X).
\end{aligned}$$

Then, using the definition or Proposition 2.3.1, one gets

$$\begin{aligned}
[a\Delta, b\Delta]_\sigma &= \Delta(b)a - \Delta(a)b \\
&= X(P_b - Q_b)(X)(\mu_a + XP_a(X)) - X(P_a - Q_a)(X)(\mu_b + XP_b(X)) \\
&= X(\mu_a(P_b - Q_b)(X) - \mu_b(P_a - Q_a)(X)) + X^2(Q_aP_b - Q_bP_a)(X).
\end{aligned}$$

There are several questions coming to mind concerning this example, which will be treated in further details in a forthcoming paper. For instance, does there exists a non-zero linear subspace of $\mathcal{A}\Delta$ which is closed under the bracket and where (4) holds? For the linear subspace $B_0 = \{a = \mu_a + XP_a(X) + YQ_a(Y) \in \mathcal{A} \mid P_a = Q_a\} = \{a = \lambda_a(P_a(X) + P_a(Y)) \in \mathcal{A} \mid \lambda_a \in \mathbb{C}\}$ of $\mathcal{A}$, the linear subspace $\mathcal{B}_0\Delta$ of the linear space $\mathcal{A}\Delta$ is closed under the bracket $[\cdot,\cdot]_\sigma$, since $[a\Delta, b\Delta]_\sigma = 0$ for all $a, b \in \mathcal{B}_0$. Since the bracket is identically zero on $\mathcal{B}_0$, the Jacobi identity (4) trivially holds for any $\delta \in \mathcal{A}$. As results from the formulas above we have $\mathcal{B}_0 = \operatorname{Ker}(\Delta)$. Actually, it follows directly from (2) that the bracket is identically zero in general, on the whole space $\operatorname{Ker}(\Delta)\Delta$ for any commutative algebra $\mathcal{A}$.

2.3. *Inner σ-derivations*

We recall from Definition 2.1.2 that a σ-derivation Δ is inner if and only if there exists an element $p \in \mathcal{A}$ such that $\Delta(a) = pa - \sigma(a)p$ for all $a \in \mathcal{A}$. Because $\mathcal{A}$ is commutative, it is easy to see that if Δ is inner, then $a\Delta$ is inner for all $a \in \mathcal{A}$, so that $\mathcal{A}\Delta \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. So we mainly get interested in the case where Δ itself is not inner.

Now we prove, that when defined, the bracket $[\cdot,\cdot]_\sigma$ of two inner σ-derivations is an inner σ-derivation.

Proposition 2.3.1. *Let σ be an endomorphism of a commutative algebra $\mathcal{A}$, and $\Delta \in \mathcal{D}_\sigma(\mathcal{A})$. Then $\Delta(b)\sigma(a) - \Delta(a)\sigma(b) = \Delta(b)a - \Delta(a)b$, and hence for the bracket satisfying* (2) *the following statements hold.*

1. *The equality $[a\Delta, b\Delta]_\sigma = (\Delta(b)a - \Delta(a)b)\Delta$ holds for all $a, b \in \mathcal{A}$.*
2. *Let $a, b \in \mathcal{A}$ be such that $a\Delta$ and $b\Delta$ are inner, that is $a\Delta = p_a(\mathrm{id} - \sigma)$ and $b\Delta = p_b(\mathrm{id} - \sigma)$ for some $p_a, p_b \in \mathcal{A}$. Then $[a\Delta, b\Delta]_\sigma$ is inner and $[a\Delta, b\Delta]_\sigma = c(\mathrm{id} - \sigma)$, with $c = \Delta(b)p_a - \Delta(a)p_b$.*
3. *Let $a \in \mathcal{A}$ be such that $a\Delta$ is inner, that is $a\Delta = p_a(\mathrm{id} - \sigma)$ for some $p_a \in \mathcal{A}$ and let $b \in \mathcal{A}$. Then $\sigma(a)\Delta(b) = (p_a - \Delta(a))(b - \sigma(b))$.*

Proof. 1. The equality $\Delta(b)\sigma(a) - \Delta(a)\sigma(b) = \Delta(b)a - \Delta(a)b$ is equivalent to the equality $(a - \sigma(a))\Delta(b) = (b - \sigma(b))\Delta(a)$ from Lemma 2.1.4, and both are just two different ways of rewriting the equality

$$\Delta(a)b + \sigma(a)\Delta(b) = \Delta(ab) = \Delta(ba) = \Delta(b)a + \sigma(b)\Delta(a)$$

using the σ-Leibniz rule for Δ and commutativity of $\mathcal{A}$.

2. So, if $a\Delta = p_a(\mathrm{id} - \sigma)$ and $b\Delta = p_b(\mathrm{id} - \sigma)$ for some $p_a, p_b \in \mathcal{A}$, then

$$\begin{aligned}[a\Delta, b\Delta]_\sigma &= \big(\Delta(b)a - \Delta(a)b\big)\Delta = \Delta(b)a\Delta - \Delta(a)b\Delta \\ &= \Delta(b)p_a(\mathrm{id} - \sigma) - \Delta(a)p_b(\mathrm{id} - \sigma) = \big(\Delta(b)p_a - \Delta(a)p_b\big)(\mathrm{id} - \sigma).\end{aligned}$$

3. If $b \in \mathcal{A}$ and $a\Delta = p_a(\mathrm{id} - \sigma)$ with $p_a \in \mathcal{A}$, then using the equality $(b - \sigma(b))\Delta(a) = (a - \sigma(a))\Delta(b)$ from Lemma 2.1.4 and commutativity of $\mathcal{A}$ one gets $\sigma(a)\Delta(b) = a\Delta(b) - (b - \sigma(b))\Delta(a) = p_a(b - \sigma(b)) - (b - \sigma(b))\Delta(a) = (p_a - \Delta(a))(b - \sigma(b))$. $\square$

Remark 2.3.2.

(1) In the UFD case, let σ, g and $\Delta = \Delta_0$ be defined as in Theorem 2.2.1, and $a\Delta = p_a(\mathrm{id} - \sigma)$ and $b\Delta = p_b(\mathrm{id} - \sigma)$ for some $p_a, p_b \in \mathcal{A}$. Then $a = gp_a$ and $b = gp_b$, and one gets $c = \sigma(g)(\Delta(p_b)p_a - \Delta(p_a)p_b)$.

(2) Note that if $\sigma(\mathrm{Ann}(\Delta)) \subseteq \mathrm{Ann}(\Delta)$, then $[\cdot,\cdot]_\sigma$ defined by (1) is a well-defined bilinear multiplication, and $(\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) \cap \mathcal{A}\Delta, [\cdot,\cdot]_\sigma)$ is a subalgebra of $(\mathcal{A}\Delta, [\cdot,\cdot]_\sigma)$ according to Proposition 2.3.1. However, we will see later that it does not need to be an ideal (see Remark 3.4.3). Note also that when $\mathcal{A}$ is UFD and $\sigma \neq \mathrm{id}$, there exists Δ such that any σ-derivation is of the form $a\Delta$ according to Theorem 2.2.1, that is $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$. Then $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) \cap \mathcal{A}\Delta = \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

(3) In the non-commutative case, with the notations of Definition 2.1.2, one can proof in the same way as in the proof of Proposition 2.3.1 that if $a\Delta = \Delta_{p_a}$ and $b\Delta = \Delta_{p_b}$ and $[a\Delta, b\Delta]_\sigma = (\Delta(b)a - \Delta(a)b)\Delta$ holds, then $[a\Delta, b\Delta]_\sigma = \Delta_t$, with $t = \Delta(b)p_a - \Delta(a)p_b$. Moreover, under the assumption that Δ sends the center $Z(\mathcal{A})$ of $\mathcal{A}$ to itself, we can still prove, that (2) implies $[a\Delta, b\Delta]_\sigma = (\Delta(b)a - \Delta(a)b)\Delta$ for $a, b \in Z(\mathcal{A})$. Actually, even

without the assumption $\Delta(Z(\mathcal{A})) \subseteq Z(\mathcal{A})$, this implication holds for all $a, b \in \mathcal{A}$ such that $\Delta(b)$ commutes with $\sigma(a)$ and $\Delta(a)$ commutes with $\sigma(b)$. Thus it holds for all $a, b \in \mathcal{A}$ also if any element of $\Delta(\mathcal{A})$ commutes with any element in $\sigma(\mathcal{A})$. Note that we assume that σ is just an endomorphism of $\mathcal{A}$ and thus might not be surjective, thus making the last condition more general than $\Delta(Z(\mathcal{A})) \subseteq Z(\mathcal{A})$.

