On the convergence of asynchronous iteration methods for nonlinear paracontractions and consistent linear systems

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Abstract

Eisner et al. (L. Eisner, I. Koltracht, M. Neumann, Numer. Math. 62(1992) 305-319) introduced the concept of paracontracting operators for fixed point problems and their solution with asynchronous iteration methods. Their results are extended with respect to the properties of the pool of operators from which a common fixed point is searched. Also, a more general kind of asynchronous iterations than theirs is presented. As an application of this theory, asynchronous iteration methods for consistent linear systems $Ax=b$ are considered, where $A$ is a singular M-matrix. Lubashevski and Mitra (B. Lubashevski, D. Mitra, J. ACM 33(1) (1986) 130-150) investigated asynchronous iteration methods for the Perron-vector problem, i.e. determining a positive solution of $Tx = \rho(T)x$, where $T$ is an irreducible nonnegative matrix and its spectral radius is known. Their result can be fortified and extended to the reducible-affine case by using a quite different approach, the developed theory of paracontractions and confluence. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

In Ref. [7] the convergence of the sequential iteration

$$x(j + 1) := G^{(j)}(x(j)), \quad j = 0, 1, \ldots,$$

(1)
and the asynchronous iteration
\[ x(j + 1) := x(j) + (1 - \alpha_j) G^{(j)}(x(s(j))), \quad j = 0, 1, \ldots, \]
(2)
is studied, where in both cases the \( G^{(j)} \)'s are chosen from a finite pool of real operators \( \mathcal{G} = \{ G^k \mid k \in \mathbb{K} \} \), here \( \mathbb{K} \) is any index-set. This pool was assumed to be paracontracting, that is: All the operators \( G : \mathbb{R}^n \to \mathbb{R}^n \) in \( \mathcal{G} \) are continuous, and for any fixed point \( \xi \) of an operator \( G \) of this pool, and for any \( x \in \mathbb{R}^n \) either
\[ \| G(x) - \xi \| < \| x - \xi \| \]
or \( x \) itself is a fixed point of \( G \). The parameters \( \alpha_j \) in Eq. (2) are chosen from a finite subset of \( [0,1) \) with the restriction that \( \alpha_j = 0 \) is only permitted, if \( s(j) = j \) (\( s(j) \). \( j = 0, 1, \ldots, \) is a sequence from \( \mathbb{N}_0 \), with \( s(j) \leq j \), \( \forall j \in \mathbb{N}_0 \), while \( k(j), j = 0, 1, \ldots, \) is a sequence from \( \mathbb{K} \)). The underlying model of parallelism in Eq. (2) is the following: Assume we have a parallel computer with \( K \) processors \( \pi_1, \ldots, \pi_K \) and shared memory. At iteration step \( s(j) \) processor \( \pi_{k(j)} \) retrieves the global approximation \( x(s(j)) \), which resides at this time in the shared memory, and computes a local iteration \( G^{(j)}(x(s(j))) \). At time \( j \), when this computation is done, the global iteration in the shared memory is updated by a convex combination as in Eq. (2).

The classical asynchronous iteration scheme was introduced for linear systems by Chazan and Miranker [5] and in a variant as an asynchronous multi-splitting-method in Ref. [4]. In Ref. [1] this concept was extended to the nonlinear case. This method was created for determining a fixed point of a single operator \( G : \mathbb{R}^n \to \mathbb{R}^n \) by the iteration
\[ x_i(j + 1) := \begin{cases} x_i(j) & \text{if } i \neq p_j, \\ G_i \begin{pmatrix} x_1(s^1(j)) \\ \vdots \\ x_n(s^n(j)) \end{pmatrix} & \text{if } i = p_j, \end{cases} \quad j = 0, 1, \ldots, \]
(4)
where again \( s^i(j), j = 0, 1, \ldots, i = 1, \ldots, n \), are sequences from \( \mathbb{N}_0 \), with \( s^i(j) \leq j, \forall i, j \), and \( p_j, j = 0, 1, \ldots, \) is a sequence from \( \{1, \ldots, n\} \). The concept of parallelism here is substantially the same as above, but here the different processors load and update only single or (as in the original, but after renumbering equivalent definition in Ref. [1]) groups of components of the system. Criteria of contraction for the analysis of such methods were introduced in Refs. [1,14]. In this paper we discover some connections between both kinds of asynchronous iterations and we will use these connections to develop convergence theorems.

In Section 2 we will define the asynchronous iteration scheme, which we shall study here. The convex combination in Eq. (2) is used to guarantee something like a coupling. Let, for example, \( G^1 \) be an operator with fixed point \( \xi \),
and $G^2$ be one with fixed point $\eta (\neq \xi)$, and let both vectors arise as iterations in Eq. (2). Then, if the $x_j$'s were zero, without the abovementioned restriction, it would be easy to construct divergent sequences, namely some, which alternate between $\eta$ and $\xi$. Even if there is only one operator which has two fixed points, the same phenomenon could arise, and the iteration process would not converge. We will formulate another concept of coupling to avoid the convex combination in Eq. (2). Asynchronous iterations which fulfill these coupling postulates will be called confluent.

In Section 3 we will collect some criteria of contraction for the pool $\mathcal{G}$, which is in our definition an infinite pool of multiple point operators. That is: All the operators are defined on (different) products of $\mathbb{R}^n$. A new criterion of contraction will perhaps be interesting for future research, because the class of operators satisfying this criterion, contains operators, which have non-convex sets of fixed points. The modification of the postulate in Eq. (3) is as follows: In Eq. (3) the norm is independent of both, the operator and the fixed point, and we do not use the latter. In this criterion of contraction we will also not use, that the operators of the pool are continuous. Therefore the new criterion will be called (e = extended)-paracontracting. We will also compare these criteria with two other criteria of contraction, and give some examples.

Results about convergence shall comprise Section 4. We will prove a theorem, which extends the results of Ref. [7] by several points: Confluent iterations, multiple point operators, infinite pools, nonconvex sets of fixed points, discontinuous operators, and arbitrary vector norms (not strictly convex as in Ref. [7]). We will add results for approximate pools. These are sequences of operators, where subsequences converge to operators of another pool.

As an application of this theory, we will analyze in Section 5 the iteration scheme (4) for affine-linear systems, i.e. $G(x) = Tx + f$, where $T$ is an irreducible, nonnegative matrix of spectral radius one. Already in the first paper about asynchronous iterations for linear systems from Chazan and Miranker in Ref. [5] one finds a criterion, which was often interpreted as a necessary and sufficient one for the general convergence of Eq. (4). The criterion given there is that the spectral radius of $|T|$, the matrix whose entries are the moduli of $T$'s entries, has to be less than one. For some reasons, this result was often interpreted to reduce the usage of asynchronous iterations to systems which are related to regular M-systems, i.e. to linear systems whose system matrix is a regular M-matrix, or related to such a matrix, like H-matrices, etc.

But an interpretation of Chazan and Miranker's result in this way would ignore a basic assumption of their counterexample for the case $\rho(|T|) > 1$. Their $T$ is derived from a splitting of a regular matrix $A$, i.e. $A = B - C$, with $\det(B) \neq 0$ and $T = B^{-1}C$. Therefore, such a $T$ cannot have one as an eigenvalue. In our convergence theorems in Section 5 of this paper, we will claim, with very few additional assumptions for the sequences $s'(j), j = 0, 1, \ldots, i = 1, \ldots, n$, and $p_j, j = 0, 1, \ldots$, which are not at all restrictive for a practical
implementation, that Eq. (4) converges in cases where $T$ is nonnegative and of unit spectral radius. We will see that our graph theoretical approach for the description of a sufficient coupling of an asynchronous iteration process relates to the irreducibility of the iteration matrix $T$. We will extend the result of Lubashevski and Mitra [8] to the case that $f \neq 0$ and that $T$ is reducible, but convergent and its Frobenius normal form has at least one positive diagonal element in each of its diagonal blocks, which are of unit spectral radius.

It should be mentioned that there are some more papers dealing with parallel algorithms for singular linear systems, respectively parallel iterative methods with nonnegative iteration matrices of spectral radius one, an area of research connected with the analysis of Markov chains. But these methods, based on multisplitting-methods of Neumann and Plemmons [3] or two-stage-methods of Migallón et al. [9], are synchronized. Therefore, their analysis relates to nonstationary serial methods, i.e., like in Eq. (1), iteration processes of the kind $x(j + 1) := T_{kl}(j)x(j) + f_{kl}(j)$, $j = 0, 1, \ldots$. As well as in this paper the theory of serial nonstationary iterative methods is extended by a new criterion of contraction, the concept of indexwise-regulated sequences, and approximate pools, in Section 5 we also develop some basic tools for the analysis of nonstationary asynchronous iterations.

2. A class of asynchronous iterations

Below we give a definition of asynchronous iterations where, different from the original [5], the whole vector is updated in every iteration step. Also, all components of vectors, which are retrieved from any memory, have the same delay.

**Definition 2.1.** Let $\mathbb{K}$ be a set of indices, $m \in \mathbb{N}$ a fixed number, and $\mathcal{G} = \{G^k | k \in \mathbb{K}\}$ be a pool of operators $G^k : D^{m_k} \subset \mathbb{R}^{m_k} \rightarrow D$, where $m_k \in \{1, \ldots, m\}$, $\forall k \in \mathbb{K}$, and $D \subset \mathbb{R}^n$ is closed. Furthermore, let $\mathcal{X}_\ell = \{x(0), \ldots, x(-M)\} \subset D$ be a set of given vectors. Then, for sequences $\mathcal{K} = k(j)(j = 0, 1, \ldots)$ of elements in $\mathbb{K}$, $\mathcal{G} = \{s^1(j), \ldots, s^{m_{k(j)}}(j)\}$, $j = 0, 1, \ldots$, of $m_k$-tuple from $\mathbb{N}_0 \cup \{-1, \ldots, -M\}$ with $s^l(j) \leq j$ for all $j \in \mathbb{N}_0$, $l = 1, \ldots, m_{k(j)}$, we call the sequence $x(j)$, $j = 0, 1, \ldots$, given by

$$x(j + 1) := G^{k(j)}(x(s^1(j)), \ldots, x(s^{m_{k(j)}}(j))), \quad j = 0, 1, \ldots,$$  

an asynchronous iteration.

We note that we intentionally did not define all the $G^k$'s on $\mathbb{R}^{m_k}$, since we need this type of definition for later studies. An asynchronous iteration corresponding to $\mathcal{G}$, starting with $\mathcal{X}_\ell$ and defined by $\mathcal{K}$ and $\mathcal{G}$ can be denoted now...
by \((\mathcal{G}, \mathcal{X}, \mathcal{H}, \mathcal{F})\). A fixed point \(\xi\) of a multiple point operator \(G : \mathbb{R}^m \rightarrow \mathbb{R}^n\) is a vector \(\xi \in \mathbb{R}^n\) which satisfies

\[ G(\xi, \ldots, \xi) = \xi, \]

and a common fixed point of a pool is a fixed point of all its operators in this sense. Sometimes, in the rest of this paper, some notations of Definition 2.1 will be used and some basic properties (like \(s^i(j) \leq j\)) will be assumed without an explanation.

**Remark 2.1.** Obviously one can embed the iteration (2) into the iteration scheme described above: Let \(\{\beta_1, \ldots, \beta_K\}\) be the set from which the \(\alpha_j\)'s come, excluding zero. Then define operators

\[ G^{(k, \beta)} : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

by \(G^{(k, \beta)}(x_1, x_2) := \beta_1 x_1 + (1 - \beta_1)G^k(x_2), \quad \forall l = 1, \ldots, K, \quad k \in \mathbb{K}, \quad (6)\)

and add them to the pool \(\mathcal{G}\), reset \(\mathbb{K}\) by \(\mathbb{K} = \mathbb{K} \cup \mathbb{K} \times \{\beta_1, \ldots, \beta_K\}\) and redefine \(\mathcal{H} = k(j)(j = 0, 1, \ldots)\) by

\[ k(j) = \begin{cases} k(j) & \text{if } \alpha_j = 0, \\ (k(j), \alpha_j) & \text{if } \alpha_j \neq 0. \end{cases} \]

Finally, define \(\mathcal{F}\) by \(s^1(j) = j, \forall j \in \mathbb{N}_0\), and \(s^2(j) = s(j), \forall j \in \{l \in \mathbb{N}_0 \mid \alpha_l \neq 0\}\), then the iterations (2) and (5) are identical.

