Nonlinear electro- and magneto-elastostatics:
Material and spatial settings

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Abstract

The material and spatial settings of the nonlinear coupling problem of electro- and magneto-elastostatics are discussed in this paper. The governing equations and variational formulations of the problem derived in these two settings using basic equations of electricity, magnetism and elasticity allow the consideration of material defects by the material force method. © 2007 Elsevier Ltd. All rights reserved.

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1. Introduction

The problem of nonlinear electro- and magneto-elastostatics has been recently the subject of many researches due to the interesting applications of electro- and magneto-elastic materials. Those applications includes robotic arms that are actuated by artificial muscles, adaptive tuned vibration absorbers, stiffness tunable mounts and suspensions and automotive bushing, see for example Kordonsky (1993), Jolly et al. (1996), Carlson and Jolly (2000), Bar-Cohen (2002), Dorfmann and Ogden (2004).

Because of the nonlinear electro- and magneto-mechanical coupling behavior, special attention is required in solving nonlinear electro- and magneto-elastic problems, especially when defects are involved. Based on the basic laws of electricity, magnetism and elasticity, the general equations and some analyses for boundary-value problems of nonlinear electro- and magneto-elasticity were addressed by, for example, Maugin (1988), Eringen and Maugin (1989, 1990), Voltaire et al. (2003), Brigdanov and Dorfmann (2003), Dorfmann and Ogden (2003, 2004, 2005), Steigmann (2004), Kankanala and Triantafyllidis (2004), Vu et al. (2006). Concerning with defects, the problem of material forces in nonlinear electro- and magneto-elasticity also received considerable attention, see for example the works of Pak and Herrmann (1986a,b), Maugin and Epstein (1991), Epstein and Maugin (1991), Maugin (1993), Huang and Batra (1996), Kalpakides and Agiasofitou (2002), Yavari et al. (2006), Trimmer (2007). In this work, the nonlinear electro- and magneto-elastostatic problems are revisited.

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The two problems of electro-elastostaticity and magneto-elastostaticity are presented separately for the practical reason that most of the current materials exhibit either electro-elastic or magneto-elastic behavior. By using a variational approach, the striking similarity between the spatial and material motion stresses as well as the similarity between the spatial and material balance equations of the three problems of elastostatics, electro-elastostatics and magneto-elastostatics are revealed. With the introduction of some material tractions and body forces to capture the energetic changes that are associated with changes in material configuration and material motions of defects relative to the ambient material, this work also shows that the two problems of material and spatial motions (and not only their balance equations) are only equivalent for defect free bodies. For details about the material force method and its applications, the readers are referred to, for example, the works of Steinmann (2000, 2002a,b,c), Steinmann et al. (2001), Liebe et al. (2003), Denzer et al. (2003), Kuhl and Steinmann (2004), Denzer (2006).

In the following sections, after recalling some notions of spatial and material motion problems, the basic laws of nonlinear elasticity, electricity and magnetism are reviewed. Next, the governing equations of nonlinear electro- and magneto-elastostatics will be derived for the spatial and material settings.

2. Spatial and material motion problems

Let us consider a body with its material (reference) configuration \( B_0 \) in the absence of electric and magnetic fields and mechanical loads. The spatial (current) configuration of the body is denoted by \( B \).

For the static cases, in the spatial motion problem, the position vector \( x \) of a point in the spatial configuration \( B \) is described by the nonlinear spatial motion map: \( x = \varphi (X) \) and the deformation is characterized by the spatial motion deformation gradient: \( F = \nabla_X \varphi \) wherein \( \nabla_X \varphi \) denotes \( \partial \varphi / \partial X \).

In the material motion problem, the position vector \( X \) of a point in the material configuration \( B_0 \) is described by the nonlinear material motion map: \( X = \Phi (x) \) and the deformation is characterized by the material motion deformation gradient \( f = \nabla_x \Phi \).

For a conservative mechanical system, the elastic material response is characterized by some internal potential energy densities per unit volume \( W_{0F} \), \( W_{0f} \) in material configuration or \( W_{fF} \), \( W_{fF} \) in spatial configuration:

\[
\begin{align*}
W_{0F} &= W_{0F}(F;X); & W_{0f} &= W_{0f}(f;\Phi) \\
W_{fF} &= W_{fF}(F;X); & W_{fF} &= W_{fF}(f;\Phi)
\end{align*}
\]

such that: \( W_{0F}|_{F,X} = W_{0f}|_{f,\Phi} \) and \( W_{fF}|_{F,X} = W_{fF}|_{f,\Phi} \).

The conservative loading is characterized by some external potential energy densities per unit volume \( V_{0F} \), \( V_{0f} \) in material configuration or \( V_{fF} \), \( V_{fF} \) in spatial configuration:

\[
\begin{align*}
V_{0F} &= V_{0F}(|\varphi;X); & V_{0f} &= V_{0f}(|\varphi;x;\Phi) \\
V_{fF} &= V_{fF}(|\varphi;X); & V_{fF} &= V_{fF}(|\varphi;x;\Phi)
\end{align*}
\]

such that: \( V_{0F}|_{|\varphi,X} = V_{0f}|_{|\varphi,\Phi} \) and \( V_{fF}|_{|\varphi,X} = V_{fF}|_{|\varphi,\Phi} \). In the above definitions “\( X \)” and “\( \Phi \)” denote possible explicit dependence of the energy densities on \( X \) and \( \Phi \), and the lower scripts \( F \) and \( f \) denote the spatial and material motion parameterizations, respectively.

The total potential energy densities characterizing the system are written as:

\[
\begin{align*}
U_{0F} &= W_{0F} + V_{0F}; & U_{0f} &= W_{0f} + V_{0f} \\
U_{fF} &= W_{fF} + V_{fF}; & U_{fF} &= W_{fF} + V_{fF}
\end{align*}
\]

in reference to the material configuration, or as:

\[
\begin{align*}
U_{fF} &= W_{fF} + V_{fF}; & U_{fF} &= W_{fF} + V_{fF}
\end{align*}
\]

in reference to the spatial configuration.