2.4. The UFD case

We present here some general statements concerning σ-derivations in the UFD case. In the next section we will give some more precise and deep results in the particular case $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$. One can first precise Theorem 2.2.1 in the following way.

Proposition 2.4.1. *Let $\sigma \neq \mathrm{id}$ be an endomorphism of a unique factorization domain $\mathcal{A}$. Set $g = \gcd((\mathrm{id} - \sigma)(\mathcal{A}))$. Then*

(1) $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$, *with* $\Delta = \frac{\mathrm{id}-\sigma}{g}$;
(2) *the σ-derivation $a\Delta$ is inner if and only if g divides a. In other words, $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$. In particular, Δ itself is inner if and only if g is a unit.*

Proof. The first point is just Theorem 2.2.1. Then for any σ-derivation $\widetilde{\Delta} = a\Delta$ with $a \in \mathcal{A}$. Obviously, if $a = bg$ then $a\Delta = b(\mathrm{id} - \sigma)$ is inner.

Conversely, assume that $a\Delta$ is inner. Then there is an element $b \in \mathcal{A}$ such that $a\Delta(r) = b(r - \sigma(r))$ for all $r \in \mathcal{A}$. Multiplying by g one obtains $ag\Delta(r) = bg(r - \sigma(r))$, i.e. $a(r - \sigma(r)) = bg(r - \sigma(r))$ by definition of Δ. Now one can choose r such that $r - \sigma(r) \neq 0$ (existing by $\sigma \neq \mathrm{id}$), and conclude that $a = bg$ using the fact that $\mathcal{A}$ is a domain.

Finally, when $a = 1$, that is $a\Delta = \Delta$, the condition that g divides $a = 1$ means precisely the same thing as g being a unit (an invertible element in $\mathcal{A}$). $\square$

We aim now to use the bracket $[\cdot,\cdot]_\sigma$ to understand what is "between" $\mathcal{D}_\sigma(\mathcal{A})$ and $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. So we define some subspaces of $\mathcal{D}_\sigma(\mathcal{A})$, which will be more precisely described in the next section for $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$.

Recall that $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$ and $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$.

Definition 2.4.2. We define the following subspaces of $\mathcal{D}_\sigma(\mathcal{A})$.

(1) $S^1 = \mathrm{Span}_\mathbb{C}[\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}), \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})]_\sigma$;
(2) $\widetilde{S}_1 = \{\widetilde{\Delta} \in \mathcal{D}_\sigma(\mathcal{A}) \mid [\widetilde{\Delta}, S^1]_\sigma \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})\}$;
(3) $S_1 = \{\widetilde{\Delta} \in \mathcal{D}_\sigma(\mathcal{A}) \mid [\widetilde{\Delta}; S^1]_\sigma \subseteq S^1\}$.

Remark 2.4.3.

(1) The space S^1 would be the usual derived Lie algebra of $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ for $\sigma = \mathrm{id}$.
(2) It follows from Proposition 2.3.1 that one always has $S^1 \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.
(3) Because $S^1 \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ it follows from Proposition 2.3.1 that $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) \subseteq \widetilde{S}_1$. As we will see in Theorem 3.5.2, in the case of q-deformed Witt algebras, $\widetilde{S}_1$ is the whole space of σ-derivations. This is why we introduce the "smaller" space S_1.

(4) It is also clear that $S_1 \subseteq \widetilde{S}_1$, and S_1 is the normalizer of S^1 in $\mathcal{D}_\sigma(\mathcal{A})$ with respect to $[\cdot,\cdot]_\sigma$. We consider this space rather than the normalizer of $\mathfrak{Inn}_\sigma(\mathcal{A})$ because it will appear in Corollary 3.4.5 of Proposition 3.4.4 that this normalizer is just the space of inner σ-derivations for the context of the main specific example considered in depth further in the article.

Lemma 2.4.4. *Let $\mathcal{A}$ be a UFD. The following inclusion $S^1 \subseteq \sigma(g)g\mathcal{A}\Delta$ holds.*

Proof. This inclusion relies on Remark 2.3.2. We are in the UFD case, and if $a, b \in \mathcal{A}$ are such that $a\Delta, b\Delta \in \mathfrak{Inn}_\sigma(\mathcal{A})$, then $a = gp_a$ and $b = gp_b$ for some $p_a, p_b \in \mathcal{A}$, and $[a\Delta, b\Delta]_\sigma = \sigma(g)(\Delta(p_b)p_a - \Delta(p_a)p_b)g\Delta$. Hence $[\mathfrak{Inn}_\sigma(\mathcal{A}), \mathfrak{Inn}_\sigma(\mathcal{A})]_\sigma \subseteq \sigma(g)g\mathcal{A}\Delta$. □

The following corollary is a direct consequence of the preceding lemma and part (2) of Proposition 2.4.1.

Corollary 2.4.5. *Let $\mathcal{A}$ be a UFD.*

(1) *The strict inclusion $\sigma(g)g\mathcal{A}\Delta \subsetneq \mathfrak{Inn}_\sigma(\mathcal{A})$ holds if and only if $\sigma(g)$ is not a unit in $\mathcal{A}$ (i.e. not invertible element in $\mathcal{A}$).*
(2) *If $\sigma(g)$ is not a unit in $\mathcal{A}$, then the strict inclusion $S^1 \subsetneq \mathfrak{Inn}_\sigma(\mathcal{A})$ holds.*

Proof. (1) Since $\mathrm{Ann}(\Delta) = \{0\}$ and there are no zero divisors in $\mathcal{A}$ for any $a, b \in \mathcal{A}$:

$$\sigma(g)ga\Delta = gb\Delta \quad \Leftrightarrow \quad \big(\sigma(g)a - b\big)g = 0 \quad \Leftrightarrow \quad \sigma(g)a = b.$$

The equality $\sigma(g)g\mathcal{A}\Delta = \mathfrak{Inn}_\sigma(\mathcal{A})$ holds if and only if a solution $a \in \mathcal{A}$ for $\sigma(g)a = b$ exists for any $b \in \mathcal{A}$, which is equivalent to $\sigma(g)$ being invertible, i.e. a unit in $\mathcal{A}$.

(2) By Lemma 2.4.4, $S^1 \subseteq \sigma(g)g\mathcal{A}\Delta$. By part (1), if $\sigma(g)$ is not a unit in $\mathcal{A}$, then $\sigma(g)g\mathcal{A}\Delta \subsetneq \mathfrak{Inn}_\sigma(\mathcal{A})$. Thus $S^1 \subsetneq \mathfrak{Inn}_\sigma(\mathcal{A})$ as well. □

We can summarize the preceding inclusions as follows.

Proposition 2.4.6. *Let $\mathcal{A}$ be a UFD. Then $S^1 \subseteq \mathfrak{Inn}_\sigma(\mathcal{A}) \subseteq S_1 \subseteq \widetilde{S}_1$.*

Proof. Only the inclusion $\mathfrak{Inn}_\sigma(\mathcal{A}) \subseteq S_1$ is left to prove. Set $\widetilde{\Delta} \in \mathfrak{Inn}_\sigma(\mathcal{A})$. Since $S^1 \subseteq \mathfrak{Inn}_\sigma(\mathcal{A})$, we have

$$\big[\widetilde{\Delta}, S^1\big]_\sigma \subseteq \big[\widetilde{\Delta}, \mathfrak{Inn}_\sigma(\mathcal{A})\big]_\sigma \subseteq \big[\mathfrak{Inn}_\sigma(\mathcal{A}), \mathfrak{Inn}_\sigma(\mathcal{A})\big]_\sigma \subseteq S^1$$

because $\widetilde{\Delta} \in \mathfrak{Inn}_\sigma(\mathcal{A})$. By definition this means that $\widetilde{\Delta} \in S_1$. □

All these spaces will be considered in more detail in the case of Laurent polynomials $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ in the last section of the paper.

The following statement is an extension of a result proved in [33] for $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ to arbitrary unique factorization domains.