For our goal, to develop convergence theorems for Eq. (5), we need to formulate some conditions for the elements of an asynchronous iteration. First investigate \(\mathcal{S}\) and \(\mathcal{H}\). In Ref. [4] the terms admissible and regulated were introduced. We add indexwise-regulated in the following.

**Definition 2.2.** Let \((\mathcal{G}, \mathcal{X}, \mathcal{H}, \mathcal{F})\) be an asynchronous iteration. Then

(i) \(\mathcal{S}\) is called **admissible**, if \(s^i(j) \rightarrow \infty\), whenever \(j \rightarrow \infty\), \(\forall j = 1, \ldots, m\).

(ii) \(\mathcal{S}\) is said to be **regulated**, if

\[ s := \max_{j,i} j - s^i(j) \quad (7) \]

exists.

(iii) \(\mathcal{H}\) is **admissible**, if for all \(j \in \mathbb{N}_0\)

\[ \{k(j)\} \cup \{k(j+1)\} \cup \cdots = \mathbb{K} \quad (8) \]

holds.

(iv) \(\mathcal{H}\) is an **indexwise-regulated** sequence, if for all \(k \in \mathbb{K}\) there is a number \(c_k \in \mathbb{N}_0\), such that for all \(j \in \mathbb{N}_0\)

\[ k \in \{k(j)\} \cup \{k(j+1)\} \cup \cdots \cup \{k(j+c_k)\} . \quad (9) \]
(v) \( \mathcal{H} \) is called regulated, if there is a number \( c \in \mathbb{N}_0 \), such that for all \( j \in \mathbb{N}_0 \)
\[
\{k(j)\} \cup \{k(j + 1)\} \cup \cdots \cup \{k(j + c)\} = \mathbb{K}.
\] (10)

It is clear that \( \mathbb{K} \) has to be finite if \( \mathcal{H} \) is regulated. If \( \mathbb{K} = \mathbb{N} \), the implicitly defined sequence
\[
k(j) = k \iff j \mod 2^k = 2^{k-1}
\]
is indexwise-regulated, because every \( k \in \mathbb{N} \) appears in a period of \( c_k = 2^k \) steps in \( \mathcal{H} \), and so \( \mathbb{K} \) can be infinite, if \( \mathcal{H} \) is indexwise-regulated or admissible.

The next step in our analysis is to formulate some coupling of an iteration process, which avoids the divergence phenomena, described in Section 1, but is not as restrictive as Eq. (2). To this end, we associate a directed graph with \((G, \mathcal{H}, \mathcal{X}, \mathcal{P})\). Every iteration, including the initial vectors, gets a vertex, so the set of vertices is \( \mathcal{V} = \mathbb{N}_0 \cup \{-1, \ldots, -M\} \). A pair \((j, k)\) is now an element of the set of edges \( \mathcal{E} \) in the directed graph \((\mathcal{V}, \mathcal{E})\), if and only if the \( j \)th iteration vector is used for the computation of the \( k \)th iteration vector. As a demonstration for the concept of coupling we have in mind, we should observe again method (2) in the modification as described in Remark 2.1. Here all edges of the form \((j, j+1)\) would lie in \( \mathcal{V} \). This means that there is a directed path in \((\mathcal{V}, \mathcal{E})\) from every vertex \( j \in \mathcal{V} \) to every vertex \( k \in \mathcal{V} \), if \( k \geq j + 1 \). Our modified postulate will be that there are constants \( b, n_0 \in \mathbb{N} \) and a sequence \( b_j(j = n_0, n_0 + 1, \ldots) \) in \( \mathbb{N} \), such that \( j - b_j(j = n_0, n_0 + 1, \ldots) \) is bounded by \( b \), and such that there is a directed path in \((\mathcal{V}, \mathcal{E})\) from every vertex \( b_j \) to every vertex \( k \in \mathcal{V} \) with \( k \geq j \). Finally, we also need some criterion, which guarantees that every operator is sufficiently involved in the iteration process. In Ref. [7] this criterion was, that \( \mathcal{H} \) be regulated.

**Definition 2.3.** Let \((G, \mathcal{H}, \mathcal{X}, \mathcal{P})\) be an asynchronous iteration. Then the **graph** of \((G, \mathcal{H}, \mathcal{X}, \mathcal{P})\) is the directed graph \((\mathcal{V}, \mathcal{E})\), whose vertices \( \mathcal{V} \) are \( \mathbb{N}_0 \cup \{-1, \ldots, -M\} \), and whose edges \( \mathcal{E} \) are given by

\[
(j, j_0) \in \mathcal{E}, \text{ iff there is an } 1 \leq l \leq m_{k(j_0 - 1)}, \text{ such that } s^l(j_0 - 1) = j.
\]

\((G, \mathcal{H}, \mathcal{X}, \mathcal{P})\) is called **confluent**, if there are numbers \( n_0 \in \mathbb{N} \), \( b \in \mathbb{N} \) and a sequence \( b_j(j = n_0, n_0 + 1, \ldots) \) in \( \mathbb{N} \), such that for all \( j \geq n_0 \) the following is true:

(i) For every vertex \( j_0 \geq j \) there is a directed path from \( b_j \) to \( j_0 \) in \((\mathcal{V}, \mathcal{E})\),

(ii) \( j - b_j \leq b \),

(iii) \( \mathcal{P} \) is regulated,

(iv) for every \( k \in \mathbb{K} \) there is a \( c_k \in \mathbb{N} \) so that for all \( j \geq n_0 \) there is a vertex \( w^l_j \) in \( \mathcal{V} \), which is a successor of \( b_j \) and a predecessor of \( b_{j+c_k} \), and for which is \( k(w^l_j - 1) = k \).
Confluent asynchronous iterations could be useful for analyzing a variation of method (2), if there is no shared, but only local memory. In this case, one could use an implementation, where every processor takes the approximation of a common fixed point lying in its own memory, and, to guarantee some communication, computes a convex combination like in Eq. (2) with an approximation lying in the local memory of one of its neighbour processors. This would lead to an iteration process of the form
\[
x(j + 1) := G^{k(j)}(x(s^1(j)), x(s^2(j))) = \alpha_j x(s^1(j)) + (1 - \alpha_j) G^{k(j)}(x(s^2(j)))
\]
for \( j = 0, 1, \ldots \). Here, \( x(j) \) is in general not involved in the \( j \)th iteration step. We do not want to analyze exactly when such methods are confluent, but an exact view on Definition 2.3 shows, that there are simple ways to make such implementations confluent.

3. Criteria of contraction

In this section we want to investigate criteria for the pool \( \mathcal{G} \), from which a common fixed point is searched. For the mentioned new criterion of contraction, we need the following generalization of equicontinuity for multiple point operators.

**Definition 3.1.** Let \( K \) be any set of indices, \( \| \cdot \| \) a norm on \( \mathbb{R}^n \), and \( m_k \) be positive integers such that \( m_k \leq m, \forall k \in K \), for an \( m \in \mathbb{N} \). A set of functionals \( \{ \omega^k : D^{m_k} \subset \mathbb{R}^{m_k} \rightarrow \mathbb{R} \mid k \in K \} \) is called **equicontinuous at** \( X = (x^1, \ldots, x^m) \in D^m \), if for all \( \varepsilon > 0 \) there is a \( \delta > 0 \), such that
\[
\max_{1 \leq i \leq m} \| x^i - x' \| < \delta
\]
implies that
\[
|\omega^k(y^1, \ldots, y^{m_k}) - \omega^k(x^1, \ldots, x^{m_k})| < \varepsilon \quad \forall k \in K.
\]

If \( \{ \omega^k | k \in K \} \) is equicontinuous at all \( X \in D^m \), then it is said to be **equicontinuous on** \( D \).

We note that the \( \delta \)'s, but not the equicontinuity, depend on the norm used. For convenience, we shall use throughout the rest of the paper the notation \( X = (x^1, \ldots, x^{m_k}) \) for an element of \( \mathbb{R}^{m_k} \), i.e. unless confusion arises, we will not use indices to distinguish different dimensions.

**Definition 3.2.** Let \( \mathcal{G} \) be a pool of operators as in Definition 2.1.
(i) If for all \( k \in K, X, Y \in D^{m_k} \), an \( 0 \leq \omega < 1 \) and a norm \( \| \cdot \| \)
\[
\| G^k(X) - G^k(Y) \| \leq \omega \max_j \| x^j - y^j \|.
\]
then \( \mathcal{G} \) is called **contractive** on \( D \).

(ii) If for all \( k \in \mathbb{K}, X, Y \in D^m \) and a norm \( \| \cdot \| \)

\[
\|G^k(X) - G^k(Y)\| < \max_j \|x^j - y^j\|
\]

or \( G^k(X) - G^k(Y) = x^j - y^j \ \forall j \in \{1, \ldots, m_k\} \),

then \( \mathcal{G} \) is called **strictly nonexpansive** on \( D \).

(iii) If for all \( k \in \mathbb{K}, X \in D^m \) and a norm \( \| \cdot \| \), \( G^k \) is continuous on \( D^m \), then \( \mathcal{G} \) is **paracontracting** on \( D \), if for any fixed point \( \xi \in \mathbb{R}^n \) of \( G^k \),

\[
\|G^k(X) - \xi\| < \max_j \|x^j - \xi\|
\]

or \( X = (x, \ldots, x) \) and \( x \) is a fixed point of \( G^k \).

(iv) Let there be a norm \( \| \cdot \|_k \) on \( \mathbb{R}^n \) for every vector \( x \in D \), such that there is a constant \( c \geq 1 \), so that \( \|z\|_k \leq c\|z\|_k \), \( \forall x, y \in D, z \in \mathbb{R}^n \). Then, if \( \xi \in \mathbb{R}^n \) is a fixed point of all operators in \( \mathcal{G} \) with index in \( \mathbb{K}_\xi \neq \emptyset \), but of no other operator, and if this entails that for all \( k \in \mathbb{K}_\xi \) and \( X \in D^m \)

\[
\|G^k(X) - \xi\| \leq \omega^k_\xi(X) \max_j \|x^j - \xi\|,
\]

where the set of functionals

\[
\{\omega^k_\xi : D^m \to [0, 1] \mid k \in \mathbb{K}_\xi \}
\]

is equicontinuous on \( D \), and that for all \( X \in D^m \) for which

\[
\mathbb{K}_{x, \xi} := \{k \in \mathbb{K}_\xi \mid \exists j \in \{1, \ldots, m_k\} : G^k(x^1, \ldots, x^m) \neq x^j\}
\]

is not empty,

\[
\sup_{k \in \mathbb{K}_{x, \xi}} \omega^k_\xi(x^1, \ldots, x^m) < 1 \quad (11)
\]

validates, then \( \mathcal{G} \) is called **(e)-paracontracting** on \( D \).