3. Nonlinear elastostatics

In order to facilitate the derivation of the governing equations of nonlinear electro- and magneto-elastostatics in spatial and material settings we recall here some results in the case of nonlinear elastostatics. For more details, see for example the work of Steinmann (2005).
3.1. Spatial motion problem in nonlinear elastostatics

For a conservative system, the governing equations can be derived using Dirichlet principle. For the total potential energy density $U_{0F}$, Dirichlet principle states that $\varphi$ minimizes the following functional:

$$ I_0' = \int_{B_0} U_{0F} \, dV \to \inf_{\varphi} $$

Following this condition, the first variation of $I_0'$ at fixed material placement $X$ vanishes and leads to:

$$ \nabla_X \cdot P + b_0 = 0 \quad \text{in } B_0 \quad \text{and} \quad \sigma \cdot N = 0 \quad \text{on } \partial B_0 $$

(6.1-2)

where $N$ is the outward pointing unit normal at the boundary $\partial B_0$, $\nabla_X \cdot \{ \bullet \}$ denotes $\partial \{ \bullet \} / \partial X$, $P$ is the total nominal stress tensor (or the spatial motion Piola stress tensor) and $b_0$ is the body force:

$$ P := \partial_F U_{0F} \quad \text{and} \quad b_0 := -\partial_\varphi U_{0F} $$

(7.1-2)

If we consider the total potential energy density $U_{1F}$, according to Dirichlet principle, $\varphi$ minimizes the functional:

$$ I_1' := \int_{B_1} U_{1F} \, dv \to \inf_{\varphi} $$

(8)

The stationary condition for $I_1'$ gives:

$$ \nabla_X \cdot \sigma + b_i = 0 \quad \text{in } B_i \quad \text{and} \quad \sigma \cdot n = 0 \quad \text{on } \partial B_i $$

(9.1-2)

where $n$ is the outward pointing unit normal at the boundary $\partial B_i$, and the stress tensor $\sigma$ and the body force $b_i$ are defined as:

$$ \sigma := U_{1F} I + \partial_F U_{1F} \cdot F^t \quad \text{and} \quad b_i := -\partial_\varphi U_{1F} $$

(10.1-2)

where the upper script $t$ denotes transpose.

Note that the system (9) is actually equivalent to the system (6), where the stress tensor $\sigma$ and the body force $b_i$ are related to their counterparts $P$ and $b_0$ by:

$$ \sigma = J^{-1} P \cdot F^t \quad \text{and} \quad b_i = J^{-1} b_0 $$

(11.1-2)

where $J = \det F$.

3.2. Material motion problem in nonlinear elastostatics

In the material motion problem, by considering the total potential energy functional:

$$ I_m' = \int_{B_i} U_{mF} \, dv $$

(12)

Steinmann (2005) noted that taking a variation of this functional at fixed spatial placement $x$ only leads to a stationary point $\delta I_m' = 0$ for the case of configurational equilibrium. In more general cases, configurational or rather material tractions $T_i^d$ acting on the boundary $\partial B_i$ and material forces, acting on defects such as vacancies, interfaces, dislocations, cracks and the like must be considered. These material forces capture the energetic changes that go along with material motions of the defects relative to the ambient material. If for simplicity we only consider distributed defects, then energetic changes will be captured by distributed configurational forces $B_i^d$ and material tractions $T_i^d$:

$$ \delta I_m' := \int_{B_i} B_i^d \cdot \delta \Phi \, dv + \int_{\partial B_i} T_i^d \cdot \delta \Phi \, ds \leq 0 $$

(13)

Note that the inequality sign is a reminder of the second law of thermodynamics, since changes in configuration $\delta \Phi$ are only admissible if potential energy is released.
From (13), the condition: \( \delta I^m_t - \int_{B_t} \mathbf{B}_t^d \cdot \delta \mathbf{F} \, dv - \int_{\partial B_t} \mathbf{T}_t^d \cdot \delta \mathbf{F} \, ds = 0 \) leads to the following balance equation and boundary condition:

\[
\nabla_x \cdot \mathbf{p} + \mathbf{B}_t^d = - \mathbf{B}_t^d \quad \text{in } B_t \quad \text{and} \quad \mathbf{p} \cdot \mathbf{n} = \mathbf{T}_t^d \quad \text{on } \partial B_t
\]

(14.1-2)

where the material motion Piola stress tensor \( \mathbf{p} \) and the material body force \( \mathbf{B}_t \) are defined as:

\[
\mathbf{p} := \partial_{\mathbf{r}} U_{\mathbf{r}} \quad \text{and} \quad \mathbf{B}_t := - \partial_{\mathbf{r}} U_{\mathbf{r}}
\]

(15.1-2)

From the definitions of the stress tensors \( \mathbf{\sigma} \) and \( \mathbf{p} \), one has the relationships:

\[
\mathbf{p} = U_{\mathbf{r}} f^{-1} - f^{-1} \cdot \mathbf{\sigma} \quad \text{and} \quad \mathbf{\sigma} = U_{\mathbf{r}} I - f^{-1} \cdot \mathbf{p}
\]

(16.1-2)

which lead to the following formulation of the material tractions \( \mathbf{T}_t^d \) in the presence of the boundary condition \( \mathbf{\sigma} \cdot \mathbf{n} = 0 \):

\[
\mathbf{T}_t^d = U_{\mathbf{r}} f^{-1} \cdot \mathbf{n}
\]

(17)

This formulation can be considered as the definition of \( \mathbf{T}_t^d \) in the sense that for a defect free body (\( \mathbf{B}_t^d = 0 \)) if \( \mathbf{p} \), \( \mathbf{B}_t \), and \( \mathbf{T}_t^d \) are defined by (15.1), (15.2) and (17), respectively, then the two systems (9) and (14) are completely equivalent. Consequently the material and spatial motion problems are equivalent, and they are only equivalent for defect free bodies since, as noted above, for a body with defects the material forces \( \mathbf{B}_t^d \) or other material forces (other than \( \mathbf{T}_t^d \)) acting on defects must be taken into account.