Proposition 2.4.7. *Let $\mathcal{A}$ be a UFD. Then $\widetilde{S}_1 = \mathcal{D}_\sigma(\mathcal{A})$.*

Proof. Since $g = \gcd((\mathrm{id} - \sigma)(\mathcal{A}))$ divides $g - \sigma(g)$ it also divides $\sigma(g)$. Hence, $\sigma(g) = gw$ for some $w \in \mathcal{A}$, and thus $\sigma(\sigma(g)) = \sigma(g)\sigma(w) = gw\sigma(w)$, in particular meaning that g divides $\sigma(\sigma(g))$. Therefore, for any $\widetilde{\Delta} = a\Delta \in \mathcal{D}_\sigma(\mathcal{A})$ and any $\Delta_1 = \sigma(g)gb\Delta \in S^1$, using the part (1) of Proposition 2.3.1 and commutativity of $\mathcal{A}$, it follows that

$$\begin{aligned}
[\widetilde{\Delta}, \Delta_1]_\sigma &= \big[a\Delta, \sigma(g)gb\Delta\big]_\sigma = \big(\Delta\big(\sigma(g)gb\big)a - \Delta(a)\sigma(g)gb\big)\Delta \\
&= \big(g\big(\Delta\big(\sigma(g)b\big)a - \Delta(a)\sigma(g)b\big) + \sigma\big(\sigma(g)\big)\Delta(g)\sigma(b)a\big)\Delta \\
&= g\big(\Delta\big(\sigma(g)b\big)a - \Delta(a)\sigma(g)b + w\sigma(w)\Delta(g)\sigma(b)a\big)\Delta \in \mathfrak{Inn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta,
\end{aligned}$$

which proves that $\widetilde{S}_1 = \mathcal{D}_\sigma(\mathcal{A})$ for any unique factorization domain $\mathcal{A}$. □

3. Non-linearly q-deformed Witt algebras

We develop now the preceding framework for a particular algebra $\mathcal{A}$, namely $\mathbb{C}[t^{\pm 1}]$, in order to obtain some more deep and precise results. In this case, $\mathcal{D}_\sigma(\mathcal{A})$ with the twisted bracket is the deformation of the Witt algebra in the sense of [16]. The deformations of the Witt algebra are of importance in mathematical physics (see [16] and references therein, and the introduction of the present work). Note that as $\mathbb{C}[t^{\pm 1}]$ is a UFD, Proposition 2.4.1 and Proposition 2.4.6 apply.

Most of the articles on σ-derivations of $\mathbb{C}[t^{\pm 1}]$ are concerned with the case where σ is an automorphism (see for instance [21,32]). We do not assume here that σ is an automorphism, so it involves some power s of t, which as we will see plays a crucial role in the study of $\mathcal{D}_\sigma(\mathcal{A})$.

3.1. Some notations

In this subsection we review some notations and definitions that will be used throughout this article.

Degree, valuation. For a Laurent polynomial $f(t) = \sum_{n=n_0}^{n_1} \alpha_n t^n$ with $\alpha_n \in \mathbb{C}$, $\alpha_{n_0} \neq 0$, $\alpha_{n_1} \neq 0$ we denote $\nu(f) = n_0$ the valuation of f and $\deg(f) = n_1$ its degree.

The endomorphism σ. Because an endomorphism of the algebra $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ sends units to units, the image of t by σ must be a monomial. So denote $\sigma(t) = qt^s$, with $q \in \mathbb{C}^*$ and $s \in \mathbb{Z}$. Note that σ is injective if and only if $s \neq 0$, and surjective if and only if $s = 1$ or $s = -1$.

Generators of $\mathcal{D}_\sigma(\mathcal{A})$. If $\sigma \neq \mathrm{id}$ then by Theorem 2.2.1 one has $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$ with $\Delta(f) = (f - \sigma(f))/g$, where $g = \gcd((\mathrm{id} - \sigma)(\mathcal{A}))$. Then one can check (see [16, Example 3.2]) that $g = \alpha^{-1}t^{k-1}(t - qt^s)$ with $\alpha \in \mathbb{C}^*$ and $k \in \mathbb{Z}$. Since g is defined up to a unit, then α and k are arbitrary. If $s \geqslant 1$ then choose $k = 0$ and $\alpha = 1$, so that $g(t) = 1 - qt^{s-1}$. If $s \leqslant 0$ then choose $k = -s + 1$ and $\alpha = -q$, so that $g(t) = -q^{-1}(t^{1-s} - q) = 1 - q^{-1}t^{1-s}$. To avoid repetition of these two cases in future claims, we denote ϵ the sign of $s - 1$, with the convention that $\epsilon = 1$ if $s = 1$. Then $g(t) = 1 - q^\epsilon t^d$ is a usual polynomial ($\nu(g) = 0$) of degree $d = |s - 1|$ such that $g(0) = 1$. For $s \neq 1$ one has $d \geqslant 1$. Note that with our conventions, for $\sigma = \mathrm{Id}$ (i.e. $s = 1 = q$), one gets $g = 0$. It follows easily from the choices we made while defining Δ that

$$\Delta(t) = \begin{cases} t & \text{if } s \geqslant 1; \\ -qt^s & \text{if } s < 1. \end{cases} \tag{5}$$

The monomial $T = qt^{s-1} = qt^{\epsilon d}$ will play a crucial role in the following. Note that $g = 1 - q^{\epsilon} t^d = 1 - q^{\epsilon} t^{(s-1)\epsilon} = 1 - (qt^{(s-1)})^{\epsilon} = 1 - T^{\epsilon}$ which is equal to $1 - T$ if $s \geqslant 1$, and $1 - T^{-1} = -T^{-1}(1 - T)$ if $s < 1$. Thanks to Proposition 2.4.1 this means that $\mathfrak{Inn}_\sigma(\mathcal{A})$ is generated by $(1 - T)\Delta$ as an $\mathcal{A}$-module.

We consider the basis $(\delta_n)_{n\in\mathbb{Z}}$ of $\mathcal{D}_\sigma(\mathcal{A})$ defined by $\delta_n = -t^n \Delta$ for all $n \in \mathbb{Z}$. Note also that $\sigma(T) = T^s$, and that T "acts" on $\mathcal{D}_\sigma(\mathcal{A})$ as $T\delta_n = q\delta_{n+s-1}$. Last, since $\Delta(t) \neq 0$ and $\mathcal{A}$ is a domain, we have $\operatorname{Ann}(\Delta) = \{0\}$.

3.2. Decomposition of $\mathcal{D}_\sigma(\mathbb{C}[t^{\pm 1}])$

If $s = 1$ then two cases may occur. Firstly, if $s = 1$ and $q = 1$ then σ is the identity map, so one gets the usual Witt algebra, and we will not consider this case here. Secondly, if $s = 1$ and $q \neq 1$ then $g = 1 - q$ is a unit, and Proposition 2.4.1 implies that all σ-derivations are inner, so that $\mathcal{D}_\sigma(\mathcal{A}) = \mathfrak{Inn}_\sigma(\mathcal{A}) = \widetilde{S}_1 = S_1$. Also T is a unit, and we can deduce from Proposition 3.3.3 below and Proposition 2.4.1 that $S^1 = \mathcal{D}_\sigma(\mathcal{A})$.

As we are interested in the study of what happens "between" $\mathcal{D}_\sigma(\mathcal{A})$ and $\mathfrak{Inn}_\sigma(\mathcal{A})$, we shall assume that $s \neq 1$. Note that then g is not a unit in $\mathcal{A}$, so thanks to Corollary 2.4.5 we have $S^1 \subsetneq \mathfrak{Inn}_\sigma(\mathcal{A})$.

The vector space $\mathcal{D}_\sigma(\mathcal{A})$ is made a non-associative algebra thanks to the bracket $[\cdot,\cdot]_\sigma$ defined in Theorem 2.2.3. Since g is not a unit, the set $\mathfrak{Inn}_\sigma(\mathcal{A})$ of inner σ-derivations is a proper subalgebra of $(\mathcal{D}_\sigma(\mathcal{A}), [\cdot,\cdot]_\sigma)$. Moreover, thanks to Proposition 2.4.1 we know that a σ-derivation $f\Delta$ is inner if and only if g divides f, that is $\mathfrak{Inn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$. This leads to the following result.