The criteria (i) and (ii) are variations of standard definitions (see for example Ref. [11]) extended to pools and multiple data operators. (iii) corresponds to paracontracting pools as defined in Ref. [7] (compare Eq. (3)). If \( \mathcal{G} \) contains only one operator \( G : \mathbb{R}^m \to \mathbb{R}^n \), (i) is that criterion, where a unique fixed point has to exist, this is a consequence of Banach's Fixed Point Theorem, applied to the operator \( \bar{G} : \mathbb{R}^n \to \mathbb{R}^n \) with \( \bar{G}(x) := G(x, \ldots, x) \). In the case, that \( D \) is convex, we have as a consequence of the Theorem of Schauder, that operators, which fulfill the criteria (ii) and (iii), have fixed points, because they are continuous. It is easy to see that the sets of fixed points in these cases are convex. Let us consider some examples, the first one will show that the latter is not true anymore in the (e)-paracontracting case.
Example 3.1. Observe for given $C > 1$ the function $G : [1/C, C]^2 \rightarrow [1/C, C]^2$ defined by

$$G\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) := \begin{pmatrix} \min\{\frac{1}{x}, x\} \\ \min\{\frac{1}{y}, y\} \end{pmatrix}. \quad (12)$$

Now $G(\xi) = \xi$, iff $\xi_1 < 1/\xi_2$, and the set of fixed points is not convex. Define for every $x \in [1/C, C]^2$ a weighted maximum-norm $\| \cdot \|_x$ by

$$\|v\|_x := \max\left\{ \frac{v_1}{x_1}, \frac{v_2}{x_2} \right\}, \quad \forall v \in \mathbb{R}^2. \quad (13)$$

We will show that, if $\xi$ is a fixed point of $G$, then

$$\omega_x(\xi) := \begin{cases} \frac{\|G(x) - \xi\|}{\|x - \xi\|} & \text{if } x \neq \xi, \\ 1 & \text{if } x = \xi, \end{cases} \quad (14)$$

is less than one, if $G(x) \neq x$. Then $\{G\}$ is $(\varepsilon)$-paracontracting, because $C^2$ is a common equivalent constant for the defined norms and $\omega_x(\xi)$ is continuous. So, for $x_1 > 1/x_2$, we have to show

$$\max\left\{ \frac{|\frac{1}{x_1} - \xi_1|}{\xi_1}, \frac{|\frac{1}{x_2} - \xi_2|}{\xi_2} \right\} < \max\left\{ \frac{|x_1 - \xi_1|}{\xi_1}, \frac{|x_2 - \xi_2|}{\xi_2} \right\}.$$

But it is

$$\left| \frac{1}{x_1} - \xi_1 \right| = \begin{cases} \frac{1}{x_1} - \xi_1 < x_1 - \xi_1 & \text{if } \frac{1}{x_1} > \xi_1, \\ \xi_1 - \frac{1}{x_2} \leq \frac{1}{x_2} (x_2 - \xi_2) & \text{if } \frac{1}{x_1} < \xi_1, \end{cases} \quad (15)$$

and likewise

$$\left| \frac{1}{x_1} - \xi_2 \right| = \begin{cases} \frac{1}{x_1} - \xi_2 < x_2 - \xi_2 & \text{if } \frac{1}{x_1} > \xi_2, \\ \xi_2 - \frac{1}{x_1} \leq \frac{1}{x_1} (x_1 - \xi_1) & \text{if } \frac{1}{x_1} < \xi_2. \end{cases} \quad (16)$$
Therefore, in the case $1/x_2 < \zeta_1$ and $1/x_1 \geq \zeta_2$, we have $|1/x_1 - \zeta_2|/\zeta_2 < |x_2 - \zeta_2|/\zeta_2$ and
\[
\frac{|1/x_1 - \zeta_1|}{\zeta_1} < \frac{|x_2 - \zeta_2|}{\zeta_2} < \frac{|x_2 - \zeta_2|}{\zeta_2},
\]
since $\zeta_1 x_2 > 1$. The case $1/x_1 < \zeta_2$ and $1/x_2 \geq \zeta_1$ is similar. The case $1/x_1 < \zeta_2, 1/x_2 > \zeta_1$ follows directly from Eqs. (15) and (16) and in the case $1/x_1 > \zeta_2, 1/x_2 < \zeta_1$ we would have
\[
\frac{1}{x_2} < \zeta_1 \leq \frac{1}{\zeta_2} < x_1,
\]
and, therefore, $x$ itself would be a fixed point.

Example 3.2. An example of a paracontracting pool, given in Ref. [7], is as follows: Let \( \{q_k : \mathbb{R}^n \to \mathbb{R} \mid k \in \mathbb{K}\} \) be a finite set of differentiable convex functionals, then the operators
\[
G^k(x) = \begin{cases} 
q^k(x) & \text{if } f^k \leq q^k(x), \\
x & \text{else,}
\end{cases}
\]
where $\gamma_k \in (0, 2), \forall k \in \mathbb{K}$, form a paracontracting pool with respect to the $\| \cdot \|_2$-norm. The fixed points $\eta$ of the operators here are those with $q^k(\eta) \leq f^k$. In a former paper [6] the authors took linear $q_k$'s in order to use the asynchronous algorithm (2) for the problem of tomographic reconstruction from incomplete data, i.e., rectangular linear systems. In general, pools of such operators can be useful to solve constrained optimization problems asynchronously, if the constrains are convex and the objective function is also convex and its optimum is known.

Example 3.3. The function $g : [-1 + \varepsilon, 1 - \varepsilon] \to [-1 + \varepsilon, 1 - \varepsilon]$ with $g(x) := x^2$ is for every $1/2 > \varepsilon > 0$ paracontracting, since $|g(x)| \leq |x||x|$. But this function is not strictly nonexpansive, because we have for $1 > x > y > 1/2$ that $|g(x) - g(y)| = |x + y||x - y| > |x - y|$. On the other hand it is easy to see, that every strictly nonexpansive pool is paracontracting.

Example 3.4. If one defines functions like in Eq. (14) (with $\| \cdot \|_2 = \| \cdot \|, \forall \xi$), it is also easy to see that every finite paracontracting pool is (e)-paracontracting. This statement does not hold in the infinite case as the example \( \{g^n(x) := x^{1+1/n} \mid n \in \mathbb{N}\}, D = [-1 + \varepsilon, 1 - \varepsilon], 1 > \varepsilon > 0 \) shows. Here, we would have $\omega_0^0(x) = x^{1/n}, \forall n \in \mathbb{N}$ and the supremum condition in Eq. (11) is not satisfied. Indeed, in Ref. [7] only finite paracontracting pools were investigated, except asymptotical pools (see below).
Example 3.5. A further difference between paracontracting and (e)-paracontracting pools is that the operators of an (e)-paracontracting pool do not need to be continuous. If in an asynchronous method, implemented with the pool described in Example 3.2, we wish that the descendence, this means the difference between the values $q_k(y) - f^k$ of a new iteration $y$ and $q_k(x) - f^k$ of the old iteration $x$, fulfills a given criterion, then we could apply the operator $G^k$ onto the iteration $x$ as often as necessary, until this criterion is reached. The iterations computed in this manner can be considered as an evaluation of another, in general discontinuous, but (e)-paracontracting operator: The iteration operator for the convex function $q(x) = x^2$, $f = 0$, and $\gamma = 1$ would be $g(x) = (1/2)x$. Now define another iteration function $\hat{g}$ by the following algorithm:

$$y := x$$
$$\text{while } y > \min \left\{ x^2, \frac{1}{2} x \right\} \text{ do } y := g(y)$$

$$\hat{g}(x) := y.$$

It is obvious that $\hat{g}$ is (e)-paracontracting, with $\omega_0(x) = 1/2$, $\forall x \in \mathbb{R}$, but $\hat{g}$ is discontinuous at $1/2$.

Example 3.6. If $T$ is a nonnegative substochastic $K \times m$-matrix, and $t_{km_{k,j,j}}$ is the $j$th of $m_k$ nonzero entries in its $k$th row, the operators $\{ T^k : \mathbb{R}_{+0}^{m_k} \rightarrow \mathbb{R}_{+0} \mid k = 1, \ldots, K \}$ defined by

$$T^k(x^1, \ldots, x^{m_k}) := \sum_{j=1}^{m_k} t_{km_{k,j,j}} x^j, \quad k = 1, \ldots, K.$$

form a strictly nonexpansive pool of operators. If $T$ is stochastic, all nonnegative numbers are common fixed points, if $T$ is not stochastic, then zero is the one and only common fixed point.

Assume that one wants to solve asynchronously a convex optimization problem with operators as described in Example 3.2, but the optimum $f^0$ of its objective function $q^0$ is not known. A way out can be, to adapt successively and approximately estimates for this optimum. That is one example for which the following definition, which we will use later on in a different context, could be helpful.

Definition 3.3. An infinite pool of operators $\mathcal{F} = \{ F^j \mid j \in \mathbb{N}_0 \}$

with $F^j : D^{\tilde{m}_i} \subset \mathbb{R}^{n_i} \rightarrow D$ and $\tilde{m}_i \in \{ 1, \ldots, m \}$, $\forall j \in \mathbb{N}_0$, is said to approximate the pool $\mathcal{G} = \{ G^k \mid k \in \mathbb{K} \}$ with $G^k : D^{m_k} \subset \mathbb{R}^{n_k} \rightarrow D$, $\forall k \in \mathbb{K}$. 
if for all \( j \in \mathbb{N}_0 \) there are \( k_j \in \mathbb{K} \) and a norm \( \cdot \), such that \( m_j = m_{k_j} \), \( \forall j \in \mathbb{N}_0 \), and

\[
\lim_{j \to \infty} \| F^j(x^1, \ldots, x^{m_j}) - G^j(x^1, \ldots, x^{m_j}) \| = 0
\]

(18)

uniformly for all \( X \in D^m \). If, in addition,

\[
\sum_{j=0}^{\infty} \| F^j(x^1, \ldots, x^{m_j}) - G^j(x^1, \ldots, x^{m_j}) \| < \infty
\]

(19)

uniformly for all \( X \in D^m \), then \( \mathcal{F} \) is called asymptotical to \( \mathcal{G} \).

4. Convergence theorems

In this section we give the first of our main results. The convergence theorem in Ref. [7] was:

**Theorem 4.1.** Let \( \| \cdot \|_{sc} \) be a strictly convex vector norm on \( \mathbb{R}^n \), \( x_0 = \{ x(0) \} \subset \mathbb{R}^n \), and \( \mathcal{G} \) be a finite paracontracting pool on \( \mathbb{R}^n \) and with respect to \( \| \cdot \|_{sc} \). Further, let \( k(j), j = 0, 1, \ldots \), and \( s(j), j = 0, 1, \ldots \), with \( s(j) < j \), \( \forall j \in \mathbb{N}_0 \) be regulated sequences from \( \mathbb{K} \) and \( \mathbb{N}_0 \), respectively. Finally, let \( x_j \in \{ \beta_1, \ldots, \beta_K \} \subset [0,1), x_j = 0 \Rightarrow s(j) = j, \forall j \in \mathbb{N}_0 \). Then the iteration (2) converges if and only if a common fixed point of \( \mathcal{G} \) exists. Moreover, if it converges, it converges to a common fixed point of \( \mathcal{G} \).

When we return to the pool defined by a one point-operator-pool in Remark 2.1, and if we assume that the latter one is paracontracting with respect to a strict convex vector norm, then we have for

\[
\omega^{(k, \beta_j)}_\xi (x^1, x^2) := \frac{\| \beta_j x^1 + (1 - \beta_j) G^j(x^2) - \xi \|_{sc}}{\max_{i=1,2} \| x^i - \xi \|_{sc}}
\]

by the strict convexity of \( \| \cdot \|_{sc} \) and the triangle inequality, that

\[
\omega^{(k, \beta_j)}_\xi (x^1, x^2) < 1, \quad \text{if} \ G^j(x^2) \neq x^2 \quad \text{or} \quad x^1 \neq x^2.
\]

Hence, the new pool is paracontracting and also (e)-paracontracting, as defined in Definition 3.2 (iii) and (iv), and, furthermore, the asynchronous iteration scheme (2) is confluent. Therefore, Theorem 4.1 can also be deduced from the following result.

**Theorem 4.2.** Let \( \mathcal{G} \) be an (e)-paracontracting pool on \( D \subset \mathbb{R}^n \), and assume that \( \mathcal{G} \) has a common fixed point \( \xi \in D \), then a confluent asynchronous iteration \((\mathcal{G}, \mathcal{X}_c, \mathcal{X}, \mathcal{F})\) converges to a common fixed point of \( \mathcal{G} \).
\textbf{Proof.} We divide the proof into six parts.

1. (Notations and statements): (a) All terms and notations are as defined or used above. Furthermore, we need the set

\[ D_0 := \{ x \in D \mid \| x - \xi \| \leq C \}. \]

where \( C \) is given by

\[ C := \max_{-M \leq j \leq 0} \| x(j) - \xi \| \xi. \]

Since \( G \) is (e)-paracontracting and \( \xi \) is a common fixed point of \( G \), all iterations \( x(j)(j = 0, 1, \ldots) \) stay in the compact set \( D_0 \), thus

\[ \lambda := \limsup_{j \to \infty} \| x(j) - \xi \| \xi \]

exists.