Consider now the material motion problem in material configuration. A variation of the total potential energy functional:

\[
I^m_0 = \int_{B_0} U_{\mathbf{r}} \, dV
\]

(18)

at fixed spatial placement \( x \) leads to the definition:

\[
\delta I^m_0 := \int_{B_0} \mathbf{B}_0^d \cdot \delta \mathbf{F} \, dV + \int_{\partial B_0} \mathbf{T}_0^d \cdot \delta \mathbf{F} \, dS \leq 0
\]

(19)

where \( \mathbf{B}_0^d \) is defined as the material body forces acting in \( B_0 \) and \( \mathbf{T}_0^d \) as material tractions acting on \( \partial B_0 \).

The condition (19) gives us:

\[
\nabla_x \cdot \mathbf{\Sigma} + \mathbf{B}_0^d = - \mathbf{B}_0^d \quad \text{in } B_0 \quad \text{and} \quad \mathbf{\Sigma} \cdot \mathbf{N} = \mathbf{T}_0^d \quad \text{on } \partial B_0
\]

(20.1-2)

where \( \mathbf{\Sigma} \) is the material motion stress and \( \mathbf{B}_0 \) is the material body force:

\[
\mathbf{\Sigma} := U_{\mathbf{r}} I + \partial_{\mathbf{r}} U_{\mathbf{r}} \cdot f^{-1} \quad \text{and} \quad \mathbf{B}_0 := - \partial_{\mathbf{r}} U_{\mathbf{r}}
\]

(21.1-2)

It can be seen that this system is equivalent to the system (14), where \( \mathbf{\Sigma}, \mathbf{B}_0, \mathbf{B}_0^d \) and \( \mathbf{T}_0^d \) are related to their counterparts \( \mathbf{p}, \mathbf{B}_t, \mathbf{B}_t^d \) and \( \mathbf{T}_t^d \) by:

\[
\mathbf{\Sigma} = \mathbf{J} \mathbf{p} \cdot f^{-1} \quad \mathbf{B}_0 = \mathbf{J} \mathbf{B}_t \quad \mathbf{B}_0^d = \mathbf{J} \mathbf{B}_t^d \quad \mathbf{T}_0^d = \mathbf{J} \left[ \mathbf{N} \cdot \mathbf{C}^{-1} \cdot \mathbf{N} \right]^{1/2} \mathbf{T}_t^d
\]

(22.1-3)

where \( \mathbf{C} = \mathbf{F}^t \cdot \mathbf{F} \). The definitions of the stress tensors \( \mathbf{\Sigma} \) and \( \mathbf{P} \) give us the relationship:

\[
\mathbf{\Sigma} = U_{\mathbf{r}} I - \mathbf{F}^t \cdot \mathbf{P}
\]

(23)

which is actually the Eshelby’s stress tensor (Eshelby, 1951) and has a similar format as (16.2). The relationship (23) leads to the following formulation of the material tractions \( \mathbf{T}_0^d \) in the presence of the boundary condition \( \mathbf{P} \cdot \mathbf{N} = 0 \):

\[
\mathbf{T}_0^d = U_{\mathbf{r}} \mathbf{N}
\]

(24)

By introducing (23) into (20), it can be shown that for a defect free body (\( \mathbf{B}_t^d = 0 \)) if \( \mathbf{\Sigma}, \mathbf{B}_0 \) and \( \mathbf{T}_0^d \) are defined by (21.1), (21.2) and (24), respectively, then the two systems (6) and (20) are completely equivalent. This leads to the conclusion that the material and spatial motion problems are equivalent, and they are only equivalent for defect free bodies since for a body with defects the material forces \( \mathbf{B}_t^d \) or other material forces (other than \( \mathbf{T}_0^d \)) acting on defects must be taken into account.
4. Nonlinear electro-elastostatics

In the first part of this section, a variational formulation for the spatial motion problem of electro-elastostatics is built from the basic equations of electrostatics. The material motion problem is then considered based on the formulations of the spatial one.

4.1. Basic equations in electrostatics

Under the assumption of nonlinear electro-elastostatics, in the absence of magnetic fields, free currents and electric charges, the electric field is governed by Faraday’s law:

\[ \nabla \times \mathbf{e} = 0 \]  

(25)

and by electric Gauss’ law:

\[ \nabla \cdot \mathbf{d} = 0 \]  

(26)

where \( \mathbf{e} \) and \( \mathbf{d} \) denote respectively the electric field vector and the electric displacement vector in spatial configuration.

At the boundary of the considered body or across a surface of discontinuity within the body, in the absence of surface charges, the electric field vector and the electric displacement vector must satisfy the jump conditions:

\[ \mathbf{n} \times \llbracket \mathbf{e} \rrbracket = 0 \quad \text{and} \quad \llbracket \mathbf{d} \rrbracket \cdot \mathbf{n} = 0 \]  

(27.1-2)

where at the boundary: \( \llbracket \mathbf{e} \rrbracket = \mathbf{e}_{\text{outside}} - \mathbf{e}_{\text{inside}} \).

Because the electric field vector \( \mathbf{e} \) is conservative, this vector can be expressed as the gradient of some scalar electric potential \( \psi \):

\[ \mathbf{e} = -\nabla \psi \]  

(28)

Besides, the electric displacement \( \mathbf{d} \) can be computed from the electric field vector \( \mathbf{e} \) by the relationship:

\[ \mathbf{d} = \varepsilon_0 \mathbf{e} + \mathbf{p} \]  

(29)

where \( \mathbf{p} \) is the electric polarization density and \( \varepsilon_0 \) is the vacuum electric permittivity. Note that in vacuo: \( \mathbf{d} = \varepsilon_0 \mathbf{e} \). The first term on the right-hand side of (29) is the contribution of free space and the second term is the contribution of condensed matter.

