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Note

Finite isomorphically complete systems

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Abstract

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In the theory of finite automata it is an important problem to characterize such systems of automata from which any automaton can be built under a given composition and representation. Such systems are called complete with respect to the fixed composition and representation. From practical point of view, it is useful to determine those compositions and representations for which there are finite complete systems. In this paper we show that the existence of finite complete systems implies the unboundedness of the feedback dependency of the composition.

By *an automaton* we mean a system $A=(X,A,\delta)$ where X and A are nonempty finite sets, the set of *inputs* and the set of *states*, respectively, and $\delta: A\times X\to A$ is the *transition function*.

Since each automaton can be considered a unoid (a universal algebra with unary operational symbols), the notions such as *isomorphism*, *embedding* and *subautomata* can be introduced in a natural way.

The concept of a composition can be defined in different ways. If we impose some restrictions on the feedback dependency of the general product (see [4,5]), then we obtain special compositions. Such restrictions as "the ith feedback function may depend on some arguments only" can be given by functions $\gamma: N\to\mathfrak{P}(N)$, where $N=\{1,2,\dots\}$ and $\mathfrak{P}(N)$ denotes the power set of N. Taking a suitable nonempty set of such functions, we obtain a composition. This approach was used in [6], where a decomposition theorem was proved.

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We get another approach to the same notion by considering a composition as a network of automata. In this case each vertex of the network denotes an automaton, and the input of an automaton may depend only on those automata which have direct connections to the given one. In [2,7] this approach is used to study completeness under the isomorphic simulation as representation. In this paper we are also using this approach.

Let $D=(E,V)$ be a directed graph consisting of a nonempty finite set of vertices $V=\{1,\dots,n\}$ and edges $E\subseteq V\times V$. Consider an arbitrary nonempty set $\mathscr{D}$ of such finite directed graphs. Moreover, let $A_i=(X_i,A_i,\delta_i)$ $(i=1,\dots,n)$ be a system of automata, X a finite nonempty set and φ a mapping of $A_1\times\cdots\times A_n\times X$ into $X_1\times\cdots\times X_n$. It is said that the automaton $A=(X,A,\delta)$ is a ***$\mathscr{D}$-product*** of A_i $(i=1,\dots,n)$ with respect to X and φ if the following conditions are satisfied:

(1) $A=\prod_{i=1}^n A_i$,

(2) there exists a graph $D=(\{1,\dots,n\},E)$ in $\mathscr{D}$ such that the mapping φ can be given in the form

$$\varphi(a_1,\dots,a_n,x)=(\varphi_1(a_1,\dots,a_n,x),\dots,\varphi_n(a_1,\dots,a_n,x)),$$

where $(a_1,\dots,a_n)\in A$, $x\in X$ and each φ_i $(1\le i\le n)$ is independent of any a_j with $(j,i)\notin E$,

(3) for arbitrary $x\in X$ and $a_i\in A_i$ $(i=1,\dots,n)$

$$\delta(a_1,\dots,a_n,x)=(\delta_1(a_1,\varphi_1(a_1,\dots,a_n,x)),\dots,\delta_n(a_n,\varphi_n(a_1,\dots,a_n,x))).$$

For this product we shall use the notation

$$\prod_{i=1}^{n} A_i(X,\varphi,D).$$

Now let Γ be a system of automata. It is said that Γ is *isomorphically complete* with respect to the $\mathscr{D}$-product if any automaton can be embedded isomorphically into a $\mathscr{D}$-product of automata from Γ.

We shall use the following special automata. For every natural number $m\ge 1$ let $T_m=(T_m,\{1,\dots,m\},\delta_m)$ be the automaton for which T_m is the set of all transformations of $\{1,\dots,m\}$ and $\delta_m(j,t)=t(j)$ for arbitrary $j\in\{1,\dots,m\}$ and $t\in T_m$.

Now we are ready to prove the following statement:

Theorem 1. *Let $\mathscr{D}$ be an arbitrary nonempty set of finite directed graphs. If there exists a finite isomorphically complete system of finite automata with respect to the $\mathscr{D}$-product, then for every integer $k\ge 1$ there is a graph $D\in\mathscr{D}$ such that D has a subgraph for which the indegree of each vertex is at least k.*

Proof. Let us suppose that a finite system Γ of finite automata is isomorphically complete with respect to the $\mathscr{D}$-product. Then there is an integer $s\ge 1$ such that $|C|\le s$ holds for every automaton $C=(X,C,\delta_C)$ in Γ. Now let $k\ge 1$ be an arbitrary fixed integer and $m=s^{k+1}$. Consider the automaton T_m. Since Γ is isomorphically complete with respect to the $\mathscr{D}$-product, there exist a graph $D=(\{1,\dots,n\},E)\in\mathscr{D}$

and automata $A_1, \ldots, A_n \in \Gamma$ such that T_m can be embedded isomorphically into a $\mathscr{D}$-product

$$B = (T_m, B, \delta_B) = \prod_{i=1}^{n} A_i(T_m, \varphi, D).$$

Let us denote by $\mu : \{1, \ldots, m\} \to B$ a suitable isomorphism, and for arbitrary $r \in \{1, \ldots, m\}$ let $(a_{r1}, \ldots, a_{rn})$ be the image of r under μ.