(b) On \( D_0 \) there is for all \( \delta > 0 \) some \( \varepsilon > 0 \), which is independent of \( k \), such that

\[ 0 < 1 - \lambda < 1 - \xi \]

implies that there is a fixed point \( \eta \) of \( G^k \), satisfying

\[ \| x^k - \eta \| \xi < \frac{1}{\varepsilon^2} \delta \quad \forall l = 1, \ldots , m. \]

\textbf{Proof.} W.l.o.g. we can assume that \( m_k = m \), \( \forall k \in \mathbb{K} \). Otherwise we could prove this statement for all of the subsets \( \mathbb{K}_i := \{ k \in \mathbb{K} \mid m_i = i \}, i = 1, \ldots , m \). Assume there would be sequences \( X_j, j = 0, 1, \ldots \) in \( D_0^m \) and \( k_j, j = 0, 1, \ldots \), in \( \mathbb{K}_0 \), a non-empty subset of \( \mathbb{K} \), for which

\[ \lim_{j \to \infty} \omega^k_{\xi}(X_j) = 1 \]

and (since \( D_0 \) is compact)

\[ \lim_{j \to \infty} X_j = X = (x^1, \ldots , x^m). \]

but for all \( k \in \mathbb{K}_0 \) there would be some \( l(k) \in \{ 1, \ldots , m \} \) such that

\[ G^k(X) \neq X^{l(k)}. \]

This would contradict

\[ \sup_{k \in \mathbb{K}_0} \omega^k_{\xi}(X) < 1 \]
and

$$\lim_{j \to \infty} \left( \omega^k_j(X_j) - \omega^k(X) \right) = 0 \quad \text{(cf. Definition 3.2(iv))}. $$

(c) **Vice versa:** Let \( \eta \in D_0 \) be not a fixed point all \( G^k \) with \( k \in \mathbb{K}_0 \). Then, there is an \( \tilde{\varepsilon} > 0 \) such that for all \( \tilde{\varepsilon} \in (0, \tilde{\varepsilon}) \) there is a \( \tilde{\delta} > 0 \) so that for all \( k \in \mathbb{K}_0 \)

$$\|x^l - \eta\| < \tilde{\delta} \quad \text{for some } l \in \{1, \ldots, m\}, \ X \in D_0^m$$

implies that

$$\omega^k(x^1, \ldots, x^m) < 1 - \tilde{\varepsilon}.$$ 

**Proof.** Again assume \( m_k = m \). Say, there would be sequences \( X_j = (x_j^1, \ldots, x_j^m), j = 0, 1, \ldots \), in \( D_0^m, k_j, j = 0, 1, \ldots, \) in \( \mathbb{K}_0 \), and \( l_j, j = 0, 1, \ldots, \) in \( \{1, \ldots, m\} \) such that

$$\lim_{j \to \infty} \omega^k_j(X_j) = 1$$

and

$$\lim_{j \to \infty} x_j^{l_j} = \eta.$$ 

Since \( D_0^m \) is compact, \( X_j, j = 0, 1, \ldots \), would have a point of accumulation \( X = (x^1, \ldots, x^m) \in D_0^m \) with \( x^l = \eta \) for some \( l \in \{1, \ldots, m\} \). Due to this point of accumulation we would have, by Eq. (11), that

$$\sup_{k, \ X_0} \omega^k(X) \leq \omega,$$

for some \( \omega < 1 \). Hence, there would be a sequence \( j, \ k = 0, 1, \ldots \) in \( \mathbb{N}_0 \) for which

$$1 = \lim_{k \to \infty} \omega^k(X_0)$$

\( \leq \lim_{k \to \infty} |\omega^k(X_0) - \omega^k(X)| + \omega \)

\( = \omega. \)

**II.** Let \( \tilde{\varepsilon}, \tilde{\delta}, x \in \mathbb{R}, x < \lambda \). A sequence of positive numbers \( x_L L = 0, 1, \ldots \) and a sequence of pairs \( (\delta_L, \epsilon_L)L = 0, 1, \ldots \) of them satisfying some \((\delta, \epsilon)\)-criterion as in I.(b) can be defined as follows:

$$x_0 := \frac{\lambda - x}{2},$$

$$\delta_0 := \min \left\{ \frac{\lambda - x_0}{2}, \frac{\lambda - x_0}{4} \right\}.$$
\( \varepsilon_0 \) corresponding to \( \delta_0 \) as described in I.(b) and satisfying \( 0 < \varepsilon_0 < \tilde{\varepsilon} \).

\[
\alpha_{L+1} := \max \left\{ \frac{1 - \varepsilon_L}{1 - \varepsilon_L \cdot 0.9} \lambda, \ 2 \cdot \delta_L + \alpha_L \right\},
\]

\[
\delta_{L+1} := \min \left\{ \frac{\lambda - \alpha_{L+1}}{2.1}, \frac{\delta_L}{4} \right\}
\]

\( \varepsilon_{L+1} \) corresponding to \( \delta_{L+1} \), satisfying \( 0 < \varepsilon_{L+1} < \varepsilon_L < 1 \), \( \forall L \geq 0 \).

Then, we have

\[
0 < \alpha_L < \alpha_{L+1} \quad \text{and} \quad \alpha_L < \tilde{\alpha}, \quad \forall L \geq 0,
\]

thus these sequences are well defined. Furthermore, it holds that

\[
\sum_{r=0}^{\infty} 2 \delta_r \leq \tilde{\delta}.
\]

**III.** By induction, we can show that for arbitrary \( \alpha, \tilde{\alpha}, \tilde{\delta} \) the sequences defined in II. satisfy the following assertion: Assume \( J \in \mathbb{N}_0 \) is a vertex such that there is an \( L \in \mathbb{N}_0 \) for which

\[
\| x(s^j(j)) - \tilde{z} \|_z < \frac{\lambda}{1 - \varepsilon_L \cdot 0.9} \quad \forall j \geq J, \ l = 1, \ldots, m_k(j).
\]  \( \text{(20)} \)

Then

\[
\| x(j) - \tilde{z} \|_z < \alpha_L
\]

implies, that for all \( j_0 \), for which there is a directed path in \( (\mathcal{V}, \mathcal{E}) \) of length less than \( L + 1 \) from \( J \) to \( j_0 \),

\[
\| x(j_0) - \tilde{z} \|_z < \alpha_L
\]

is validated.

**Proof.** This is no question for \( L = 0 \). Now assume the assertion is proved for all

\( 0 \leq L_0 < L \). Then we have for all \( j_0 \), for which there is a path from \( J \) to \( j_0 \) of length \( L \) in \( (\mathcal{V}, \mathcal{E}) \), that there is an \( l_0 \in \{1, \ldots, m_k(j_0-1)\} \), such that

\[
\| x(s^h(j_0-1)) - \tilde{z} \|_z < \alpha_L.
\]

Since

\[
\frac{\lambda}{1 - \varepsilon_L \cdot 0.9} < \frac{\hat{\lambda}}{1 - \varepsilon_{L+1} \cdot 0.9},
\]

this is a consequence of the assumption we made. Furthermore, due to Eq. (20), we have

\[
\| x(j_0) - \tilde{z} \|_z = \| G^{k(j_0-1)}(X_{j_0-1}) - \tilde{z} \|_z \leq \alpha_k^{k(j_0-1)}(X_{j_0-1}) \cdot \frac{\hat{\lambda}}{1 - \varepsilon_L \cdot 0.9}.
\]

Using I.(b), the latter term is less than
\[
\frac{1 - \varepsilon L}{1 - \varepsilon L \cdot 0.9} \lambda \leq x_{L+1}
\]

or there is a fixed point \( \eta \) of \( G^{k(j_0-1)} \), satisfying

\[
\| x^l(s^l(j_0 - 1)) - \eta \|_\zeta < \frac{1}{c^2} \delta_L, \quad \forall l = 1, \ldots, m_{k(j_0-1)}.
\]

But in the latter case, we have

\[
\| x(j_0) - \eta \|_\zeta \leq c \| G^{k(j_0-1)}(X_{j_0-1}) - \eta \|_\eta
\]

\[
\leq c^2 \omega^{k(j_0-1)}(X_{j_0-1}) \max_l \| x^l(s^l(j_0 - 1)) - \eta \|_\zeta < \delta_L. \tag{21}
\]

Thus

\[
\| x(j_0) - \zeta \|_\zeta \leq \| x(j_0) - \eta \|_\zeta + \| \eta - x(s^\kappa(j_0 - 1)) \|_\zeta + \| x(s^\kappa(j_0 - 1)) - \zeta \|_\zeta
\]

\[
< 2\delta_L + \lambda_L \quad \text{(Assumption)}
\]

\[
\leq \lambda_{L+1}.
\]

\textbf{IV.} The following equality holds

\[
\lim_{j \to \infty} \| x(b_j) - \zeta \|_\zeta = \lambda. \tag{22}
\]

\textbf{Proof.} Assume there is a subsequence \( x(b_{\kappa}) \), \( \kappa = 0, 1, \ldots \) of \( x(b_j) \), \( j = n_0, n_0 + 1, \ldots \), for which

\[
\lim_{\kappa \to \infty} \| x(b_{\kappa}) - \zeta \|_\zeta = \lambda
\]

and

\[
\lambda < \lambda
\]

Now define sequences as in II. by this \( \lambda \). The assumption (23) then implies the existence of a \( \kappa_0 \in \mathbb{N} \), for which at the same time

\[
\| x(b_{\kappa_0}) - \zeta \|_\zeta < \tilde{x}_0
\]

and

\[
\| x(s^l(j)) - \zeta \|_\zeta < \frac{\lambda}{1 - \varepsilon b_{\kappa_0} \cdot 0.9}, \quad \forall j \geq b_{\kappa_0}, \quad l = 1, \ldots, m_{l(j)} \tag{24}
\]

hold. Since in \( \langle \gamma, \delta \rangle \) there is for all \( j \geq j_{\kappa_0} \) a directed path of length less or equal than \( b + j - j_{\kappa_0} \) from \( b_{\kappa_0} \) to \( j \), we have by III.

\[
\| x(j) - \zeta \|_\zeta < \tilde{x}_0, \quad \forall j \in \{ j_{\kappa_0}, \ldots, j_{\kappa_0} + s \}.
\]

But this means \( \| x(j) - \zeta \|_\zeta < \tilde{x}_0, \), for all \( j > j_{\kappa_0} + s \), since \( s \) is an upper bound for the delay, compare Eq. (7). Hence, \( \lambda \) cannot be the limit superior defined in I.(a) and Eq. (22) holds.
V.: W.l.o.g., we can assume: There is at least one \( k \in \mathbb{K} \), such that there is a fixed point \( \eta \) of \( G^k \) and such that
\[
\lim_{k \to \infty} x(b_j, k) = \eta
\]  
(25)
for an infinite subsequence \( b_j, k = 0, 1, \ldots \), of \( b_j, j = n_0, n_0 + 1, \ldots \), with \( k(b_j, k) = k \), \( \forall k \geq 0 \).

**Proof.** For any fixed \( k \in \mathbb{K} \) the sequence \( w_k^j (j = n_0 + c_k, n_0 + c_k + 1, \ldots) \) satisfies the criteria demanded in Definition 2.3 for the sequence \( b_j (j = n_0, n_0 + 1, \ldots) \) (cf. (i), (ii), (iv)). Thus, we can assume that there is a \( k \) for which
\[
k(b_j, k) = k \quad \forall k \geq 0,
\]
for a subsequence \( b_j, \tau = 0, 1, \ldots \), of \( b_j, j = n_0, n_0 + 1, \ldots \). Because of IV., this sequence holds
\[
\lambda = \lim_{\tau \to \infty} \| x(b_j, \tau) - \xi \| \leq \limsup_{\tau \to \infty} \omega_{\xi, \tau}^k (X_{b_{n_\tau} - 1}) \max_{\ell} \| x_{b_{n_\tau - 1}}^\ell - \xi \| \leq \lambda.
\]
Hence, there is a subsequence \( X_{b_{n_\tau} - 1}, \kappa = 0, 1, \ldots \) of \( X_{b_{n_\tau} - 1}, j = n_0, n_0 + 1, \ldots \), converging to the \( m_k \)-time product of a fixed point \( \eta \) of \( G^k \). Because, if \( \lambda = 0 \), there has to be a subsequence \( b_j, \kappa = 0, 1, \ldots \), such that
\[
\omega_{\xi, \tau}^k (X_{b_{n_\tau} - 1}), \kappa = 0, 1, \ldots,
\]
converges to one and, therefore,
\[
\lim_{k \to \infty} \| x(b_j, k) - \eta \| = \limsup_{k \to \infty} \omega_{\eta}^k (X_{b_{n_\tau} - 1}) \max_{\ell} \| x_{b_{n_\tau - 1}}^\ell - \eta \| = 0.
\]

VI.: Conclusion.