Theorem 3.2.1. *Let σ be the endomorphism of $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ defined by $\sigma(t) = qt^s$. From Proposition* 2.4.1 *we have $\mathcal{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$. We denote $d = |s - 1|$, and $\delta_n = -t^n \Delta$ for all $n \in \mathbb{Z}$. Assume that $s \neq 1$. Then*

$$\mathcal{D}_\sigma(\mathcal{A}) = \mathbb{C}\delta_0 \oplus \mathbb{C}\delta_1 \oplus \cdots \oplus \mathbb{C}\delta_{d-1} \oplus \mathfrak{Inn}_\sigma(\mathcal{A}) \tag{6}$$

$$= \mathbb{C}\Delta \oplus \mathbb{C}t\Delta \oplus \cdots \oplus \mathbb{C}t^{d-1}\Delta \oplus \mathfrak{Inn}_\sigma(\mathcal{A}). \tag{7}$$

Hence any $\widetilde{\Delta} \in \mathcal{D}_\sigma(\mathcal{A})$ can be uniquely decomposed as $\widetilde{\Delta} = f(t)\Delta + h(t)(\mathrm{id} - \sigma)$ so that $f \in \mathbb{C}[t]$ with $\deg(f) < d$ *and $h(t) \in \mathbb{C}[t^{\pm 1}]$.*

Proof. Note that for all $n \in \mathbb{N}$ one has $\mathbb{C}\delta_n = \mathbb{C}t^n\Delta$. We first show that $\mathcal{D}_\sigma(\mathcal{A}) = \mathbb{C}\Delta + \mathbb{C}t\Delta + \cdots + \mathbb{C}t^{d-1}\Delta + \mathfrak{Inn}_\sigma(\mathcal{A})$. Take any non-zero $\widetilde{\Delta} = f(t)\Delta \in \mathcal{D}_\sigma(\mathcal{A})$, with $f(t) = \sum_{n=n_0}^{n_1} \alpha_n t^n$, with $\nu(f) = n_0$ and $\deg(f) = n_1$. Up to an inner σ-derivation we can always assume that $\nu(f) \geqslant 0$. If not, consider $f_1 = f - \alpha_{n_0} t^{n_0} g$: we have $f\Delta = f_1\Delta + \alpha_{n_0} t^{n_0} g\Delta$, $\alpha_{n_0} t^{n_0} g\Delta \in \mathfrak{Inn}_\sigma(\mathcal{A})$ and $\nu(f_1) > \nu(f)$. If $\nu(f_1) \geqslant 0$ we are done, else we repeat this operation with f_1. Then after at most $\nu(f)$ iterations we have a polynomial $\tilde{f}$ such that $\nu(\tilde{f}) \geqslant 0$ and g divides $f - \tilde{f}$, that is $(f\Delta - \tilde{f}\Delta) \in \mathfrak{Inn}_\sigma(\mathcal{A})$. So assume that $f \in \mathbb{C}[t]$. Now we can make the usual Euclidian division of f by g in $\mathbb{C}[t]$, and we obtain $f = q(t)g(t) + r(t)$, with $\deg(r) < \deg(g) = d$. Since $\mathfrak{Inn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$ we are done.

Now we prove that this sum is a direct sum. Set $\alpha_0, \ldots, \alpha_{d-1}$ in $\mathbb{C}$ and $f \in \mathbb{C}[t^{\pm 1}]$ such that $\sum_{i=0}^{d-1} \alpha_i t^i \Delta + fg\Delta = 0$. Since $\operatorname{Ann}(\Delta) = \{0\}$ this implies $\sum_{i=0}^{d-1} \alpha_i t^i + fg = 0$. First we prove that f must be a non-Laurent polynomial, that is $\nu(f) \geqslant 0$. Assume on the contrary that $\nu(f) = n_0 < 0$, and f has lowest degree term $\beta_{n_0} t^{n_0} \neq 0$. Then because $g = 1 - \lambda t^d$ with $d > 0$, the term of lowest degree of $\sum_{i=0}^{d-1} \alpha_i t^i + fg$ is $\beta_{n_0} t^{n_0}$, a contradiction since $\sum_{i=0}^{d-1} \alpha_i t^i + fg = 0$.

Now the latest equality is nothing else but the Euclidian division of the 0 polynomial by g in $\mathbb{C}[t]$. By uniqueness we have $\alpha_i = 0$ for all i and $f = 0$. $\square$

In Subsection 3.4 we will give a description of the bracket in $\mathcal{D}_\sigma(\mathcal{A})$ in terms of this decomposition, thanks to the brackets computed in [16]. But first we re-interpret these in terms of the element $T = qt^{s-1}$ defined in Subsection 3.1, and show that the algebra $\mathcal{D}_\sigma(\mathcal{A})$ is graded by a finite cyclic group.

3.3. Grading of $\mathcal{D}_\sigma(\mathcal{A})$

We recall first the following result.

Theorem 3.3.1. *(See Hartwig, Larsson, Silvestrov, [16, Theorem 31].) When $\mathcal{A}$ is $\mathbb{C}[t^{\pm 1}]$, the $\mathbb{C}$-linear space $\mathcal{D}_\sigma(\mathcal{A}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot d_n$, where $D = \alpha t^{-k+1}\frac{id-\sigma}{t-qt^s}$, $d_n = -t^n D$ and $\sigma(t) = qt^s$, $s \in \mathbb{Z}$ can be equipped with the bracket product*

$$\langle\cdot,\cdot\rangle_\sigma : \mathcal{D}_\sigma(\mathcal{A}) \times \mathcal{D}_\sigma(\mathcal{A}) \longrightarrow \mathcal{D}_\sigma(\mathcal{A})$$

defined on generators according to (1) *as*

$$[d_n, d_m]_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n. \tag{8}$$

This bracket satisfies defining commutation relations

- $[d_n, d_m]_\sigma = \alpha \operatorname{sign}(n-m) \displaystyle\sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} d_{s(n+m-1)-(k-1)-l(s-1)}$ *for* $n, m \geqslant 0$;
- $[d_n, d_m]_\sigma = \alpha \left(\displaystyle\sum_{l=0}^{-m-1} q^{n+m+l} d_{(m+l)(s-1)+ns+m-k} + \sum_{l=0}^{n-1} q^{m+l} d_{(s-1)l+n+ms-k} \right)$ *for* $n \geqslant 0,\ m < 0$;
- $[d_n, d_m]_\sigma = -\alpha \left(\displaystyle\sum_{l_1=0}^{m-1} q^{n+l_1} d_{(s-1)l_1+m+ns-k} + \sum_{l_2=0}^{-n-1} q^{m+n+l_2} d_{(n+l_2)(s-1)+n+ms-k} \right)$ *for* $m \geqslant 0,\ n < 0$;
- $[d_n, d_m]_\sigma = \alpha \operatorname{sign}(n-m) \displaystyle\sum_{l=\min(-n,-m)}^{\max(-n,-m)-1} q^{n+m+l} d_{(m+n)s+(s-1)l-k}$ *for* $n, m < 0$.

Furthermore, this bracket satisfies skew-symmetry $[d_n, d_m]_\sigma = -[d_m, d_n]_\sigma$ and a twisted Jacobi identity (*written explicitly in* [16, *Theorem* 31]).

Remark 3.3.2.

(1) If $s = 1$ then (8) becomes $[d_n, d_m]_q = q^n d_n d_m - q^m d_m d_n$ which is the bracket for the usual q-Witt algebra associated to Jackson q-derivative [16, Theorem 27], reducing further to the usual commutator for Witt Lie algebra if $s = 1$ and $q = 1$.

(2) Note that D and generators d_n are more general than Δ and generators δ_n we defined here, because we have specially chosen $g = \gcd((\mathrm{id} - \sigma)(\mathcal{A}))$ so that g becomes a usual (non-Laurent) polynomial in t for any $s \in \mathbb{Z}$ both when $s - 1 \geqslant 0$ and when $s - 1 < 0$, in contrast to Theorem 3.3.1 not making any distinction of this kind or polynomial requirement on g.

Now for arbitrary $s \in \mathbb{Z}$, recall that $T = qt^{s-1}$, and $\sigma(T) = T^s$. For all $n \in \mathbb{Z}$ define the T-integer as a geometric sum, more precisely one has $\{0\}_T = 0$, $\{n\}_T = \sum_{k=0}^{n-1} T^k$ for $n > 0$ and $\{n\}_T = -\sum_{k=n}^{-1} T^k$ for $n < 0$. Note that with this definition $(1 - T)\{n\}_T = 1 - T^n$ for all $n \in \mathbb{Z}$. So, if $T \neq 1$ (that is $\sigma \neq \mathrm{id}$), then the formula $\{n\}_T = \frac{1-T^n}{1-T} \in \mathbb{C}[T^{\pm 1}]$ can be used in computations for all $n \in \mathbb{Z}$ with this quotient meaning the geometric sums $\{n\}_T$. We also will use the following notation

$$\begin{aligned}
\{n\}_{T;\epsilon} &= \frac{1 - T^n}{1 - T^\epsilon} = \frac{1 - T^n}{1 - T}\frac{1 - T}{1 - T^\epsilon} = \frac{1 - (T^\epsilon)^{\epsilon n}}{1 - T^\epsilon} \\
&= \{\epsilon n\}_{T^\epsilon} = \frac{\{n\}_T}{\{\epsilon\}_T} = \begin{cases} \{0\}_{T;\epsilon} = 0, & \text{for } \epsilon = \pm 1; \\ \sum_{k=0}^{n-1} T^k,\ n > 0, & \text{for } \epsilon = 1; \\ -\sum_{k=1}^{n} T^k,\ n > 0, & \text{for } \epsilon = -1; \\ -\sum_{k=n}^{-1} T^k,\ n < 0, & \text{for } \epsilon = 1; \\ \sum_{k=n+1}^{0} T^k,\ n < 0, & \text{for } \epsilon = -1; \end{cases}
\end{aligned}$$

which turns out to be handy when treating all $s \in \mathbb{Z}$ and $n \in \mathbb{Z}$ in the same time.