In material configuration, the Faraday’s law (25) and the electric Gauss’ law (26) can be written as:

\[ \nabla \times \mathbf{E} = 0 \]  

(30)

and:

\[ \nabla \cdot \mathbf{D} = 0 \]  

(31)

where \( \mathbf{E} \) and \( \mathbf{D} \) denote respectively the electric field vector and the electric displacement vector in material configuration, which are the pull-back versions of \( \mathbf{e} \) and \( \mathbf{d} \):

\[ \mathbf{E} = \mathbf{F}^{-1} \cdot \mathbf{e} \quad \text{and} \quad \mathbf{D} = J \mathbf{F}^{-1} \cdot \mathbf{d} \]  

(32.1-2)

At the boundary of the considered body or across a surface of discontinuity within the body, the jump conditions for the electric field vector \( \mathbf{E} \) and the electric displacement vector \( \mathbf{D} \) in the absence of surface charges are:

\[ \mathbf{N} \times \llbracket \mathbf{E} \rrbracket = 0 \quad \text{and} \quad \llbracket \mathbf{D} \rrbracket \cdot \mathbf{N} = 0 \]  

(33.1-2)

Similar to \( \mathbf{e} \), the electric field vector \( \mathbf{E} \) can be expressed as the gradient of some scalar electric potential \( \Psi \):

\[ \mathbf{E} = -\nabla \mathbf{X} \Psi \]  

(34)

where \( \Psi \) can be computed as the composition of \( \psi \) and \( \varphi \): \( \Psi = \psi \circ \varphi \)

In material configuration, the relationship (29) between the electric displacement and the electric polarization may be written as, Dorfmann and Ogden (2005):
\[
D = \varepsilon_0 J C^{-1} \cdot E + \mathbb{P}
\]

where \( \mathbb{P} = J F^{-1} \cdot \mathbb{p} \)

The electric field exerts on matter a body force \( \mathbf{b}_e \), which can be computed by, Pao (1978):

\[
\mathbf{b}_e^J = [\nabla \varepsilon] \cdot \mathbb{p}
\]

or by using (25), (26) and (29):

\[
\mathbf{b}_e^J = \nabla_x \cdot [\varepsilon \otimes \mathbb{d} - \frac{1}{2} \varepsilon_0 [\varepsilon \cdot \varepsilon] I]
\]

wherein \( \otimes \) denotes the dyadic product. In reference to the material configuration this body force can be written as:

\[
\mathbf{b}_e^0 = J \mathbf{b}_e^J = \nabla_x [F^{-t} \cdot E] \cdot \mathbb{P}
\]

or:

\[
\mathbf{b}_e^0 = \nabla_X \cdot \left[ F^{-t} \cdot E \otimes D - \frac{1}{2} \varepsilon_0 J [E \cdot C^{-1} \cdot E] F^{-t} \right]
\]

4.2. Spatial motion problem in nonlinear electro-elastostatics

Let us examine the case of conservative mechanical loading where spatial tractions are ignored and spatial body forces are conservative so that (7.2) and (10.2) are applied. Due to the existence of the body force \( \mathbf{b}_e \), the balance equation (9.1) has the form:

\[
\nabla_x \cdot \mathbf{\sigma} + \mathbf{b}_t + \mathbf{b}_e = 0
\]

or by using Eq. (37):

\[
\nabla_x \cdot \mathbf{\hat{\sigma}} + \mathbf{b}_t = 0
\]

where the total stress tensor \( \mathbf{\hat{\sigma}} \) is defined as:

\[
\mathbf{\hat{\sigma}} = \mathbf{\sigma} + \varepsilon \otimes \mathbb{d} - \frac{1}{2} \varepsilon_0 [\varepsilon \cdot \varepsilon] I
\]

which is actually the sum of Cauchy and Maxwell stresses.

In reference to the material configuration, the counterpart of the balance equation (41) is:

\[
\nabla_X \cdot \mathbf{\hat{P}} + \mathbf{b}_0 = 0
\]

where \( \mathbf{\hat{P}} \) is the counterpart of the spatial motion stress tensor \( \mathbf{P} \) in nonlinear elastostatics and is considered as the pull-back version of the total stress tensor \( \mathbf{\hat{\sigma}} \):

\[
\mathbf{\hat{P}} = J \mathbf{\hat{\sigma}} \cdot F^{-t}
\]

At the boundary of the considered body or across a surface of discontinuity within the body, the jump conditions for \( \mathbf{\hat{\sigma}} \) and \( \mathbf{\hat{P}} \) are:

\[
\left[ \mathbf{\hat{\sigma}} \right] \cdot n = 0 \quad \text{and} \quad \left[ \mathbf{\hat{P}} \right] \cdot N = 0
\]

Note that these jump conditions for \( \mathbf{\hat{\sigma}} \) and \( \mathbf{\hat{P}} \) take into account both electrical and mechanical contributions and therefore are preferred over the jump conditions for \( \mathbf{\sigma} \) and \( \mathbf{P} \). This is due to the fact that on the one hand the Cauchy stress difference \( \left[ \mathbf{\sigma} \right] \) across a surface must balance both the electrical and mechanical surface tractions. On the other hand, any traction measured by mechanical means is related to the total stress \( \mathbf{\hat{\sigma}} \), since there are no experiments that can separate the effects of the Cauchy and Maxwell stresses unambiguously. For more details, see for example Toupin (1956), Landau and Lifschitz (1960), Eringen (1963), McMeeking and Landis (2005).

For the sake of simplicity, let us assume that we only have the jump conditions (27), (33) and (45) at the boundary of the body under consideration. Furthermore, assume that these conditions are homogeneous such that: \( \mathbb{D} \cdot N = 0, \mathbb{d} \cdot n = 0, \mathbb{P} \cdot N = 0, \mathbf{\hat{\sigma}} \cdot n = 0, \mathbf{\hat{P}} \cdot N = 0 \). The spatial motion problem in nonlinear electro-elastostatics can in these conditions be set as the following system of equations:
\[ \nabla_X \cdot \mathbf{P} + b_0 = 0 \quad \text{and} \quad \nabla_X \cdot \mathbf{D} = 0 \quad \text{in} \: \mathcal{B}_0 \]
\[ \mathbf{P} \cdot N = 0 \quad \text{and} \quad \mathbf{D} \cdot N = 0 \quad \text{on} \: \partial \mathcal{B}_0 \] (46)
in reference to the material configuration, or:
\[ \nabla_X \cdot \mathbf{\hat{s}} + \mathbf{b}_i = 0 \quad \text{and} \quad \nabla_X \cdot \mathbf{d} = 0 \quad \text{in} \: \mathcal{B}_i \]
\[ \mathbf{\hat{s}} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{d} \cdot \mathbf{n} = 0 \quad \text{on} \: \partial \mathcal{B}_i \] (47)
in reference to the spatial configuration.