If we can show that this \( \eta \in D_0 \) is a common fixed point of \( G \), the theorem is proved, because after repeating the whole argument, substituting \( \xi \) by \( \eta \), the assertion is a consequence of IV. Let \( \mathbb{K}_0 \) contain the indices of all those operators for which \( \eta \) is not a fixed point, and let \( k_0 \in \mathbb{K}_0 \). Using I.(c), we can choose an \( \delta > 0 \), such that there is a \( \tilde{\delta} > 0 \), which satisfies for all \( X \in D_0^n, k \in \mathbb{K}_0 \), that
\[
\| x^\ell - \eta \| \leq \tilde{\delta}
\]
for an \( l \in \{1, \ldots, m_k\} \) implies that
\[
\omega_{\eta}^k (x^1, \ldots, x^{m_k}) < 1 - \tilde{\delta}.
\]
For an arbitrary \( 0 < a < \lambda \), let \( x_L, L = 0, 1, \ldots, \delta_L, L = 0, 1, \ldots \), and \( \omega_L, L = 0, 1, \ldots \), be the sequences defined in II. by \( \tilde{\delta}, \delta, a \). Then we have, by II., \( x_{\omega_L, \tau} a, b < \lambda \) and there is a \( j_0 \) satisfying at the same time
\[ \|x(b_j) - \eta\|_z < 2\delta_0, \quad \text{(by V.)} \]
\[ \|x(b_j) - \xi\|_z > \alpha_{c_{k_0} + b}, \quad \forall j \geq j_0, \quad \text{(by IV.)} \]

and
\[ \|x(j) - \xi\|_z < \frac{\lambda}{1 - \varepsilon c_{k_0} + b \cdot 0.9}, \quad \forall j \geq b_{j_0} - s. \quad (26) \]

Further, there is a path \( b_{j_0} = w^0, \ldots, w^{k_0 - 1} = b_{j_0 + c_{k_0}} \) in \((\gamma', \delta')\), which leads across a vertex \( w^{k_0}_{j_0} \), for which is \( k(w^{k_0}_{j_0} - 1) = k_0 \) and which is of length less or equal than \( c_{k_0} + b \). By induction on the vertices \( w^L, L = 0, \ldots, (j_0, k_0) \), we will show that for every \( k(w^L - 1), L = 0, 1, \ldots, (j_0, k_0) \), \( \eta \) is a fixed point of \( G^k(w^L - 1) \) and we will conclude therefore that \( \Omega_{k_0} \) is empty. For the case \( L = 0 \) the assertion can be assumed, because of V. Now let the assertion and also
\[ \|x(w^q) - \eta\|_z < \sum_{r=0}^{q} 2\delta_r \]
be shown for all \( 0 \leq q < L \leq (j_0, k_0) \). Suppose \( k(w^L - 1) \in \Omega_{k_0} \), then, by
\[ \|x(w^L - 1) - \eta\|_z < \delta, \]
we would have
\[ \|x(w^L) - \xi\|_z = \|G^{k(w^L - 1)}(X_{w^L - 1}) - \xi\|_z < \frac{(1 - \varepsilon)\lambda}{1 - \varepsilon c_{k_0} + b \cdot 0.9} < \frac{(1 - \varepsilon_0)\lambda}{1 - \varepsilon_0 \cdot 0.9} \leq \alpha_1. \]

But, by III., this would mean that
\[ \|x(b_{j_0 + c_{k_0}}) - \xi\|_z < \alpha_{c_{k_0} + b}, \]
and therefore contradicting our assumption. Not only \( \|x(w^L) - \xi\|_z \geq \alpha_1 \), but also
\[ \|x(w^L) - \xi\|_z \geq \alpha_L, \quad \forall L \geq 0, \]
since otherwise, also by III., \( \|x(b_{j_0 + c_{k_0}}) - \xi\|_z < \alpha_{c_{k_0} + b} \). Thus
\[ \|x(w^L) - \xi\|_z = \|G^{k(w^L - 1)}(X_{w^L - 1}) - \xi\|_z \leq \omega^k_{z(w^L - 1)}(X_{w^L - 1}) \frac{\lambda}{1 - \varepsilon c_{k_0} + b \cdot 0.9} \quad \text{(cf. Eq. (26))} \]
\[ < \omega^k_{z(w^L - 1)}(X_{w^L - 1}) \frac{\lambda}{1 - \varepsilon_L \cdot 0.9} \]
implies, by II., that
\[ \omega^k_{z(w^L - 1)}(X_{w^L - 1}) > 1 - \varepsilon_L. \]

Hence, (b) guarantees the existence of a fixed point \( \eta^0 \) of \( G^{k(w^L - 1)} \) with
\[ \|x_{w^L - 1} - \eta^0\|_z < \delta_L \frac{1}{c^2}, \quad \forall l = 1, \ldots, m_{k(w^L - 1)}. \]
This means (cf. Eq. (21))
\[ \|x(w^t) - \eta^0\|_\xi < \delta_t, \]
and, therefore,
\[ \|x(w^t) - \eta\|_\xi \leq \|x(w^t) - \eta^0\|_\xi + \|x(w^t - 1) - \eta^0\|_\xi + \|x(w^t - 1) - \eta\|_\xi \]
\[ < \delta_t + \delta_t \frac{1}{c^2} + \sum_{r=0}^{t-1} 2\delta_r < \sum_{r=0}^{t} 2\delta_r. \]

This completes the proof. \( \Box \)

We note that in the case of (e)-paracontracting pools it is not possible to prove the sufficient condition in Theorem 4.1, i.e. that an asynchronous iteration converges only, if a common fixed point exists: The operators of such a pool are in general not continuous. But it is obvious that the continuity of the operators and the admissibility of \( \mathcal{H} \) are in any case the only assumptions, we need to claim this.

Next, we turn to approximate pools. We will begin with an according result for pools, which are asymptotical to strictly nonexpansive pools and in the infinite case also (e)-paracontracting.

**Theorem 4.3.** Let \( \mathcal{G} = \{G^k \mid k \in \mathbb{K}\} \) be an (e)-paracontracting and at the same time strictly nonexpansive pool on \( D \) with respect to \( \| \cdot \| \), and let there be a common fixed point \( \xi \in D \) of \( \mathcal{G} \). Further, let \( \mathcal{F} \) be asymptotical to \( \mathcal{G} \), therefore, let
\[ \|F^j(X) - G^k(X)\| \leq \epsilon_j, \quad \forall X \in D^{m_1}, \quad j \in \mathbb{N}_0. \]
for a sequence \( \mathcal{K} = k_j, j = 0, 1, \ldots, \) in \( \mathbb{K} \) and a null sequence \( \epsilon_j, j = 0, 1, \ldots, \) in \( \mathbb{R}_+ \) so that
\[ \epsilon_\Sigma := \sum_{j=0}^{\infty} \epsilon_j \in \mathbb{R}. \]
Additionally, let, the maximal ball around \( \xi \) of distance \( \epsilon_\Sigma \) to the border of \( D \), be
\[ D_0 := \{x \in D \mid \forall y \in \mathbb{R}^n \text{ and } \|y - \xi\| \leq \|x - \xi\| + \epsilon_\Sigma \text{ then } y \in D\}, \quad (27) \]
\( \mathcal{X}_\epsilon \subset D_0 \) and denote the sequence \( 0, 1, \ldots \) by \( \mathcal{N}_\epsilon \). Then, if the asynchronous iteration \( (\mathcal{G}, \mathcal{X}_\epsilon, \mathcal{V}_\epsilon, \mathcal{F}) \) is confluent, the asynchronous iteration \( (\mathcal{F}, \mathcal{X}_\epsilon, \mathcal{V}_\epsilon, \mathcal{F}) \) is well defined and converges to a common fixed point of \( \mathcal{G} \).

**Proof.** Let \( x(j), j = 0, 1, \ldots, \) and \( y^0(j), j = 0, 1, \ldots, \) be the asynchronous iterations \( (\mathcal{F}, \mathcal{X}_\epsilon, \mathcal{V}_\epsilon, \mathcal{F}) \), respectively \( (\mathcal{G}, \mathcal{X}_\epsilon, \mathcal{H}, \mathcal{F}) \). We will show, by induction,
\[ ||x(j) - y^0(j)|| \leq \sum_{q=0}^{j-1} \epsilon_q, \quad \forall j \geq 0, \]  
(28)

and thus
\[ ||x(j) - \xi|| \leq \max_{0 \leq l \leq M} ||x(-l) - \xi|| + \epsilon_0, \quad \forall j \in \mathbb{N}_0, \]

which means that \((\mathcal{F}, \mathcal{X}_\xi, \mathcal{N}_\xi, \mathcal{F}_j)\) is well-defined, since \(x(j), j = 0, 1, \ldots,\) stays in \(D_0\). Inequality (28) holds for \(j = 0\) and if it is shown for all \(0 \leq j_0 \leq j\), then (cf. Definitions 3.2(ii) and 3.3)
\[ ||x(j+1) - y^0(j+1)|| \leq \epsilon_j + \max_{1 \leq l \leq m_k} ||x(s^l(j)) - y(s^l(j))|| \leq \sum_{q=0}^{j} \epsilon_q. \]

Now, define for all \(j_0 \in \mathbb{N}_0\) sequences \(\mathcal{X}^{j_0} = k^{j_0}(j), j = 0, 1, \ldots,\) in \(\mathbb{K}\), \(\mathcal{N}^{j_0} = \{s_{j_0}^l(j), \ldots, s_{m_k(j_0-1)-1}^{m_k(j_0)-1}(j)\}, j = 0, 1, \ldots,\) of \(m_{k(j_0-1)}\)-tuple of integers in \(\mathbb{N} \cup \{0, \ldots, -M\}\) and subsets \(\mathcal{X}^{j_0}_c = \{x^{j_0}(-s), \ldots, x^{j_0}(0)\}\) from \(\mathbb{R}^n\) by
\[ k^{j_0}(j) := k_{j-j_0}, \]
\[ s_{j_0}^l(j) := s^l(j-j_0), \quad \forall l = 1, \ldots, m_{k(j-j_0)}, \]
\[ x^{j_0}(-l) := x(j_0 - l), \quad \forall l = 0, \ldots, s, \quad \forall j \geq 0. \]

Because of Theorem 4.2, the confluent asynchronous iterations \((\mathcal{G}, \mathcal{X}^{j_0}_c, \mathcal{N}^{j_0}_c, \mathcal{F}^{j_0}_c)\), which we denote by \(y^{j_0}(j), j = 0, 1, \ldots,\) converge for all \(j_0 \in \mathbb{N}_0\) to common fixed points \(\eta^j, j = 0, 1, \ldots,\) of \(\mathcal{G}\). Moreover, all of the common fixed points lie in the compact set \(D_0\) (compare the induction at the beginning of the proof). Therefore, \(\eta^j, j = 0, 1, \ldots,\) has an accumulation point in \(D_0\), say \(\eta\). Now assume the sequence \(\eta^{k_j}, j = 0, 1, \ldots,\) converges to \(\eta\), then for all \(\epsilon > 0\) there are some \(j_0, j_1 \in \mathbb{N}\) such that
\[ ||\eta^{k_{j_0}} - \eta|| \leq \frac{\epsilon}{3}, \]
\[ \sum_{q=k_{j_0}}^{\infty} \epsilon_q \leq \frac{\epsilon}{3}, \]
\[ ||y^{k_{j_0}}(j) - \eta^{k_{j_0}}|| \leq \frac{\epsilon}{3}, \quad \forall j \geq j_1. \]

Hence, for all \(j \geq j_1\), we have, by Eq. (28).
\[ ||x(j + k_{j_0}) - \eta|| \leq ||x(j + k_{j_0}) - y^{k_{j_0}}(j)|| + ||y^{k_{j_0}}(j) - \eta^{k_{j_0}}|| + ||\eta^{k_{j_0}} - \eta|| \leq \epsilon. \]