Thanks to these notations we shall rewrite the preceding formulas in the following way.

Proposition 3.3.3. *For all $n, m \in \mathbb{Z}$, the following relations hold*:

$$[\delta_n, \delta_m]_\sigma = T^n \delta_n \delta_m - T^m \delta_m \delta_n = \big(\{n\}_{T;\epsilon} - \{m\}_{T;\epsilon}\big)\delta_{n+m}. \tag{9}$$

Proof. Recall that $g = 1 - T^\epsilon$. For all $n, m \in \mathbb{Z}$ we have

$$\begin{aligned}
[\delta_n, \delta_m]_\sigma &= \big(\Delta\big(-t^m\big)\big(-t^n\big) - \Delta\big(-t^n\big)\big(-t^m\big)\big)\Delta \\
&= \left(\frac{-t^m + (qt^s)^m}{g}\big(-t^n\big) - \frac{-t^n + (qt^s)^n}{g}\big(-t^m\big)\right)\Delta \\
&= \left(\frac{t^{m+n} - (T)^m t^{m+n}}{1 - T^\epsilon} - \frac{t^{n+m} - T^n t^{n+m}}{1 - T^\epsilon}\right)\Delta \\
&= \left(\frac{1 - (T)^n}{1 - T^\epsilon} - \frac{1 - T^m}{1 - T^\epsilon}\right)\big(-t^{n+m}\Delta\big) \\
&= \big(\{n\}_{T;\epsilon} - \{m\}_{T;\epsilon}\big)\delta_{n+m}. \qquad \square
\end{aligned}$$

Let us remark here that the second expression in formula (9) shows that these non-linearly deformed Witt algebras, constructed *a priori* in [16] by taking any endomorphism of $\mathbb{C}[t^{\pm 1}]$ instead of the automorphism defined by $t \mapsto qt$, really “look like” the q-Witt algebra. More precisely, if one takes for σ the automorphism $t \mapsto qt$, then $s = 1$, so $\epsilon = 1$, $d = 0$ and $T = q$.

Then $\{n\}_{T;\epsilon} = \{n\}_T = \{n\}_q$ are the usual q-integers, and (9) is the usual bracket of the q-Witt algebra, as defined for instance in [16] or [21], and leading for $q = 1$ to the classical Witt algebra.

Note that $\mathcal{A} = \bigoplus_{i=0}^{d-1} t^i \mathbb{C}[T^{\pm 1}]$ is naturally a $\mathbb{Z}/d\mathbb{Z}$-graded ring, and that the natural map $\mathcal{A} \to \mathcal{A}\Delta = \mathcal{D}_\sigma(\mathcal{A})$ is an isomorphism of graded $\mathbb{Z}$-modules. Concerning the quasi-Lie structure, we show that $\mathcal{D}_\sigma(\mathcal{A})$ admits a $\mathbb{Z}/d\mathbb{Z}$-gradation with coefficients in $\mathbb{C}[T^{\pm 1}]$. This echoes the $\mathbb{Z}$-gradation of the q-Witt algebra (case $d = 0$) with coefficients in $\mathbb{C}$.

Theorem 3.3.4. *Let σ be the endomorphism of $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ defined by $\sigma(t) = qt^s$, with $q \in \mathbb{C}^*$ and $s \in \mathbb{Z}$. Recall that $\mathcal{D}_\sigma(\mathcal{A})$, the space of σ-derivations of $\mathcal{A}$, is endowed with the bracket $[\cdot,\cdot]_\sigma$ defined in Theorem* 2.2.3. *Define $d = |s-1|$, and denote $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$, in particular for $s = 1$ one has $\mathbb{Z}/\{0\} = \mathbb{Z}$. For any $k \in \mathbb{Z}$ note $\bar{k} = k + d\mathbb{Z} \in \mathbb{Z}_d$.*

The non-associative algebra $(\mathcal{D}_\sigma(\mathcal{A}), [\cdot,\cdot]_\sigma)$ is $\mathbb{Z}_d$-graded: $\mathcal{D}_\sigma(\mathcal{A}) = \bigoplus_{\bar{k}\in\mathbb{Z}_d} \mathcal{D}_{\bar{k}}$, with $\mathcal{D}_{\bar{k}} = \mathbb{C}[T^{\pm 1}]\delta_k$ for any $k \in \bar{k}$.

Proof. The case $s = 1$ is straightforward. So assume $s \neq 1$. Then $T = qt^{\epsilon d}$, with $d \geqslant 1$. So $\mathbb{C}[t^{\pm 1}] = \bigoplus_{i=0}^{d-1} t^i \mathbb{C}[T^{\pm 1}]$ as vector spaces. Now the direct sum in the theorem follows from this and from the fact that $\operatorname{Ann} \Delta = \{0\}$. The grading results directly from formulas (9) and from the fact that $t^{n+m} = t^{n+m-d} q^{-\epsilon} T^\epsilon$. $\square$

Remark 3.3.5. The homogeneous part of degree 0 is a quasi-Lie subalgebra, with linear basis $(-T^n\Delta,\ n \in \mathbb{Z})$, and the relations

$$\left[-T^n\Delta, -T^m\Delta\right]_\sigma = \left(\left\{(s-1)n\right\}_{T;\epsilon} - \left\{(s-1)m\right\}_{T;\epsilon}\right)\left(-T^{n+m}\Delta\right),$$

proven as follows:

$$\begin{aligned}
\left[-T^n\Delta, -T^m\Delta\right]_\sigma &= \left(\sigma\left(-T^n\right)\Delta\left(-T^m\right) - \sigma\left(-T^m\right)\Delta\left(-T^n\right)\right)\Delta \\
&= \left(T^{sn}\frac{T^m - T^{sm}}{1-T^\epsilon} - T^{sm}\frac{T^n - T^{sn}}{1-T^\epsilon}\right)\Delta = \frac{T^{sn+m} - T^{sm+n}}{1-T^\epsilon}\Delta \\
&= \left(\left\{(s-1)n\right\}_{T;\epsilon} - \left\{(s-1)m\right\}_{T;\epsilon}\right)\left(-T^{n+m}\Delta\right).
\end{aligned}$$

3.4. The bracket "modulo inner σ-derivations"

Motivated by the previous results we are interested next in obtaining a more detailed description of what the relations for the bracket in Theorem 3.3.1 become modulo inner σ-derivations.

Notation. We will use the following notation for congruence of two σ-derivations modulo inner σ-derivations: $\forall \widetilde{\Delta}, \widehat{\Delta} \in \mathcal{D}_\sigma(\mathcal{A})$, the expression $\widetilde{\Delta} \equiv \widehat{\Delta}$ means that $\widetilde{\Delta} - \widehat{\Delta} \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

Motivated by Theorem 3.2.1, and since $[\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}), \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})]_\sigma \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ and the bracket is skew-symmetric, computation modulo $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ of the brackets of the following two types are of a special interest:

$$\begin{aligned}
&[\delta_n, \delta_m]_\sigma, \quad \text{with } 0 \leqslant n < m \leqslant d-1; \\
&[\delta_n, g\delta_m]_\sigma, \quad \text{with } m \in \mathbb{Z} \text{ and } 0 \leqslant n \leqslant d-1. \qquad (10)
\end{aligned}$$

Recall we denote by ϵ the sign of $s-1$ with convention $\epsilon = 1$ if $s = 1$. As noticed in Subsection 3.1, for all $s \in \mathbb{Z}$ one has $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = g\mathcal{D}_\sigma(\mathcal{A}) = (1-T^\epsilon)\mathcal{D}_\sigma(\mathcal{A}) = (1-T)\mathcal{D}_\sigma(\mathcal{A})$.

Lemma 3.4.1. *For all $n, m \in \mathbb{Z}$, one has*:

(1) $[\delta_n, \delta_m]_\sigma \equiv \epsilon(n-m)\delta_{n+m}$;
(2) $\delta_m \equiv q^{-\epsilon}\delta_{m-d}$.

Proof. (1) We know that $[\delta_n, \delta_m]_\sigma = (\{n\}_{T;\epsilon} - \{m\}_{T;\epsilon})\delta_{n+m}$ according to Proposition 3.3.3. Since $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}) = (1-T)\mathcal{A}\Delta$, we must compute the remainder of the Euclidian division of the polynomial $F(T) = \{n\}_{T;\epsilon} - \{m\}_{T;\epsilon}$ by $1-T$. This reminder is $F(1) = \epsilon(n-m)$.