Now define the set V' as follows:

$$V' = \{i \in \{1, \ldots, n\} \mid \text{there are } 1 \le u \ne v \le m \text{ with } a_{ui} \ne a_{vi}\}.$$

We shall show that the subgraph $D' = (V', E \cap (V' \times V'))$ has the required properties. To this end let i be an arbitrarily fixed vertex of D and $\{i_1, \ldots, i_l\} = \{v \in V \mid (v, i) \in E\}$. (Especially, if i has a loop edge, then $i \in \{i_1, \ldots, i_l\}$.) Now let us suppose that there are indices $u \ne v$ $(1 \le u, v \le m)$ with $a_{ui} = a_{vi}$ and $a_{uj} = a_{vj}$ $(j = i_1, \ldots, i_l)$. Let $1 \le w \le m$ be an arbitrary integer and t_w such a transformation of $\{1, \ldots, m\}$ for which $t_w(u) = u$, $t_w(v) = w$. Then $\delta_m(u, t_w) = u$ and $\delta_m(v, t_w) = w$. From this, using the fact that μ is an isomorphism, it follows that

$$\delta_B(\mu(u), t_w) = \mu(u)$$

and

$$\delta_B(\mu(v), t_w) = \mu(w)$$

hold in the $\mathscr{D}$-product B. But then, by the definition of the $\mathscr{D}$-product,

$$\delta_i(a_{ui}, \varphi_i(a_{u1}, \ldots, a_{un}, t_w)) = a_{ui}$$

and

$$\delta_i(a_{vi}, \varphi_i(a_{v1}, \ldots, a_{vn}, t_w)) = a_{wi}.$$

Again, by the definition of the $\mathscr{D}$-product, we obtain

$$\varphi_i(a_{u1}, \ldots, a_{un}, t_w) = \varphi_i(a_{ui_1}, \ldots, a_{ui_l}, t_w)$$

and

$$\varphi_i(a_{v1}, \ldots, a_{vn}, t_w) = \varphi_i(a_{vi_1}, \ldots, a_{vi_l}, t_w).$$

Therefore

$$\delta_i(a_{ui}, \varphi_i(a_{ui_1}, \ldots, a_{ui_l}, t_w)) = a_{ui}$$

and

$$\delta_i(a_{vi}, \varphi_i(a_{vi_1}, \ldots, a_{vi_l}, t_w)) = a_{wi}.$$

According to our assumption $a_{ui} = a_{vi}$, $a_{ui_j} = a_{vi_j}$ $(j = 1, \ldots, l)$, and so, the arguments of δ_i are the same in both equations. But then $a_{ui} = a_{wi}$. On the other hand, $1 \le w \le m$ is arbitrary, which results $a_{ui} = a_{wi}$ $(w = 1, \ldots, m)$.

The above observation yields that for arbitrary $i \in V'$, the elements

$$(a_{ri}, a_{ri_1}, \ldots, a_{ri_l}) \quad (r = 1, \ldots, m)$$

are pairwise different, where $i_1, \ldots, i_l$ are all the ancestors of i in D. Now, if $i_j \in V - V'$ for some $1 \le j \le l$, then $a_{ri_j} = a_{i_j}$ $(r = 1, \ldots, m)$ holds for a fixed state

$a_{i_l} \in A_{i_l}$. But then for arbitrary $i \in V'$ the elements $(a_{ri}, a_{ri_1}, \ldots, a_{ri_l})$ $(r = 1, \ldots, m)$ are pairwise different, where $i_1, \ldots, i_l$ denote all the ancestors of i in D'. On the other hand, $|A_i| \leq s$ and $|A_{i_j}| \leq s$ $(j = 1, \ldots, l)$, and so, the number of the pairwise different $(l+1)$-tuples $(a_{ri}, a_{ri_1}, \ldots, a_{ri_l})$ $(a_{ri} \in A_i,\ a_{ri_j} \in A_{i_j}$ $(j = 1, \ldots, l))$ is not greater than s^{l+1}. Therefore, $m \leq s^{l+1}$. But $m = s^{k+1}$. Thus $k \leq l$. This means that the indegree of i is at least k for arbitrary $i \in V'$, which completes the proof of our statement. □

Remark 2. From the above theorem it follows that for the well-known notions of composition as the quasi-direct product, the α_i-product [3,4], the ν_i-product [1], and the star-product [7] no finite isomorphically complete systems exist.