Since for all \(k \in \mathbb{K}\), in the (e)-paracontracting formulation of the contraction criterion,
\[ 1 = \lim_{j \to \infty} \omega_j^k(\eta^{k_{j+1}}, \ldots, \eta^{k_{j+1}}) = \omega_j^k(\eta, \ldots, \eta), \quad \text{if } \eta \neq \xi, \]
\(\eta\) is a common fixed point of \(\mathcal{G}\). \(\square\)
Example 4.1. A simple example shows that this theorem is not generally true anymore, if $\mathcal{F}$ only approximates $\mathcal{G}$: The paracontracting function $g : \mathbb{R}_{+0} \to \mathbb{R}_{+0}$,

$$g(x) := x \frac{x^4}{x^4 + 1},$$

with fixed point $0$, is approximated by the pool

$$f^j(x) := g(x) + \frac{1}{h_j}, \quad j = 0, 1, \ldots,$$

if $h_j \to \infty$, whenever $j \to \infty$. For $h = x^5 + x^4 + x^2 + 1$, one obtains for $x \neq 0$

$$g(x) + \frac{1}{h} = x + \frac{1}{x}.$$

Hence, for $x(0) > 0$ and

$$h_j := \frac{x(j)^5 + x(j)}{x(j)^4 + x(j)^2 + 1}, \quad j = 0, 1, \ldots,$$

the sequential iteration

$$x(j + 1) := f^j(x(j)), \quad j = 0, 1, \ldots,$$

is diverging as

$$x(j) \geq \sum_{n=1}^{c_j} \frac{1}{n}, \quad \forall j \in \mathbb{N}_0, \quad (31)$$

for an ascending sequence $c_j, j = 0, 1, \ldots, \text{from } \mathbb{N}_0.$

To prove a convergence theorem for approximate pools we will assume that $\mathcal{G}$ is contractive:

Theorem 4.4. Let $\mathcal{G}$ be a contractive pool on $D \subset \mathbb{R}^n$, with respect to some norm $\| \cdot \|$. Assume that $\mathcal{G}$ has a common fixed point $\xi \in D$, and that the pool $\mathcal{F}$ approximates $\mathcal{G}$ i.e. there is a null sequence $c_j, j = 0, 1, \ldots, \text{in } \mathbb{R}_{+0}$ and a sequence $\mathcal{K} = k_j, j = 0, 1, \ldots \text{in } \mathbb{K}$, so that

$$\|F^j(x^1, \ldots, x^{m_j}) - G^{k_j}(x^1, \ldots, x^{m_{k_j}})\| \leq c_j, \quad \forall X \in D^n, \ j \in \mathbb{N}_0.$$ 

Further, let, respectively,
\[ p_0 := -M - 1. \]

\[ p_i := \min\{p \in \mathbb{N}_0 \mid s'(j) > p_{i-1}, \forall j \geq p, i = 1, \ldots, m_k\}, \quad \forall r \in \mathbb{N}. \]

\[ \delta_j := \sum_{l=p_i}^{p_{i-1}-1} \varepsilon_j, \quad \forall j \in \mathbb{N}, \]

\[ \delta_{\max} := \max_{j \geq 1} \delta_j. \quad (32) \]

Let \( \omega \) be the constant belonging to the contractive pool \( \mathcal{G} \), as required in Definition 3.2, and

\[ D_0 := \{ x \in D \mid \forall y \in \mathbb{R}^n \text{ and } \| y - \xi \| \leq \| x - \xi \| + \frac{\delta_{\max}}{1 - \omega} \text{ then } y \in D \} \quad (33) \]

then, for \( \mathcal{X}_\varepsilon \subset D_0 \), the asynchronous iteration \((\mathcal{F}, \mathcal{X}_\varepsilon, \mathcal{N}_\varepsilon, \mathcal{S})\) is well-defined and converges to \( \xi \). Moreover, if \( \mathcal{F} \) is asymptotical to \( \mathcal{G} \) and \( \mathcal{S} \) is regulated, then \((\mathcal{F}, \mathcal{X}_\varepsilon, \mathcal{N}_\varepsilon, \mathcal{S})\) converges to \( \xi \) and

\[ \sum_{j = -M}^{\infty} \| x(j) - \xi \| \]

exists.

**Proof.** Let \( \tilde{e}_0 := \max_{-M \leq l < 0} \| x(l) - \xi \| \). We will prove that

\[ \max_{p_i \leq l < p_{i-1}} \| x(p_i + l) - \xi \| \leq \sum_{q=0}^{j} \tilde{e}_q \omega^j, \quad \forall j \in \mathbb{N}_0. \quad (35) \]

The latter term is for all \( j \in \mathbb{N} \) less than \( \tilde{e}_0 + \delta_{\max}/1 - \omega \), and, therefore, \( x(j), j = 0, 1, \ldots \), is well defined. The definition of \( \tilde{e}_0 \) is just (35) for \( j = 0 \). Now let Eq. (35) hold for all \( j_0 = 0, \ldots, j - 1 \), then

\[ \| x(p_i + 1) - \xi \| = \| F'(X_{p_i}) \xi - \xi \| \leq \| G^b(X_{p_i}) \xi - \xi \| + \varepsilon_{p_i} \]

\[ \leq \omega \max_l \| x_{p_i}^l - \xi \| + \varepsilon_{p_i} \]

\[ \leq \omega \sum_{q=0}^{j-1} \tilde{e}_q \omega^j + \varepsilon_{p_i}, \quad \text{since } s'(p_i) \geq p_{i-1} \forall l. \quad (36) \]

Continuing this argumentation for \( l := p_i + 2, \ldots, p_{j-1} \), we get

\[ \| x(p_{j-1}) - \xi \| \leq \omega \sum_{q=0}^{j-1} \tilde{e}_q \omega^{j-1} + \sum_{l=p_i}^{p_{j-1}} e_l. \]

In the approximate case, the theorem follows now from the so-called **Toeplitz lemma** (see Ref. [15]): Let \( \omega_{j,l}, j = 0,1, \ldots, l = 0,1, \ldots, j \), and \( C \) be real numbers such that
\[
\lim_{j \to \infty} a_{ij}^{(l)} = 0, \quad \forall l \in \mathbb{N}_0.
\]

and
\[
\sum_{l=0}^{j} |a_{ij}^{(l)}| \leq C, \quad \forall j \in \mathbb{N}_0.
\]

Further, let \( \epsilon_j, j = 0, 1, \ldots \) be a null sequence, then
\[
\alpha_j := \sum_{l=0}^{j} \epsilon_l a_{ij}^{(l)}, \quad j = 0, 1, \ldots,
\]
is also a null sequence. Replace \( a_{ij}^{(l)} \) by \( a_{ij}^{(l-1)} \) in inequality (35) and the first assertion is proved. If \( S \) is regulated, then \( p_{j+1} - p_j, j = 0, 1, \ldots \) is bounded by some \( s \) and we have
\[
\sum_{j=0}^{\infty} \|x(j) - \xi\| \leq \sum_{j=0}^{\infty} \left( \sum_{q=0}^{j} \frac{\epsilon_q a_{ij}^{(q)}}{1 - a_{ij}} \right)
\]
which exists, if \( \mathcal{F} \) is asymptotical to \( \mathcal{G} \), and the proof is completed.

5. Asynchronous iterations for singular linear systems and the Frobenius vector problem

As a first application, the theory developed in the previous sections will now be used for the analysis of asynchronous iterative methods for singular linear systems and the Frobenius vector problem, i.e. determining nonnegative eigenvectors of nonnegative matrices, whose spectral radii are known. In the case of irreducible matrices, these are the Perron vectors and they are positive and unique after normalization. To compute them asynchronously, one can use the following result of Lubashewsky and Mitra (see Ref. [8], the notations are as usual).

**Theorem 5.1.** Let \( T \in L(\mathbb{R}^n) \) be a nonnegative, irreducible matrix of unit spectral radius and let there be an \( i_0 \in \{1, \ldots, n\} \) such that
\[
t_{bb_0} > 0.
\]

then the iteration
\[
x_i(j + 1) := \begin{cases} 
\sum_{l=1}^{n} x_i(s^l(j)) t_{ll} & \text{if } i = p_j, \\
x_i(j) & \text{otherwise},
\end{cases} \quad j = 0, 1, \ldots, \quad (38)
\]
converges to a positive left eigenvector of $T$, if

(i) $\mathcal{X}_c = \{x(0)\}$,

(ii) $x(0) \in \{x \in \mathbb{R}^n \mid x \geq 0 \land x_{i_0} > 0\}$,

(iii) $j - s^l(j) \leq s, \forall l, j$, for an $s \in \mathbb{N}_0$,

(iv) $\{p_j\} \cup \{p_{j+1}\} \cdots \cup \{p_{j+c}\} = \{1, \ldots, n\}, \forall j \geq 0$, for a $c \in \mathbb{N}_0$,

(v) $s^{i_0}(j) = j$, if $p_j = i_0, \forall j \geq 0$. \hfill (39)

**Remark 5.1.** Algorithm (38) is naturally a method for computing Perron vectors of any nonnegative irreducible matrix $T$ for which the spectral radius $\rho(T)$ is known. If there is no $i_0$, which has the required property (37), one can use the matrix $(1 - \alpha)I + \alpha T$, where $\alpha \in (0, 1)$, which has the same eigenvectors as $T$. Practically, condition (39) (v) is not restrictive. For example, if only one processor is used to compute the $i_0$th component of the system, then it has only to save this component in its local memory, until its next updating.

While in Ref. [8] the authors preferred for their proof a somewhat more constructive approach from the related theory of Markov chains, we will embed the analysis of these iteration methods into the theory developed in the previous sections. We will consider an equivalent problem, which allows us to analyze the affine linear and also the reducible case. Interpreting the different rows of a stochastic matrix as multiple data operators, as described in Example 3.6, leads to strictly nonexpansive pools of operators:

**Theorem 5.2.** (i) Let $T \in L(\mathbb{R}^n)$ be a matrix, $b_0, z(0) \in \mathbb{R}^n$, let $s^l(j), j = 0, 1, \ldots, l = 1, \ldots, n$, be sequences from $\mathbb{N}_0$, and $p_j, j = 0, 1, \ldots, n$, be a sequence from \{1, \ldots, n\}. Then the iteration

$$z_i(j + 1) := \begin{cases} \sum_{l=1}^{n} t_{ii} z_{s^l(j)} + b_i & \text{if } i = p_j, \\ z_i(j) & \text{otherwise}, \end{cases} \quad j = 0, 1, \ldots, \tag{40}$$

converges, if and only if for any permutation or nonsingular diagonal matrix $P \in L(\mathbb{R}^n)$ the iteration

$$x_i(j + 1) := \begin{cases} \sum_{l=1}^{n} (PTP^{-1})_{ii} x_{s^l(j)} + (Ph)_i & \text{if } i = p_j, \\ x_i(j) & \text{otherwise}, \end{cases} \quad j = 0, 1, \ldots,$$

with $x(0) := Pz(0)$, converges.

(ii) Moreover, if there is a $\zeta \in \mathbb{R}^n$ such that $(I - T) \zeta = b$, then the iteration (40) converges, if and only if

$$x_i(j + 1) := \begin{cases} \sum_{l=1}^{n} t_{ii} x_{s^l(j)} & \text{if } i = p_j, \\ x_i(j) & \text{otherwise}, \end{cases} \quad j = 0, 1, \ldots. \tag{41}$$
with \( x(0) := z(0) - \xi \), converges.