(2) For any $m \in \mathbb{Z}$ we have

$$\begin{aligned}\delta_m &= -t^m\Delta = -t^d t^{m-d}\Delta = t^d\delta_{m-d} = t^{\epsilon(s-1)}\delta_{m-d} = q^{-\epsilon}\big(qt^{s-1}\big)^\epsilon\delta_{m-d}\\ &= q^{-\epsilon}T^\epsilon\delta_{m-d} = q^{-\epsilon}\delta_{m-d} - \big(1-T^\epsilon\big)q^{-\epsilon}\delta_{m-d} = q^{-\epsilon}\delta_{m-d} - gq^{-\epsilon}\delta_{m-d} \equiv q^{-\epsilon}\delta_{m-d}\end{aligned}$$

modulo inner σ-derivations $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. □

The following properties follow directly from Lemma 3.4.1.

Proposition 3.4.2. *The following statements hold.*

(1) *For all $n, m \in \mathbb{Z}$ such that $0 \leqslant n, m < d$ and $n+m < d$*:

$$\begin{aligned}&[\delta_n, \delta_m]_\sigma \equiv \epsilon(n-m)\delta_{n+m};\\ &[\delta_0, \delta_m]_\sigma = -\epsilon m\delta_m + \widetilde{\Delta}, \quad \textit{where } \widetilde{\Delta} \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}).\end{aligned}$$

(2) *For all $n, m \in \mathbb{Z}$ such that $0 \leqslant n, m < d$ and $d \leqslant n+m < 2d$*:

$$\begin{aligned}&[\delta_n, \delta_m]_\sigma \equiv \epsilon(n-m)q^{-\epsilon}\delta_{n+m-d};\\ &[\delta_0, \delta_m]_\sigma = -\epsilon mq^{-\epsilon}\delta_{m-d} + \widetilde{\Delta}, \quad \textit{where } \widetilde{\Delta} \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}).\end{aligned}$$

(3) *For $n, m, p \in \mathbb{Z}$ such that $pd \leqslant n+m < (p+1)d$*:

$$\begin{aligned}&[\delta_n, \delta_m]_\sigma \equiv \epsilon(n-m)q^{-\epsilon p}\delta_{n+m-pd};\\ &[\delta_0, \delta_m]_\sigma \equiv -\epsilon mq^{-\epsilon p}\delta_{m-pd};\\ &[\delta_0, \delta_m]_\sigma = -\epsilon mq^{-\epsilon p}\delta_{m-pd} + \widetilde{\Delta}, \quad \textit{where } \widetilde{\Delta} \in \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A}).\end{aligned}$$

Remark 3.4.3. Note that in the first formula, there is no more q appearing, just like in the classical Witt algebra. This however does not induce such a formula on the quotient space $\mathcal{D}_\sigma(\mathcal{A})/\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ because $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$ is not an ideal for the $[\cdot,\cdot]_\sigma$ bracket, as results from the next proposition.

For the other type of bracket in (10) we have the following properties.

Proposition 3.4.4. *For all $n, m \in \mathbb{Z}$ the following formulas hold.*

$$[\delta_n, g\delta_m]_\sigma = [\delta_n, \delta_m]_\sigma - q^\epsilon[\delta_n, \delta_{m+d}]_\sigma; \tag{11}$$

$$[\delta_n, g\delta_m]_\sigma \equiv \epsilon d\delta_{n+m}, \qquad [g\delta_n, \delta_m]_\sigma \equiv -\epsilon d\delta_{n+m}; \tag{12}$$

$$\big[\delta_n, g^r\delta_m\big]_\sigma \equiv 0, \qquad \big[g^r\delta_n, \delta_m\big]_\sigma \equiv 0 \quad \text{for all } r, n, m \in \mathbb{Z} \text{ such that } r \geqslant 2; \tag{13}$$

$$\big[g^u\delta_n, g^v\delta_m\big]_\sigma \equiv 0 \quad \text{for all } u, v, n, m \in \mathbb{Z} \text{ such that } u, v \geqslant 1; \tag{14}$$

$$[\delta_n, g\delta_m]_\sigma \equiv \epsilon d\delta_{n+m} \equiv \epsilon d q^{-\epsilon p}\delta_{n+m-pd} = (s-1)q^{-\epsilon p}\delta_{n+m-p|s-1|}$$
$$\text{for all } n, m, p \in \mathbb{Z} \text{ such that } pd \leqslant n+m < (p+1)d; \tag{15}$$

$$[\delta_n, g\delta_0]_\sigma = -[\delta_n, g\Delta]_\sigma \equiv -\epsilon d\delta_n \equiv -\epsilon d q^{-\epsilon p}\delta_{n-pd} = -(s-1)q^{-\epsilon p}\delta_{n-p|s-1|}$$
$$\text{for all } n, p \in \mathbb{Z} \text{ such that } pd \leqslant n < (p+1)d; \tag{16}$$

$$[\delta_0, g\delta_m]_\sigma = -[\Delta, g\delta_m]_\sigma \equiv -\epsilon d\delta_m \equiv -\epsilon d q^{-\epsilon p}\delta_{m-pd} = -(s-1)q^{-\epsilon p}\delta_{m-p|s-1|}$$
$$\text{for all } m, p \in \mathbb{Z} \text{ such that } pd \leqslant m < (p+1)d; \tag{17}$$

$$[\delta_0, g\delta_0]_\sigma = [\Delta, g\Delta]_\sigma \equiv \epsilon d\delta_0 = -\epsilon d\Delta. \tag{18}$$

Proof. The formula (11) follows from bilinearity of the bracket, $g = 1 - T^\epsilon$ and $T^\epsilon\delta_m = q^\epsilon\delta_{m+d}$. The formulas (12) and (13) are obtained using part (1) of Proposition 3.4.2, part (2) of Lemma 3.4.1 and binomial theorems as follows:

$$\begin{aligned}
\big[\delta_n, g^r\delta_m\big]_\sigma &= \big[\delta_n, \big(1 - T^\epsilon\big)^r\delta_m\big]_\sigma = \sum_{k=0}^{r}(-1)^k\binom{r}{k}\big[\delta_n, T^{\epsilon k}\delta_m\big]_\sigma \\
&= \sum_{k=0}^{r}(-1)^k\binom{r}{k}q^{\epsilon k}[\delta_n, \delta_{m+dk}]_\sigma \equiv \sum_{k=0}^{r}(-1)^k\binom{r}{k}q^{\epsilon k}\epsilon(n-m-dk)\delta_{n+m+dk} \\
&\equiv \sum_{k=0}^{r}(-1)^k\binom{r}{k}q^{\epsilon k}\epsilon(n-m-dk)q^{-\epsilon k}\delta_{n+m} \\
&= \left(\sum_{k=0}^{r}(-1)^k\binom{r}{k}\epsilon(n-m-dk)\right)\delta_{n+m} \\
&= \left(\epsilon(n-m)\left(\sum_{k=0}^{r}(-1)^k\binom{r}{k}\right) - \epsilon d\left(\sum_{k=0}^{r}(-1)^k\binom{r}{k}k\right)\right)\delta_{n+m} \\
&= \begin{cases} 0, & \text{for } r \geqslant 2, \\ \epsilon d\delta_{n+m}, & \text{for } r = 1. \end{cases}
\end{aligned}$$

The other two formulas in (12) and (13) follow from the first ones by the skew-symmetry of the bracket. Then (14) follows from the fact that the bracket of two inner σ-derivations is still inner.

All the other formulas follow by (12) and Lemma 3.4.1. □

Corollary 3.4.5. $\{\widetilde{\Delta} \in \mathcal{D}_\sigma(\mathcal{A}) \mid [\widetilde{\Delta}, \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})]_\sigma \subseteq \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})\} = \mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$.