It is unknown yet whether the converse of Theorem 1 is true. To end this paper we give two classes of examples in which the conditions of Theorem 1 are sufficient. In the rest of the paper A_2 will denote the automaton $A_2 = (\{x, y\}, \{0, 1\}, \delta_2)$ with $\delta_2(0, x) = \delta_2(1, y) = 1$ and $\delta_2(0, y) = \delta_2(1, x) = 0$. It is well known that A_2 forms an isomorphically complete class for the general product (see [5]).

Example 3. Let d be a fixed nonnegative integer. Moreover, let $\mathscr{D}$ be a set of directed graphs. If for arbitrary positive integer n there is a graph D in $\mathscr{D}$ with n vertices which has a subgraph D' such that the indegree of each vertex in D' is at least k and $n - k \leq d$, then A_2 is isomorphically complete with respect to the $\mathscr{D}$-product.

For arbitrary $D = (E, V) \in \mathscr{D}$ $(E = \{1, \ldots, n\})$ and i $(1 \leq i \leq n)$ set $\mathrm{in}(i) = \{j \mid (j, i) \in V\}$. Let D_n be a graph in $\mathscr{D}$ which for a positive integer n with $n > 2d$ satisfies the above conditions under a subgraph D_n' and integer k. Moreover, let $\{1, \ldots, m\}$ be the set of all vertices of D_n'. Let t be maximal with $d < [m/t]$. Of course, such a t exists, since $k > d$ and $m \geq k$. Take the following subset S of $\{0, 1\}^n$: an $(s_1, \ldots, s_n) \in \{0, 1\}^n$ is in S iff the next two conditions are satisfied:

(1) for all i, j $(1 \leq i \leq j \leq [m/t]t)$, $s_i = s_j$ if $j - i \equiv 0 \pmod t$,

(2) $s_i = 0$ if $[m/t]t < i \leq n$.

Let $B = (X, B, \delta)$ be an automaton with at most 2^t states. Moreover, let τ be a one-to-one mapping of B into S. Now define the functions

$$\varphi_i : \underbrace{\{0, 1\} \times \cdots \times \{0, 1\}}_{n \text{ times}} \times X \to \{x, y\}$$

in the following way:

(i) for arbitrary $z \in X$ and $(s_{11}, \ldots, s_{1n}) \in S$,

$$\delta_2(s_{1i}, \varphi_i(s_{11}, \ldots, s_{1n}, z)) = s_{2i}$$

if $1 \leq i \leq [m/t]t$ and there are $b_1, b_2 \in B$ with $\tau(b_j) = (s_{j1}, \ldots, s_{jn})$ $(j = 1, 2)$ and $\delta(b_1, z) = b_2$,

(ii) for arbitrary $z \in X$ and $(s_1, \ldots, s_n) \in S$, $\varphi_i(s_1, \ldots, s_n, z) = y$ if $[m/t]t < i \leq n$,

(iii) in all other cases φ_i is defined arbitrarily in accordance with the definition of the $\mathscr{D}$-product.
Then φ_i is well defined since, by $d<[m/t]$ and $m-k\le d$, for all i $(1\le i\le [m/t]t)$ there is an l $(0\le l<[m/t])$ such that $\{lt+1,\dots,(l+1)t\}\subseteq \mathrm{in}(i)$.

Let us denote the resulting $\mathscr{D}$-product by $\boldsymbol{C}=(X,C,\delta')$. Then τ is an isomorphism of $\boldsymbol{B}$ into $\boldsymbol{C}$. Moreover, if n is unboundedly increasing, then t is unboundedly increasing, too. Therefore, A_2 is isomorphically complete with respect to the $\mathscr{D}$-product.

The above example shows that if the difference of n and k is under a fixed bound, then the converse of Theorem 1 is true. By the next example the difference between n and k can be arbitrary.

Example 4. Let K be a set of pairs (k,n), where $k=1,2,\dots$ and n (>1) is an arbitrary integer with $k\mid n$. For every $(k,n)\in K$ take a directed graph $D_{(k,n)}=(\{1,\dots,n\},V_{(k,n)})$ with

$$V_{(k,n)}=\{(i\ominus_n j,i)\mid i=1,\dots,n,\ j=1,\dots,k\},$$

where $i\ominus_n j$ denotes the least positive residue of $i-j$ modulo n. Set $\mathscr{D}=\{D_{(k,n)}\mid (k,n)\in K\}$. For each pair $(k,n)\in K$, let $S_{(k,n)}$ denote the following subset of $\{0,1\}^n$: $(s_1,\dots,s_n)\in\{0,1\}^n$ is in $S_{(k,n)}$ iff for all i, j $(1\le i,j\le n)$, $s_i=s_j$ if $i\equiv j$ (mod k). Take an automaton $\boldsymbol{B}=(X,B,\delta)$ with at most 2^k states and a one-to-one mapping τ of B into $S_{(k,n)}$. One can show, in a way similar to that in the previous example, the existence of a $\mathscr{D}$-product $\boldsymbol{C}=(X,C,\delta')$ of A_2 such that τ is an isomorphism of $\boldsymbol{B}$ into $\boldsymbol{C}$.

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