(iii) Let \( t_{m_i(j)} \) be for all \( i \in \{1, \ldots, n\} \) the \( j \)th of \( m_i \) nonzero entries in \( T \)'s \( i \)th row, or let \( m_i(j) = j, \forall j = 1, \ldots, n, \) and \( m_i = n, \) if this row is zero. Then the pool \( \mathcal{F} = \{ T^i \mid i = 1, \ldots, n \} \), defined by

\[
T^i : \mathbb{R}^m_i \rightarrow \mathbb{R},
\]

\[
T^i(y^1, \ldots, y^m_i) := \sum_{j=1}^{m_i} t_{m_i(j)} y^j, \quad i = 1, \ldots, n,
\]

is strictly nonexpansive on all closed intervals \( D \subset \mathbb{R} \), if \( T e = e. \) \( \mathcal{F} \) is contractive on all closed intervals \( D \subset \mathbb{R} \), containing zero, if \( T e < e. \)

(iv) Assume without loss of generality that the numbering of \( s^l(j), \) \( j = 0, 1, \ldots, \) is chosen in such a manner that all components \( x_i(s^l(j)) \) in Eq. (41) themselves are updated at time step \( s^l(j) \), i.e.

\[
p_{s^l(j)-1} = i \quad \text{for all } j \in \mathbb{N}, \ i \in \{1, \ldots, n\} \ \text{with } s^l(j) \geq 1.
\]

and, also w.l.o.g., that all initial vectors are multiples of \( e, \) therefore define

\[
x(-j) := x_j(0)e, \quad \forall j = 1, \ldots, n,
\]

and renumber in this way the elements of the sequences of \( s^l(j), \) \( j = 0, 1, \ldots, l = 1, \ldots, n, \) for which \( s^l(j) = 0. \) Then the asynchronous iteration \( (\mathcal{T}, \mathcal{H}, \mathcal{H}^c, \mathcal{F}) \), given by

\[
y(j + 1) := T^{p_j}(y(s^l(j)), \ldots, y(s^{m_j}(j))), \quad j = 0, 1, \ldots,
\]

where \( \mathcal{T} \) is as in (iii), \( \mathcal{H}^c = p_j, j = 0, 1, \ldots, \mathcal{H}^c = \{ s^l(j) \mid j = 0, 1, \ldots, i = 1, \ldots, m_j \} \) is given by

\[
s^l(j) := s^{m_j}(j), \quad \forall j \in \mathbb{N}_0, \ i = 1, \ldots, m_j,
\]

and \( \mathcal{H} \) by \( y(-l) := x_1(-l), \ l = 1, \ldots, n, \) generates

\[
y(j + 1) = x_{p_j}(j + 1), \quad \forall j \in \mathbb{N}_0.
\]

(v) Now let \( T \) be irreducible, \( i_0 \) be an index such that \( t_{i_0,i_0} \neq 0, \) and assume

(a) \( \mathcal{H} = p_j, j = 0, 1, \ldots, \) is regulated,

(b) \( s^0(j) = \max \{ j_0 \leq j | p_{j_0-1} = i_0 \} \) for all \( j > \min \{ j_0 \in \mathbb{N}_0 | p_{j_0} = i_0 \} \) with \( p_j = i_0. \)

(c) \( j - s^l(j) \leq s, \ \forall j \in \mathbb{N}_0, \ l = 1, \ldots, n, \) for an \( s \in \mathbb{N}_0. \)

Then \( (\mathcal{T}, \mathcal{H}^c, \mathcal{H}, \mathcal{F}) \) of Eq. (44) is confluent.

\[ \textbf{Proof.} \] (i) and (ii) can be shown by an induction on \( j, \) i.e. it is \( x(j) = Pz(j), \) respectively \( x(j) = z(j) - \xi, \) for all \( j \in \mathbb{N}_0. \) (iii) is also clear, since, for all \( i \in \{1, \ldots, n\}, x, y \in \mathbb{R}^m, \)
\[ |T'(x) - T'(y)| \leq \sum_{i=1}^{n} t_{ii} |x^i - y^i| \leq \max_{1 \leq i \leq m} |x^i - y^i| \quad (48) \]

and for equality one needs \( x^i - y^i = r = T'(x - y), \forall i = 1, \ldots, m \), for some \( r \in \mathbb{R} \). In the case \( T'e = 1 \) we have also

\[ |T'(x) - T'(y)| \geq \min_{1 \leq i \leq m} |x^i - y^i| \quad \forall x, y \in \mathbb{R}^m, \]

and \( T'(x) \) is self-mapping on every closed interval of \( \mathbb{R} \).

(iv) follows by induction on \( j \).

(v) Since Eq. (47)(b), in the graph of \( (\mathcal{T}, \mathcal{M}, \mathcal{N}, \mathcal{S}) \) all the vertices \( j \geq 1 \) with \( p_{j-1} = i_0 \) are connected by a directed path. Since \( T \) is irreducible there is an \( n_1 \in \mathbb{N} \), such that all vertices \( j \geq n_1 \) are successors of vertices \( a_j \), for which \( p_{a_j-1} = i_0 \). Because of the regularity of \( \mathcal{N} \) and Eq. (47)(c) (i.e. the regularity of \( \mathcal{S} \)) we can assume that \( j - a_j, j = n_1, n_1 + 1, \ldots \) is bounded by some \( a \in \mathbb{N} \).

Hence, there is an \( n_0 \in \mathbb{N} \), such that for all \( j \geq n_0 \)

\[ b_j := \max \{ j_0 \in \mathbb{N} \mid j_0 \leq j - a \land p_{b_j-1} = i_0 \} \]

exists. Then, for all \( j_0 \geq j, j_0 \) is a successor of \( b_j \) and the sequence \( j - b_j, j = n_0, n_0 + 1, \ldots \) is bounded by some \( b \in \mathbb{N} \).

Continuing this argument, by the irreducibility of \( T \), the regularity of \( \mathcal{N} \) and \( \mathcal{S} \), there is a \( c \in \mathbb{N} \), independent of \( i \) and \( j \) so that for all \( j \geq n_0, i \in \{1, \ldots, n\} \), there is a path from \( b_j \) to \( b_{j,c} \) containing a vertex \( w^i_j \) with \( w^i_j - 1 = i \). That is assertion (v). \( \square \)

**Remark 5.2.** Theorem 5.2 confirms Theorem 5.1 by Theorem 4.2. If \( T \) is a nonnegative irreducible matrix of unit spectral radius, and \( \text{diag}(v) \in L(\mathbb{R}^n) \) is a diagonal matrix, whose diagonal entries are the components of a Perron vector \( v \) of \( T \), in the same order, then we have

\[ \text{diag}(v)^{-1} T \text{ diag}(v) e = e. \]

But by (i), the convergence of the iteration (41) is independent of scaling, and since \( T \) is irreducible, the corresponding method (44) is convergent. Therefore, \( y(j) \to \lambda \) and, hence, \( x(j) \to \lambda e \), for some \( \lambda \in \mathbb{R}, 0 \). Because of condition (ii) in (39), the iterations \( y(j) \) will be positive after some first iteration steps, which means by the self-mapping of the operators in (iii) of the previous theorem, that the limit of \( x(j), j = 0, 1, \ldots \) cannot vanish.

With help of Theorem 5.2, we can state a result on asynchronous iterations for linear systems of equations \( Ax = b \). Asynchronous iterations were investigated first by Chazan and Miranker [5] (as chaotic relaxations). They considered for \( \mathcal{S}_\epsilon = \{x(0)\} \) the iteration
where the pair of matrices \((B, C)\) is a spliting of \(A\), i.e. \(A = B - C\) and \(\det(B) \neq 0\). The other notations are as usual.

A typical condition for convergence theorems for such methods is, to assume, that \(A \in L(\mathbb{R}^n)\) is an M-matrix. That is a matrix which has a decomposition \(A = sI - T\), such that \(T\) is nonnegative, and its spectral radius \(\rho(T)\) is less or equal to \(s\). Further, to assume that the splitting is weak regular, that is: \(B^{-1}\) and \(B^{-1}C\) are nonnegative. There is a lot of literature, how to get weak regular splittings of M-matrices. Here, we just mention Ref. [2]. Using Perron and Frobenius theory, one gets a \(v \in \mathbb{R}^n\), such that

\[
\begin{align*}
(a) \quad & Av = (sI - T)v > 0, v > 0 \quad \text{if } A \text{ is regular,} \\
(b) \quad & Av = (sI - T)v = 0, v > 0 \quad \text{if } A \text{ is singular and irreducible,} \\
(c) \quad & Av = (sI - T)v \gg 0, v \gg 0 \quad \text{if } A \text{ is singular.}
\end{align*}
\]

Therefore, if the splitting is weak regular we have in case (b) that

\[
B^{-1}Cv = (I - B^{-1}A)v = v.
\]

Now, due to (i) of Theorem 5.2, we can assume w.l.o.g. that \(B^{-1}C\) is stochastic. Because of (ii), we can assume that a consistent linear system \(Ax = b\) is homogeneous, and by (iv) we can observe the equivalent one-dimensional asynchronous iteration (44), produced by a strictly nonexpansive pool, due to (iii). Thus, from Theorem 4.2, it suffices to carry over the conditions of (v) to the original iteration method (49).

It is already shown in Ref. [5] that Eq. (49) converges if \(A\) is a regular M-matrix, but we have an extension for singular M-matrices. In Ref. [5] one finds an often cited necessary and sufficient condition for general convergence of method (49), which is that \(\rho(B^{-1}C)\) has to be less than one. But this criterion based on regular matrices \(A\) and, on the whole, unsteered sequences \(s^l(j)(j = 0, 1, \ldots)\). Under the above considerations, we can give the following theorem without a proof.

**Theorem 5.3.** Let \(b \in \mathbb{R}^n\) and \((B, C)\) a splitting of \(A \in L(\mathbb{R}^n)\) such that \(B^{-1}C\) is nonnegative, irreducible, and of unit spectral radius. Let there be an \(i_0\) so that the \(i_0\)th diagonal entry of \(B^{-1}C\) is positive, and

\[
\begin{align*}
(i) \quad & j - s^l(j) \leq s, \quad \forall l, j, \text{ for an } s \in \mathbb{N}, \\
(ii) \quad & \mathcal{N} = p_j, \quad j = 0, 1, \ldots, \text{ be regulated},
\end{align*}
\]
(iii) \[ s^n(j) = \max\{j_0 \leq j \mid p_{j_0-1} = i_0\} \text{ for all } j > \min\{j_0 \in \mathbb{N}_0 \mid p_{j_0} = i_0\} \]
with \( p_j = i_0 \).

Then, if \( Ax = b \) is consistent, the iteration (49) converges to a solution of it.

Our last topic is the reducible case. It is well known (see for example Ref. [2]) that in this case the corresponding sequential methods need not to converge, even if the system is consistent. The condition here is that \( T = B^{-1}C \) is convergent, i.e. \( \lim_{j \to \infty} T^j \) exists. Now, let us transform \( T \) first into its Frobenius normal form, that is

\[
PTP^{-1} = \begin{pmatrix}
T^{11} & 0 & \ldots & 0 \\
0 & T^{22} & \ldots & \\
& \ddots & \ddots & \ddots \\
0 & \ldots & 0 & T^{pp-1} \\
T^{p1} & \ldots & T^{p(p-1)} & T^{pp}
\end{pmatrix},
\]

where \( P \) is a permutation matrix such that all \( T^{ii} \in L(\mathbb{R}^{n_i}) \), \( n_i \in \mathbb{N}, i = 1, \ldots, p \),
are square and either irreducible or \( 1 \times 1 \), and then scale it by \( \diag(v_i) \in L(\mathbb{R}^{n_i}) \), the matrix, whose diagonal elements are the components of Perron vectors of the diagonal blocks of \( PTP^{-1} \), then

\[
T_0 := \diag(v_i)^{-1} PTP^{-1} \diag(v_i)
\]

is convergent, if and only if \( T \) is convergent. For that reason and Theorem 5.2 (i), we can assume without loss of generality that \( B^{-1}C \) is in Frobenius normal form and its diagonal blocks are either stochastic or all row sums of them are less than one.

**Theorem 5.4.** Let \( A \in L(\mathbb{R}^n) \) be a matrix, \( b \in \mathbb{R}^n \) and \( (B, C) \) be a splitting of \( A \), such that \( T := B^{-1}C \) is convergent, nonnegative, and in Frobenius normal form, given by Eq. (52). Further, let there be a subset \( I_0 \) of \( \{1, \ldots, n\} \), such that for each \( i \in \{1, \ldots, p\} \), for which \( T^{ii} \) is of unit spectral radius, there is some \( i_0 \in \{ \sum_{k=1}^{i-1} n_k + 1, \ldots, \sum_{k=1}^{i} n_k \} \) in \( I_0 \) such that \( t_{i_0i_0} \) is positive. In addition, let

(i) \( j - s^j(j) \leq s, \forall l = 1, \ldots, n, j \in \mathbb{N}_0, \) for an \( s \in \mathbb{N} \),
(ii) \( s^j(j) = p_j, j = 0,1, \ldots, \) be regulated, 
(iii) \( s^j(j) = \max\{j_0 \leq j \mid p_{j_0-1} = i_0\} \text{ for all } j > \min\{j_0 \in \mathbb{N}_0 \mid p_{j_0} = i_0\} \)

with \( p_j = i_0 \) and for all \( i_0 \in I_0 \).