By assuming the existence of some energy densities that depend on the current state of deformation, on the electric field and on the material placement \( W_{0e}(\mathbf{F}; \mathbf{e}; X) \), \( W_{0e}(\mathbf{F}; \mathbf{E}; X) \), \( W_{0e}|_{F,E,X} = W_{0e}|_{F,E,X} \) such that the stress tensor \( \mathbf{\sigma} \) and the electric polarization density \( \mathbf{p} \) can be computed by:
\[ \mathbf{\sigma} = J^{-1} \mathbf{F} \cdot \mathbf{F}' = J^{-1}[\mathbf{\vartheta}_0 W_{0e}] \cdot \mathbf{F}' \quad \text{and} \quad \mathbf{p} = -J^{-1} \mathbf{\vartheta}_0 W_{0e} \] (48.1-2)
we will show that the system (46) is actually the stationary condition of the following functional:
\[ \int_{\mathcal{B}_0} \mathbf{\dot{U}}_{0F} \cdot \mathbf{dV} \] (49)
where \( \mathbf{\dot{U}}_{0F} = \mathbf{\dot{W}}_{0F} + \mathbf{V}_{0F} \), and:
\[ \dot{\mathbf{W}}_{0F} = \dot{\mathbf{W}}_{0F}(\mathbf{F}; \mathbf{E}; X) = \dot{W}_{0e}(\mathbf{F}; \mathbf{E}; X) - \frac{1}{2} \dot{\mathbf{\vartheta}}_0 J \mathbf{C}^{-1} : [\mathbf{E} \otimes \mathbf{E}] \] (50)
in which the double dot denotes the double contraction.

By using the relationships (29), (42) and (48) together with the connections: \( \partial_e W_{0e} = \mathbf{F} \cdot \partial_e W_{0e} \) and \( \partial_F W_{0e} = \partial_F W_{0e} + [\mathbf{F}^{-1} \cdot \mathbf{E}] \otimes \partial_e W_{0e} \), we have:
\[ \mathbf{d} = -J^{-1} \mathbf{F} \cdot \partial_e \mathbf{\dot{U}}_{0F}; \quad \mathbf{\hat{s}} = J^{-1}[\partial_F \mathbf{\dot{U}}_{0F}] \cdot \mathbf{F}' \] (51.1-2)
and:
\[ \mathbf{P} = -\partial_e W_{0e}; \quad \mathbf{D} = -\partial_F \mathbf{\dot{U}}_{0F}; \quad \mathbf{\dot{P}} = \partial_F \mathbf{\dot{U}}_{0F} \] (52.1-3)

The stationary condition for \( \dot{I}_0 \) is:
\[ \int_{\mathcal{B}_0} \partial_F \mathbf{\dot{U}}_{0F} : \delta \mathbf{F} \, dV + \int_{\mathcal{B}_0} \partial_e \mathbf{\dot{U}}_{0F} : \delta \mathbf{E} \, dV + \int_{\mathcal{B}_0} \partial_F \mathbf{\dot{U}}_{0F} : \delta \mathbf{\vartheta} \, dV = 0 \] (53)
at fixed material placement \( X \). By using the relationships (7.2) and (52) one gets:
\[ -\int_{\mathcal{B}_0} \partial_e \mathbf{\vartheta} \cdot [\nabla_X \cdot \mathbf{\dot{P}} + b_0] \, dV + \int_{\partial \mathcal{B}_0} \partial_e \mathbf{\vartheta} \cdot \mathbf{[\mathbf{P} \cdot N]} \, dS - \int_{\mathcal{B}_0} \partial_e \mathbf{\vartheta} \cdot [\nabla_X \cdot \mathbf{D}] \, dV + \int_{\partial \mathcal{B}_0} \partial_e \mathbf{\vartheta} \cdot [\mathbf{D} \cdot N] \, dS = 0 \] (54)
which gives directly the system (46).

It can also be shown that the system (47) is the stationary condition of the functional:
\[ \dot{I}_1 := \int_{\mathcal{B}_i} \mathbf{\dot{U}}_{IF} \cdot dV \] (55)
where the total potential energy density \( \mathbf{\dot{U}}_{IF} \) is defined by: \( \mathbf{\dot{U}}_{IF} = \mathbf{\dot{W}}_{IF} + \mathbf{V}_{IF} \), \( \mathbf{\dot{W}}_{IF} = \mathbf{\dot{W}}_{IF}(\mathbf{F}; \mathbf{E}; X) \), \( \mathbf{\dot{W}}_{IF}|_{F,E,X} = J^{-1} \dot{W}_{0F}|_{F,E,X} \). By noting that from the constitutive equation (51.2) the stress tensor \( \mathbf{\sigma} \) can be expressed as:
\[ \mathbf{\sigma} = \mathbf{\dot{U}}_{IF} \mathbf{I} + \partial_F \mathbf{\dot{U}}_{IF} \cdot \mathbf{F}' \] (56)
the stationary condition of \( \dot{I}_1 \), after some manipulation, has the form:
\[ -\int_{\mathcal{B}_i} \partial_e \mathbf{\vartheta} \cdot [\nabla_X \cdot \mathbf{\hat{s}} + \mathbf{b}_i] \, dV - \int_{\mathcal{B}_i} \partial_e \mathbf{\vartheta} \cdot [\nabla_X \cdot \mathbf{d}] \, dV + \int_{\partial \mathcal{B}_i} \partial_e \mathbf{\vartheta} \cdot [\mathbf{\sigma} \cdot \mathbf{n}] \, dS + \int_{\partial \mathcal{B}_i} \partial_e \mathbf{\vartheta} \cdot [\mathbf{d} \cdot \mathbf{n}] \, dS = 0 \] (57)
which is equivalent to the system (47).
4.3. Material motion problem in nonlinear electro-elastostatics