Proof. We already know that $\mathfrak{Inn}_\sigma(\mathcal{A})$ is included in the space on the left-hand side of the equality by Proposition 2.3.1. Conversely, consider a polynomial $P \in \mathcal{A}$ such that $P\Delta$ belongs to this space. Then in particular one must have $[P\Delta, g\Delta] \in \mathfrak{Inn}_\sigma(\mathcal{A})$. But from formulas in Proposition 3.4.4 we have $[P\Delta, g\Delta]_\sigma \equiv \epsilon d P\Delta$, and so $dP\Delta \in \mathfrak{Inn}_\sigma(\mathcal{A})$, that is $P\Delta \in \mathfrak{Inn}_\sigma(\mathcal{A})$. □

3.5. The spaces S_1 and S^1

As before, let $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$, $\sigma(t) = qt^s$, and $g = 1 - (qt^{s-1})^\epsilon$, with $\epsilon = \operatorname{sign}(s-1)$ and convention $\epsilon = 1$ if $s = 1$. We have shown previously that $\mathfrak{D}_\sigma(\mathcal{A}) = \mathcal{A}\Delta$, $\mathfrak{Inn}_\sigma(\mathcal{A}) = g\mathcal{A}\Delta$ and $[\mathfrak{Inn}_\sigma(\mathcal{A}), \mathfrak{Inn}_\sigma(\mathcal{A})]_\sigma \subseteq \mathfrak{Inn}_\sigma(\mathcal{A})$. Recall also the following notations from Section 2.4.

Definition 3.5.1.

$$\begin{aligned} S^1 &= \operatorname{Span}_{\mathbb{C}}\big[\mathfrak{Inn}_\sigma(\mathcal{A}), \mathfrak{Inn}_\sigma(\mathcal{A})\big]_\sigma; \\ \widetilde{S}_1 &= \big\{\widetilde{\Delta} \in \mathfrak{D}_\sigma(\mathcal{A}) \bigm| \big[\widetilde{\Delta}, S^1\big]_\sigma \subseteq \mathfrak{Inn}_\sigma(\mathcal{A})\big\}; \\ S_1 &= \big\{\widetilde{\Delta} \in \mathfrak{D}_\sigma(\mathcal{A}) \bigm| \big[\widetilde{\Delta}; S^1\big]_\sigma \subseteq S^1\big\}. \end{aligned}$$

By Proposition 2.4.6 we have the inclusions $S^1 \subseteq \mathfrak{Inn}_\sigma(\mathcal{A}) \subseteq S_1 \subseteq \widetilde{S}_1$. Now we can describe these spaces in the case we are considering here.

Theorem 3.5.2. *The following statements hold.*

(1) *If $s = 1$ then $\mathfrak{D}_\sigma(\mathcal{A}) = \mathfrak{Inn}_\sigma(\mathcal{A}) = S^1 = \widetilde{S}_1 = S_1$.*
(2) *If $s = 0$ then $\mathfrak{D}_\sigma(\mathcal{A}) = \mathbb{C} \oplus \mathfrak{Inn}_\sigma(\mathcal{A}) = \widetilde{S}_1 = S_1$, and $S^1 = 0$.*
(3) *If $s \neq 0, 1$ then $0 \neq S^1 \subsetneq \mathfrak{Inn}_\sigma(\mathcal{A})$, and $\widetilde{S}_1 = \mathfrak{D}_\sigma(\mathcal{A})$.*

Moreover, if $s \neq -1$ then $S_1 = \mathfrak{Inn}_\sigma(\mathcal{A})$.

Proof. (1) This was already noted at the beginning of Subsection 3.2.

(2) The decomposition of $\mathfrak{D}_\sigma(\mathcal{A})$ is Theorem 3.2.1. For the rest, just note that in this case $g = 1 - q^{-1}t$ and $\sigma(g) = 0$. So, it follows from Lemma 2.4.4 that $S^1 = 0$. Then by definition of these sets one gets $\widetilde{S}_1 = S_1 = \mathfrak{D}_\sigma(\mathcal{A})$.

(3) The strict inclusion $0 \neq S^1 \subsetneq \mathfrak{Inn}_\sigma(\mathcal{A})$ follows from Corollary 2.4.5 since $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ is UFD. The equality $\widetilde{S}_1 = \mathfrak{D}_\sigma(\mathcal{A})$ was also proved for any UFD in Proposition 2.4.7 and hence holds in particular for $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$.

We already noticed (Proposition 2.4.6) that $S_1 \supseteq \mathfrak{Inn}_\sigma(\mathcal{A})$ for $s \neq 0, 1, -1$. For the inverse inclusion, we note first that g divides $\sigma(g)$. We shall remark also that considering Remark 2.3.2 for $p_b = t$ and $p_a = 1$ we get $\Delta(t)g\sigma(g)\Delta \in S^1$. Recall from (5) that $\Delta(t)$ is always a monomial. Now the proof will involve several steps, listed below.

Step 1. We first restate our problem. Consider a σ-derivation $\widetilde{P}(t)\Delta \in S_1$. Our aim is to show that $\widetilde{P}(t)\Delta \in \mathfrak{Inn}_\sigma(\mathcal{A})$. According to Theorem 3.2.1 one should write $\widetilde{P}(t) = \sum_{i=0}^{d-1} a_i t^i + g(t)R(t)$, so that $g(t)R(t)\Delta \in \mathfrak{Inn}_\sigma(\mathcal{A})$. We must show that $P = \sum_{i=0}^{d-1} a_i t^i = 0$.

Step 2. Because of Lemma 2.4.4 we have $[P(t)\Delta, \Delta(t)g\sigma(g)\Delta]_\sigma \in g\sigma(g)\mathcal{A}\Delta$. For convenience we denote $\Phi = \Delta(t)g\sigma(g)$. Since $\operatorname{Ann}\Delta = 0$, we have $g\sigma(g)$ dividing $T = \sigma(P)\Delta(\Phi) -$

$\Delta(P)\sigma(\Phi)$. But $\sigma(\Phi) = \sigma(\delta(t))\sigma(g)\sigma(\sigma(g))$ is a multiple of $g\sigma(g)$ because $\sigma(g)$ is a multiple of g. So $g\sigma(g)$ divides $\sigma(P)\Delta(\Phi)$. Since Δ is a σ-derivation we have

$$\Delta(\Phi) = \Delta\big(\Delta(t)g\sigma(g)\big) = \sigma\big(\Delta(t)\big)\Delta\big(g\sigma(g)\big) + \Delta^2(t)g\sigma(g),$$

so $g\sigma(g)$ divides $\sigma(P)\sigma(\Delta(t))\Delta(g\sigma(g))$. Now remember that $\Delta(t)$ is a monomial, so a unit, and $\sigma(\Delta(t))$ also. We conclude that $g\sigma(g)$ divides $\sigma(P)\Delta(g\sigma(g))$.

Step 3. Since $\Delta(g\sigma(g)) = \sigma(g)\Delta(\sigma(g)) + \Delta(g)\sigma(g)$, we have that g divides the polynomial $\sigma(P)(\Delta(\sigma(g)) + \Delta(g))$.

One has the following formulas concerning g:

$$g = 1 - q^\epsilon t^d = 1 - T^\epsilon, \qquad \sigma(g) = 1 - q^{\epsilon+d}t^{ds} = 1 - q^{\epsilon s}t^{ds} = 1 - T^{\epsilon s},$$

$$\begin{aligned}\sigma\big(\sigma(g)\big) = 1 - T^{\epsilon s^2} = 1 - q^{\epsilon s^2}t^{\epsilon s^2(s-1)} &= \left(\sum_{k=0}^{s^2-1}\big(q^\epsilon t^d\big)^k\right)\big(1 - q^\epsilon t^d\big)\\ &= \left(\sum_{k=0}^{s^2-1}\big(q^\epsilon t^d\big)^k\right)g = \left(\sum_{k=0}^{s^2-1}\big(T^\epsilon\big)^k\right)g.\end{aligned}$$

Step 4. Now we prove that g and $Q = \Delta(\sigma(g)) + \Delta(g) = \Delta(g + \sigma(g))$ are relatively prime, so by Gauss's Lemma g must divide $\sigma(P)$. By definition $\Delta = (\mathrm{id} - \sigma)/g$, and so $\Delta(\sigma(g) + g) = (\sigma(g) - \sigma^2(g) + g - \sigma(g))/g = 1 - (\sigma^2(g)/g)$. By the formulas above, $Q = -T^\epsilon \sum_0^{s^2-2} T^{\epsilon k}$ and it is prime with g, as any root $t_0 \in \mathbb{C}$ of g satisfies $T(t_0) = 1$, so $Q(t_0) = 1 - s^2 \neq 0$ (by hypothesis on s).

Step 5. Finally, the rest of the proof is reduced to the hypothesis that g divides $\sigma(P)$, with $P = \sum_{i=0}^{d-1} a_i t^i$. Note that $g = 1 - q^\epsilon t^d$ admits exactly d distinct roots (the dth-roots of $q^{-\epsilon}$) in $\mathbb{C}$. Let t_0 be one of these roots. Then $\sigma(P)(t_0) = \sum a_i(qt_0^s)^i = \sum a_i t_0^i$ since $qt_0^{s-1} = (q^\epsilon t_0^d)^\epsilon = 1$. So $P(t_0) = 0$, and P admits d distinct roots. Thus $P = 0$, as the degree of P is at most $d - 1$. $\square$

Remark 3.5.3.