Then the iteration method (49) converges to a solution of \( Ax = b \), if the system is consistent. If, further,
(iv) \( b = 0, \)
\[ x(0) \in \{ x \in \mathbb{R}_{+}^{n} \mid x_{i0} > 0, \forall i \in I_{0} \}, \]
and \( \rho(T) = 1, \)

then Eq. (49) converges to a nonnegative eigenvector (Frobenius vector) of \( T. \)

**Proof.** For convenience, we denote \( \sum_{k=1}^{i-1} n_k \) by \( n^i, \) for \( i = 1, \ldots, p + 1, \) we let \( x^i(j), j = 0, 1, \ldots, i = 1, \ldots, p, \) be the sequences which partition the iterations \( x(j), j = 0, 1, \ldots, \) in the same manner as \( T \)'s Frobenius normal form, assume that \( b = 0, \) and, finally, that \( T^i e \) for all \( i \in \{ 1, \ldots, p \} \) either less or equal to \( e. \)

We will show by induction on \( i: \)

(a) \( \lim_{j \to \infty} x^i(j) = z^i, \quad \text{where} \quad T^i z^i = z^i - \sum_{l=1}^{i-1} T^l z^l, \)

(b) If (iv) holds and entails \( z^i = 0, \) then
\[
\sum_{j=0}^{\infty} \| x^i(j) - z^i \| < \infty. \tag{54}
\]

For \( i = 1 \) and \( T^{11} e = e, \) this is proved by Theorem 5.1 and the previous theorem. For \( T^{11} e < e \) it is an old result in the theory of asynchronous iterations (compare for example the nonlinear case in Ref. [1]). These results guarantee the linear convergence of asynchronous iterations under the conditions above, and, therefore, Eq. (54) validates. But it can also be derived from Theorem 4.4: The pool \( \mathcal{F}, \) which belongs by Theorem 5.2(iii) to \( T^{11}, \) is contractive, its unique fixed point is zero and it is asymptotical to itself. Let (a) and (b) be shown, for all \( 1 \leq l < i, \) and define recursively: \( l_{j+1} := \min \{ l > l_j \mid p_l \in \{ n^l + 1, n^{l+1} \} \}, \quad \forall j \in \mathbb{N}_0. \)

Like in Theorem 5.2, some notational modifications and assumptions (which we can make without loss of generality) make us able to analyze, instead of \( x^i(j), j = 0, 1, \ldots, \) the convergence behaviour of an equivalent one-dimensional asynchronous iteration: For all \( j \in \mathbb{N}_0 \) we have
\[
x_{p_{l_j}}(l_j + 1) = \sum_{k=1}^{n_{l_j}} \sum_{l=1}^{i-1} p_{l_j - n^l} x^i_l(s^l + n^l(l_j)) + \sum_{k=1}^{i-1} \sum_{l=1}^{n_{l_j}} \sum_{k^l} d_{l_j - n^l} x^i_l(s^l + n^l(l_j)) - y_l^i), \tag{55}
\]

The system
\[
T^{1i} z^i = z^i - \sum_{l=1}^{i-1} T^{li} z^l
\]
is solvable for some $\zeta^i \in \mathbb{R}^n$. For $T^i e < e$ this follows from the fact that $I - T^i$ is a regular M-matrix, and therefore monotone, i.e. its inverse is nonnegative (here even positive), cf. Ref. [2]. If $T^i e = e$, assume that there are $1 \leq l < i, 1 \leq p \leq n, 1 \leq k \leq n_l$, such that $t^i_{pk} \neq 0$, but $z^i_k > 0$, then

$$
\begin{pmatrix}
T^{ii} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & T^{ii} & 0 \\
T^{ii} & \cdots & T^{ii} & T^{ii} \\
\end{pmatrix}
\begin{pmatrix}
z^i \\
\vdots \\
z^i \\
e \\
\end{pmatrix}
\geq
\begin{pmatrix}
z^i \\
\vdots \\
z^i \\
T^{ii} z^i + e \\
\end{pmatrix}
$$

and $T$ would not be convergent, so, $\zeta^i = \lambda e$ is an solution for all $\lambda \in \mathbb{R}$.

Therefore, like in Theorem 5.2(ii) and (iv), set $y(-j) = (x^i(0) - \zeta^i) e$, $j = 1, \ldots, n_l$, renumber in this way the elements $s^i(j)$ with $s^i(j) = 0$, and we have for all $j \in \mathbb{N}_0$, by induction,

$$y_{p_{j'-n}}(l+1) := \sum_{p_{i'-n} \neq l} t^i_{p_{j'-n} l} y_{l}(s^{i'+n'}(l)) + e_j = x_{p_{j}}(l+1) - \zeta^i_{p_{j'-n}}.$$

where

$$
e_j := \sum_{k=1}^{n_l} s^{m_{p_{j'-n}}} \left(x^i(s^{i'+n'}(l)) - z^i_l\right) \quad \forall j \in \mathbb{N}_0.$$

Like in Theorem 5.2(iv) assume, w.l.o.g., that $p_{i'(j)-1} = l$, for all $j \in \mathbb{N}, l \in \{1, \ldots, n\}$. Further, if $T^u \neq (0)$, let the $p$th row of $T^u$ have $m_p$ positive entries, whose $j$th be $t^u_{p_{m_p} j}$, and define sequences $s^i_j = \{s^i_j(j) \mid j = 0, 1, \ldots; l = 1, \ldots, m_{p_{j'-n}}\}$ by

$$s^i_j(j) := s^{m_{p_{j'-n}}} (l) \quad \forall j \in \mathbb{N}_0, l = 1, \ldots, m_{p_{j'-n}}.$$

or, if $T^u = (0)$, $s^i_j(j) := s^{i'+1}(l), \quad \forall j \in \mathbb{N}_0.$

Next, let $\mathcal{I}_i$ be the strictly nonexpansive pool of operators, which is belonging to the rows of $T^u$ like in Theorem 5.2(iii), i.e. $\mathcal{I}_i := \{(T^u)^j : \mathbb{R}^{m_{p_{j'-n}}} \to \mathbb{R} \mid l = 1, \ldots, n_l\}$. Let $\mathcal{I} := \{w(-j) = (p_{i'-n} - j) \mid j = 1, \ldots, n_l\}$, and, finally, define a pool $\mathcal{I}_i = \{F^i : \mathbb{R}^{m_{p_{j'-n}}} \to \mathbb{R} \mid j = 0, 1, \ldots\}$ respectively $\mathcal{I}_i = \{F^i : \mathbb{R} \to \mathbb{R} \mid j = 0, 1, \ldots\}$, if $T^u = (0)$, by

$$F^i(w^1, \ldots, w^{m_{p_{j'-n}}}) := (T^u)^{p_{j'-n}}(w^1, \ldots, w^{m_{p_{j'-n}}}) + e_j \quad \forall j \in \mathbb{N}_0,$$

respectively $F^i(w^1) := e_j \quad \forall j \in \mathbb{N}_0$.

Then the asynchronous iteration $(\mathcal{I}_i, \mathcal{I}, \mathcal{I}_i)$ produces the sequence $w(j+1), j = 0, 1, \ldots$, which we have to analyze, i.e.

$$w(j+1) = y_{p_{j'-n}}(l+1), \quad \forall j \in \mathbb{N}_0.$$
First assume $T^i e < e$, then $\mathcal{F}_i$ is contractive, and since $\mathcal{F}_i$ approximates $\mathcal{F}_i$ the assertion follows by Theorem 4.4. If (iv) holds and $x'(j), j = 0, 1, \ldots$, converges to zero, then $T^i 0 = 0$ or $z^i = 0$ for all $l < i$ and this means, by the assumption of our induction, that the sum of $v_j, j = 0, 1, \ldots$ converges. Therefore, in this case, again by Theorem 4.4, the series

$$\sum_i \|x'(j) - z^i\|, \quad j = 0, 1, \ldots$$

exists.

Now let $T^i e = e$. By the assumption of our induction, and the same argument as in Eq. (56), the sum of $\|x'(j) - z^i\|, j = 0, 1, \ldots$ converges, or $T^i 0 = 0$ for all $l < i$. Again, $\mathcal{F}_i$ is asymptotical to $\mathcal{F}_i$ and the first part of the assertion follows by Theorem 4.3. Since, in the case that (iv) holds, $x'(j) \geq v(j), \forall j \in \mathbb{N}_0$, where $v(j), j = 0, 1, \ldots$, be the sequence, which would be generated by the corresponding iteration to compute a Perron vector of $T^i$, using the initial vector $x'(0), z^i$ has to be positive, and we need not prove (b).

We note, that for determining a Frobenius vector of $T$ it suffices that there be only one $i_0 \in I_0$ with $x_{i_0}(0) > 0$. Above, we assumed just for technical reasons that this is the case for all $i_0 \in I_0$.

The theory developed so far, seems to be extendable in a lot of ways, for example:

Confluence: It is obvious, that one can develop a lot of different concepts, which guarantee confluence for the asynchronous iteration in Theorem 5.2 (iv), even if there is no positive diagonal entry of $T$, and even if it is not primitive, i.e. $T$ has no positive power. For example, a homogeneous system with

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is sequentially not convergent, but an asynchronous iteration with $p_0 = 1, p_1 = 2, s^1(1) = 1$ (i.e. local confluence!) would give a solution after two steps.

Pools: Different set-ups for serial and synchronized parallel algorithms for the solution of linear systems lead a pool of iteration matrices, i.e. nonstationary iterative methods. For example, the concept of multisplittings (cf. [4,10], several others), where the system matrix $A$ is splitted into $A = B^k - C^k, k \in \mathbb{K}$, and the iteration matrices are

$$T^k = (B^k)^{-1}C^k, \quad k \in \mathbb{K}. \quad (58)$$

Also, so-called two-stage methods (cf. for example Ref. [9], where the case $\rho(B^{-1}C) = 1$ is considered) have this structure. Here, the matrix $B$ in the splitting $A = B - C$ is again splitted into $B = F^k - G^k, k \in \mathbb{K}$, and the solution of the
system \( Bx = Cx + b \) is approximated by some iteration steps for the equations \( F^k x = G^k x + Cx + b \). The latter concept leads to iteration matrices of the form

\[
T^{(k,q)} = \left((F^k)^{-1} G^k\right)^q + \sum_{j=0}^{q-1} \left((F^k)^{-1} G^k\right)^j (F^k)^{-1} C, \quad k \in \mathbb{K}, \quad q \in \mathbb{N}.
\] (59)

In both cases, we have for the equivalent asynchronous iteration, according to Theorem 5.2 (iv), to guarantee confluence for all rows of a set of iteration matrices. Furthermore, in the case of Eq. (59) we have under suitable conditions

\[
\lim_{q \to \infty} T^{(k,q)} = B^{-1} C,
\]
so that our set-up of approximate pools can become useful for the analysis of such methods.

**Nonnegativity** (We note, that we need in Example 3.6 and Theorem 5.4 only): \(|T|\leq e\), and all rows with unit absolute row-sum are nonnegative.

**Markov-chains**: A right Frobenius vector of a nonnegative (column-) stochastic matrix, describes a stationary distribution of a finite homogeneous Markov-chain (cf., e.g., Ref. [13]). Extensions are related to all the ideas above and below, e.g.: What about infinite and/or inhomogeneous Markov-chains?

**Linearity**: Corresponding nonlinear problems are singular nonlinear M-equations, whatever that be; e.g., solve \( F(x) = 0 \), where \( F: \mathbb{R}^n \to \mathbb{R}^n \), \( F'(x) \) exists and is a (partially singular) M-matrix. (e)-paracontracting pools and their extended properties could also be useful to determine the eigenfunctions of differential equations, if the associated eigenvalue is known.

**Infinite pools**: Note, e.g., that our induction in Theorem 5.4 is finite. The question in mind is, how we can extend these results, for example, to countable infinite dimensional problems by using the idea of indexwise-regulated sequences.

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