In order to derive the balance equations of the material motion problem in spatial configuration, we consider the total potential energy functional:

\[ \hat{I}_m := \int_{B_i} \hat{U}_{gf} \, dv \]  

where the total potential energy density \( \hat{U}_{gf} \) is defined by: \( \hat{U}_{gf} = \hat{W}_{gf} + V_{gf} \), \( \hat{W}_{gf} = \hat{W}_{gf}(f, \epsilon, \Phi) \), \( \hat{W}_{gf}|_{f,\epsilon,\Phi} = \hat{W}_{gf}|_{f,\epsilon,\Phi} \). Using similar argument as that used for (13) we have:

\[ \delta \hat{I}_m := \int_{B_i} \delta \hat{B}_i^d : \delta \Phi \, dv + \int_{\partial B_i} \delta \hat{T}_i^d : \delta \Phi \, ds \leq 0 \]  

where the material body forces \( \hat{B}_i^d \) and tractions \( \hat{T}_i^d \) are the counterparts of \( B_i^d \) and \( T_i^d \) in nonlinear elastostatics.

The definition (59) gives us:

\[ \int_{B_i} \left[ \hat{e}_f \hat{U}_{gf} : \delta f + \hat{e}_\Phi \hat{U}_{gf} : \delta \Phi \right] \, dv + \int_{B_i} \partial_e \hat{U}_{gf} \, \delta \epsilon \, dv - \int_{B_i} \hat{B}_i^d : \delta \Phi \, dv \leq 0 \]  

By using the connection \( \hat{e}_\Phi \hat{U}_{gf} = JF^{-1} \cdot \partial_e \hat{U}_{gf} \), from the Eq. (51.1) one gets:

\[ d = -\hat{e}_\Phi \hat{U}_{gf} \]  

and (60) leads to:

\[ \nabla_x \cdot \hat{p} + \hat{B}_i := -\hat{B}_i^d \quad \text{and} \quad \nabla_x \cdot d = 0 \quad \text{in} \ B_i \]  

\[ \hat{p} \cdot n := \hat{T}_i^d \quad \text{and} \quad d \cdot n = 0 \quad \text{on} \ \partial B_i \]  

where the following definitions of the material motion stress tensor \( \hat{p} \) and the material body force \( \hat{B}_i \) are used:

\[ \hat{p} := \hat{U}_{gf}; \quad \hat{B}_i := -\hat{e}_\Phi \hat{U}_{gf} \]  

From the definitions of the stress tensors \( \hat{\sigma} \) and \( \hat{p} \), we have the relationships:

\[ \hat{p} = \hat{U}_{gf} f^{-1} \cdot f^{-1} \cdot \hat{\sigma} + f^{-1} \cdot \epsilon \otimes d \]  

\[ \hat{\sigma} = \hat{U}_{gf} I - f^{-1} \cdot \hat{p} + e \otimes d \]  

which lead to the following formulation of the material tractions \( \hat{T}_i^d \) in the presence of the boundary conditions \( \hat{\sigma} \cdot n = 0 \) and \( d \cdot n = 0 \):

\[ \hat{T}_i^d = \hat{U}_{gf} f^{-1} \cdot n \]  

Using this formulation, it can be shown that for a defect free body \( (\hat{B}_i^d = 0) \) if \( \hat{p}, \hat{B}_i \) and \( \hat{T}_i^d \) are defined by (63.1), (63.2) and (65), respectively, then the two systems (47) and (62) (and therefore the two problems of material and spatial motions) are equivalent.

In material configuration, the material motion problem can be considered by the total potential energy functional:

\[ \hat{I}_m := \int_{B_0} \hat{U}_{gf} \, dV \]  

where the total potential energy density \( \hat{U}_{gf} \) is defined as \( \hat{U}_{gf} = W_{gf} + V_{gf} \), \( \hat{W}_{gf} = \hat{W}_{gf}(f, \epsilon, \Phi) \), \( \hat{W}_{gf}|_{f,\epsilon,\Phi} = \hat{W}_{gf}|_{f,\epsilon,\Phi} \). A variation of this functional at fixed spatial placement \( x \) leads to the introduction of the material body forces \( \hat{B}_0^d \) applied in \( B_0 \) and tractions \( \hat{T}_0^d \) applied on \( \partial B_0 \):

\[ \delta \hat{I}_m := \int_{B_0} \delta \hat{B}_0^d : \delta \Phi \, dV + \int_{\partial B_0} \delta \hat{T}_0^d : \delta \Phi \, dS \leq 0 \]  

After some manipulation, one gets:
By defining the material motion stress \( \hat{\Sigma} \) and the material body force \( \hat{B}_0 \) as:

\[
\hat{\Sigma} := \hat{U}_{of} I + \hat{\tau}_{of} \hat{f}' \quad \text{and} \quad \hat{B}_0 := -\hat{\tau}_{of} \hat{U}_{of}
\]

the condition (68) gives us:

\[
\nabla \times \hat{\Sigma} + \hat{B}_0 =: -\hat{B}_0' \quad \text{and} \quad \nabla \times \mathbb{D} = 0 \quad \text{in} \ B_0
\]

\[
\hat{\Sigma} \cdot N =: \hat{T}_0' \quad \text{and} \quad \mathbb{D} \cdot N = 0 \quad \text{on} \ \partial B_0
\]  

(70)

It is obvious that this system is the equivalent version of the system (62) written in reference to the material configuration, where \( \hat{\Sigma}, \hat{B}_0, \hat{B}_0' \) and \( \hat{T}_0' \) are related to their counterparts \( \hat{p}, \hat{B}_i, \hat{B}_i' \) and \( \hat{T}_i' \) by:

\[
\hat{\Sigma} = J \hat{p} \cdot \hat{f}'; \quad \hat{B}_0 = J \hat{B}_i; \quad \hat{B}_0' = J \hat{B}_i'; \quad \hat{T}_0' = J [N \cdot C^{-1} \cdot N]^{1/2} \hat{T}_i'
\]