(1) The formulas we use for g in the proof also lead to

$$\begin{aligned}\Delta(g) = \frac{g - \sigma(g)}{g} = \frac{(1 - T^\epsilon) - (1 - T^{\epsilon s})}{1 - T^\epsilon} &= 1 - \frac{1 - T^{\epsilon s}}{1 - T^\epsilon}\\ = 1 - \frac{1 - (q^\epsilon t^d)^s}{1 - q^\epsilon t^d} &= 1 - \{\epsilon s\}_{T;\epsilon}.\end{aligned}$$

(2) Note that the computation of S_1 in the last case relies on the fact that $s^2 \neq 1$, and that is the reason why we assumed $s \neq -1$. When $s = -1$ things should behave mostly like when $s = 1$, but we were not able to solve the technical problems arising in this case.

(3) The "normalizer-like" sets $S_1, \widetilde{S}_1$ bear information on the relation between $\mathcal{D}_\sigma(\mathcal{A})$ and $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. It is interesting that by Theorem 3.5.2, for non-linearly q-deformed Witt algebra, this chain of sets terminates almost at the start and thus this way one does not get the chain of subalgebras that one might expect between $\mathcal{D}_\sigma(\mathcal{A})$ and $\mathcal{I}\mathrm{nn}_\sigma(\mathcal{A})$. It would be interesting to describe some broader classes of algebras that have or do not have similar properties.

Acknowledgments

This research was initiated during a course delivered by the second author at the Erasmus Intensive Program GAMAP: Geometric and Algebraic Methods of Physics and Applications in the University of Antwerp in September 2005. Both authors wish to thank Prof. Freddy van Oystaeyen for kind hospitality and fruitful discussions. The first author thanks also the Centre for Mathematical Sciences at Lund University for hospitality during his one-week stay there in February 2006. We are also grateful to the referee for many questions and remarks which helped to improve the clarity of exposition and notations in the paper.

The research was supported by the Marie Curie Research Training Network Liegrits funded by the European community, the British Engineering and Physical Sciences Research Council (EP/D034167/1), the Crafoord Foundation, the Swedish Foundation for International Cooperation in Research and Higher Education (STINT), the Royal Physiographic Society in Lund and the Royal Swedish Academy of Sciences.

References

[1] K. Brown, K. Goodearl, Lectures on Algebraic Quantum Groups, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2002.

[2] G. Cauchon, Effacement des dérivations et spectres premiers des algèbres quantiques, J. Algebra 260 (2) (2003) 476–518.

[3] M. Chaichian, D. Ellinas, Z. Popowicz, Quantum conformal algebra with central extension, Phys. Lett. B 248 (1–2) (1990) 95–99.

[4] M. Chaichian, A.P. Isaev, J. Lukierski, Z. Popowicz, P. Prešnajder, q-Deformations of Virasoro algebra and conformal dimensions, Phys. Lett. B 262 (1) (1991) 32–38.

[5] M. Chaichian, P. Kulish, J. Lukierski, q-Deformed Jacobi identity, q-oscillators and q-deformed infinite-dimensional algebras, Phys. Lett. B 237 (3–4) (1990) 401–406.

[6] M. Chaichian, Z. Popowicz, P. Prešnajder, q-Virasoro algebra and its relation to the q-deformed KdV system, Phys. Lett. B 249 (1) (1990) 63–65.

[7] T.L. Curtright, C.K. Zachos, Deforming maps for quantum algebras, Phys. Lett. B 243 (3) (1990) 237–244.

[8] E.V. Damaskinsky, P.P. Kulish, Deformed oscillators and their applications, Zap. Nauch. Semin. LOMI 189 (1991) 37–74 (in Russian); Engl. transl. in: J. Soviet Math. 62 (5) (1992) 2963–2986.

[9] C. Daskaloyannis, Generalized deformed Virasoro algebras, Modern Phys. Lett. A 7 (9) (1992) 809–816.

[10] P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, E. Witten (Eds.), Quantum Fields and Strings: A Course for Mathematicians, 2 vols., Amer. Math. Soc., 1999.

[11] P. Di Francesco, P. Mathieu, D. Sénéchal, Conformal Field Theory, Springer-Verlag, 1997, 890 pp.

[12] I. Frenkel, J. Lepowsky, A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, 1988, 508 pp.

[13] J. Fuchs, Affine Lie Algebras and Quantum Groups, Cambridge University Press, 1992, 433 pp.

[14] J. Fuchs, Lectures on Conformal Field Theory and Kac–Moody Algebras, Lecture Notes in Phys., vol. 498, 1997, pp. 1–54.

[15] K.R. Goodearl, Prime ideals in skew polynomial rings and quantized Weyl algebras, J. Algebra 150 (2) (1992) 324–377.

[16] J.T. Hartwig, D. Larsson, S.D. Silvestrov, Deformations of Lie algebras using σ-derivations, J. Algebra 295 (2) (2006) 314–361.

[17] L. Hellström, S.D. Silvestrov, Commuting Elements in q-Deformed Heisenberg Algebras, World Scientific, Singapore, ISBN 981-02-4403-7, 2000, 256 pp.

[18] N. Hu, q-Witt algebras, q-Lie algebras, q-holomorph structure and representations, Algebra Colloq. 6 (1) (1999) 51–70.
[19] H.P. Jakobsen, Matrix chain models and their q-deformations, in: Lie Theory and Its Applications in Physics V, World Scientific, River Edge, NJ, 2004, pp. 377–391, appeared first in different version as Preprint Mittag-Leffler Institute, Report No 23, 2003/2004, ISSN 1103-467X, ISRN IML-R-23-03/04-SE.
[20] H.P. Jakobsen, H.C.-W. Lee, Matrix chain models and Kac–Moody algebras, in: Kac–Moody Lie Algebras and Related Topics, in: Contemp. Math., vol. 343, Amer. Math. Soc., Providence, RI, 2004, pp. 147–165.
[21] C. Kassel, Cyclic homology of differential operators, the Virasoro algebra and a q-analogue, Comm. Math. Phys. 146 (2) (1992) 343–356.
[22] C. Kassel, Quantum Groups, Grad. Texts in Math., vol. 155, Springer-Verlag, New York, 1995, 531 pp.
[23] D. Larsson, S.D. Silvestrov, Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities, J. Algebra 288 (2005) 321–344.
[24] D. Larsson, S.D. Silvestrov, Quasi-Lie algebras, in: Noncommutative Geometry and Representation Theory in Mathematical Physics, in: Contemp. Math., vol. 391, Amer. Math. Soc., Providence, RI, 2005, pp. 241–248.
[25] D. Larsson, S.D. Silvestrov, Quasi-deformations of $sl_2(\mathbb{F})$ using twisted derivations, Comm. Algebra, in press, preprint version is available at http://arxiv.org/abs/math.RA/0506172.
[26] L. Le Bruyn, Conformal sl_2 enveloping algebras, Comm. Algebra 23 (4) (1995) 1325–1362.
[27] L. Le Bruyn, S.P. Smith, Homogenized sl_2, Proc. Amer. Math. Soc. 118 (1993) 725–730.
[28] L. Le Bruyn, S.P. Smith, M. Van den Bergh, Central extensions of three-dimensional Artin–Schelter regular algebras, Math. Z. 222 (2) (1996) 171–212.
[29] L. Le Bruyn, M. Van den Bergh, On quantum spaces of Lie algebras, Proc. Amer. Math. Soc. 119 (1993) 407–414.
[30] J.C. McConnell, J.C. Robson, Noncommutative Noetherian Rings, revised edition, Grad. Stud. Math., vol. 30, Amer. Math. Soc., Providence, RI, 2001.
[31] S. Montgomery, H.-J. Schneider, Skew derivations of finite-dimensional algebras and actions of the double of the Taft Hopf algebra, Tsukuba J. Math. 25 (2) (2001) 337–358.
[32] M.-J. Redondo, A. Solotar, α-Derivations, Canad. Math. Bull. 38 (4) (1995) 481–489.
[33] L. Richard, S. Silvestrov, Structure of σ-derivations of $C[t^{\pm 1}]$, preprints in Mathematical Sciences (2006:12), LUTFMA-5076-2006, Centre for Mathematical Sciences, Lund University, 19 pp.; arXiv: math.QA/0608196.
[34] F. Van Oystaeyen, Algebraic Geometry for Associative Algebras, Monogr. Textbooks Pure Appl. Math., vol. 232, Marcel Dekker, Inc., New York, ISBN 0-8247-0424-X, 2000, vi+287 pp.