(71.1-3)

Following the definitions of the stress tensors \( \hat{\Sigma} \) and \( \hat{p} \) one gets:

\[
\hat{\Sigma} = \hat{U}_{of} I - \hat{f}' \cdot \hat{p} + \mathbb{E} \otimes \mathbb{D}
\]

(72)

which has a similar format as (64.2) and leads to the following formulation of the material tractions \( \hat{T}_0' \) in the presence of the boundary conditions \( \hat{p} \cdot N = 0 \) and \( \mathbb{D} \cdot N = 0 \):

\[
\hat{T}_0' = \hat{U}_{of} N
\]

(73)

Again it can be proved that for a defect free body \( \hat{B}_0' = 0 \) if \( \hat{\Sigma}, \hat{B}_0 \) and \( \hat{T}_0' \) are defined by (69.1), (69.2) and (73), respectively, then the two systems (46) and (70), or correspondingly the two problems of material and spatial motions, are equivalent.

Note that by considering the formulations (42), (72) and with the help of (29), (35), (48.1) and (48.2) one has:

\[
\hat{\sigma} = J^{-1} \hat{\tau}_{of} W_{oe} \cdot \hat{F}' + \mathbb{E} \otimes \hat{p} + \varepsilon_0 [\mathbb{E} \otimes \mathbb{E} - \frac{1}{2}[\mathbb{E} \cdot \mathbb{E}] I]
\]

(74)

and:

\[
\hat{\Sigma} = [W_{oe} + V_{of}] I - \hat{f}' \cdot \hat{\tau}_{of} W_{oe} + \mathbb{E} \otimes \hat{p}
\]

(75)

The two formulations (74) and (75) reveal that in vacuum, while the spatial motion stress \( \hat{\sigma} \) survives in the form of the Maxwell stress, the material motion stress \( \hat{\Sigma} \) vanishes. Therefore, differently from spatial motion stresses, material motion stresses are defined only in the body under consideration.

5. Nonlinear magneto-elastostatics

Similar to the electro-elastostatic problem, in this section a variational formulation for the spatial motion problem of magneto-elastostatics is built from the basic equations of magnetostatics. The material motion problem is then considered based on the formulations of the spatial one.

5.1. Basic equations in magnetostatics

Under the assumption of nonlinear magneto-elastostatics, in the absence of electric current, the magnetic field is governed by Ampere's law:

\[
\nabla \times \mathbb{h} = 0
\]

(76)

and by Gauss's law for magnetism:
\[ \nabla_x \cdot \mathbf{b} = 0 \]  

(77)

where \( \mathbf{h} \) and \( \mathbf{b} \) denote respectively the magnetic field vector and the magnetic induction vector in the spatial configuration.

At the boundary of the considered body or across a surface of discontinuity within the body, in the absence of surface currents, the magnetic field vector and the magnetic induction vector must satisfy the jump conditions:

\[ \mathbf{n} \times [\mathbf{h}] = 0 \quad \text{and} \quad \mathbf{n} \cdot [\mathbf{b}] = 0 \]  

(78.1-2)

In order to solve the magnetostatic problem, a magnetic vector potential \( \mathbf{a} \) is defined such that:

\[ \mathbf{b} = \nabla_x \times \mathbf{a} \]  

(79)

which satisfies the requirement that the magnetic induction is divergence free.

The relationship between the magnetic induction \( \mathbf{b} \) and the magnetic field \( \mathbf{h} \) can be written as:

\[ \mathbf{b} = \mu_0 [\mathbf{h} + \mathbf{m}] \]  

(80)

where \( \mathbf{m} \) is the magnetization vector and \( \mu_0 \) is the vacuum magnetic permittivity. Note that in vacuo: \( \mathbf{b} = \mu_0 \mathbf{h} \).

The first term on the right-hand side of (80) is the contribution of free space and the second term is the contribution of condensed matter.

In material configuration, the Ampere’s law (76) and the Gauss’ law (77) have the form:

\[ \nabla_x \times \mathbf{H} = 0 \]  

(81)

and:

\[ \nabla_x \cdot \mathbf{B} = 0 \]  

(82)

where \( \mathbf{H} \) and \( \mathbf{B} \) denote respectively the magnetic field vector and the magnetic induction vector in material configuration, which are the pull-back versions of \( \mathbf{h} \) and \( \mathbf{b} \):

\[ \mathbf{H} = \mathbf{F}' \cdot \mathbf{h} \quad \text{and} \quad \mathbf{B} = \mathbf{J} \mathbf{F}^{-1} \cdot \mathbf{b} \]  

(83.1-2)

At the boundary of the considered body or across a surface of discontinuity within the body, in the absence of surface currents, the jump conditions for \( \mathbf{H} \) and \( \mathbf{B} \) are:

\[ \mathbf{N} \times [\mathbf{H}] = 0 \quad \text{and} \quad \mathbf{N} \cdot [\mathbf{B}] = 0 \]  

(84.1-2)

Similar to the magnetic vector potential \( \mathbf{a} \) in spatial configuration, we define a magnetic vector potential \( \mathbf{A} \) in material configuration such that:

\[ \mathbf{B} = \nabla_x \times \mathbf{A} \]  

(85)

where \( \mathbf{A} = \mathbf{F}' \cdot \mathbf{a} \).

In material configuration, the relationship (80) between the magnetic induction and the magnetic field is written as:

\[ \mathbf{J}^{-1} \mathbf{C} \mathbf{B} = \mu_0 [\mathbf{H} + \mathbf{M}] \]  

(86)

where \( \mathbf{M} \) is the magnetization vector in material configuration:

\[ \mathbf{M} = \mathbf{F}' \cdot \mathbf{m} \]  

(87)

The magnetic field exerts on matter a body force \( \mathbf{b}_m \), which can be computed as, Pao (1978):

\[ \mathbf{b}_m = [\nabla \mathbf{A}]' \cdot \mathbf{m} \]  

(88)

or by using (76), (77) and (80):

\[ \mathbf{b}_m = \nabla_x \cdot [\mu_0^{-1} [\mathbf{b} \otimes \mathbf{b} - \frac{1}{2} (\mathbf{b} \cdot \mathbf{b}) \mathbf{I}] + [\mathbf{m} \cdot \mathbf{b}] \mathbf{I} - \mathbf{m} \otimes \mathbf{b}] \]  

(89)

In reference to the material configuration, this body force can be computed as:

\[ \mathbf{b}_m^0 = \mathbf{J} \mathbf{b}_m = \mathbf{J} \mathbf{F}^{-1} \cdot [\nabla \mathbf{A} [\mathbf{J}^{-1} \mathbf{F} \cdot \mathbf{B}]]' \cdot \mathbf{F}^{-1} \cdot \mathbf{M} \]  

(90)
or equivalently:

\[ b''_0 = \nabla_X \cdot \left[ \mu_0^{-1} J^{-1} [F \cdot B \otimes B - \frac{1}{2} (B \cdot C \cdot B) F^{-1}] + [B \cdot M] F^{-1} - F^{-1} \cdot M \otimes B \right] \]  

(91)

5.2. Spatial motion problem in nonlinear magneto-elastostatics

By taking into account the body force \( b''_0 \), the balance equation (9.1) has the form:

\[ \nabla_X \cdot \sigma + b + b''_0 = 0 \]  

(92)

or by using formulation (89):

\[ \nabla_X \cdot \tilde{\sigma} + b = 0 \]  

(93)

where the total stress tensor \( \tilde{\sigma} \) is defined as:

\[ \tilde{\sigma} = \sigma + \left[ \mu_0^{-1} [b \otimes b - \frac{1}{2} (b \cdot b) I] + [m \cdot b] I - m \otimes b \right] \]  

(94)

In reference to the material configuration, the counterpart of (93) reads:

\[ \nabla_X \cdot \tilde{P} + b_0 = 0 \]  

(95)

where \( \tilde{P} \) is the counterpart of the spatial motion stress tensor \( P \) in nonlinear elastostatics. This stress tensor can be considered as the pull-back version of the total stress tensor \( \tilde{\sigma} \) in material configuration:

\[ \tilde{P} = J \tilde{\sigma} \cdot F^{-t} \]  

(96)

At the boundary of the considered body or across a surface of discontinuity within the body, the jump conditions for \( \tilde{\sigma} \) and \( \tilde{P} \) are:

\[ [\tilde{\sigma}] \cdot n = 0 \quad \text{and} \quad [\tilde{P}] \cdot N = 0 \]  

(97.1-2)

which take into account both magnetic and mechanical contributions.

Again for the sake of simplicity, let us assume that we only have the jump conditions (78), (84) and (97) at the boundary of the body under consideration such that: \( N \times H = 0, n \times h = 0, \tilde{P} \cdot N = 0, \tilde{\sigma} \cdot n = 0 \). In this case, the spatial motion problem in nonlinear magneto-elastostatics can be set as:

\[ \nabla_X \cdot \tilde{P} + b_0 = 0 \quad \text{and} \quad \nabla_X \times H = 0 \quad \text{in} \ B_0 \]

\[ \tilde{P} \cdot N = 0 \quad \text{and} \quad N \times H = 0 \quad \text{on} \ \partial B_0 \]  

(98)

in reference to the material configuration, or:

\[ \nabla_X \cdot \tilde{\sigma} + b = 0 \quad \text{and} \quad \nabla_X \times h = 0 \quad \text{in} \ B, \]

\[ \tilde{\sigma} \cdot n = 0 \quad \text{and} \quad n \times h = 0 \quad \text{on} \ \partial B \]  

(99)

in reference to the spatial configuration.

By assuming the existence of some energy densities that depend on the current state of deformation, on the magnetic induction and on the material placement \( W_{0b}(F, h; X), W_{0b}(F, B; X), W_{0b}|_{F,b,X} = W_{0b}|_{F,B,X} \), such that the stress tensor \( \sigma \) and the magnetization vector \( m \) can be computed by:

\[ \sigma = J^{-1} [\tilde{\sigma}] W_{0b} \cdot F \]  

\[ m = -J^{-1} \delta_{b} W_{0b} \]  

(100.1-2)

it can be shown that the system (98) is the stationary condition of the functional:

\[ \bar{I}_0 := \int_{B_0} \tilde{U}_{0f} dV \]  

(101)

where \( \tilde{U}_{0f} = \tilde{W}_{0f} + V_{0f}, \) and:

\[ \tilde{W}_{0f} = \tilde{W}_{0f}(F, B; X) = W_{0b}(F, B; X) + \frac{1}{2} \mu_0^{-1} J^{-1} C : [B \otimes B] \]  

(102)

Besides, one has:
\[ \mathbf{h} = F^{-1} \cdot \partial_{\mathbf{h}} \bar{U}_{0F}; \quad \bar{\sigma} = J^{-1} \left[ \partial_{\mathbf{F}} \bar{U}_{0F} \right] \cdot F^t \]  

(103.1-2)

and:
\[ \mathbf{M} = -\partial_{\mathbf{h}} W_{00}; \quad \mathbf{H} = \partial_{\mathbf{h}} \bar{U}_{0F}; \quad \bar{\mathbf{P}} = \partial_{\mathbf{F}} \bar{U}_{0F} \]  

(104.1-3)

Similarly, the system (99) is equivalent to the stationary condition of the functional:
\[ \bar{T}_i := \int_{B_i} \bar{U}_{iF} \, d\mathbf{v} \]  

(105)

where the total potential energy density \( \bar{U}_{iF} \) is defined as: \( \bar{U}_{iF} = \bar{W}_{iF} + V_{iF}, \quad \bar{W}_{iF} = \bar{W}_{iF}(\mathbf{F}, \mathbf{B}; X), \) \( \bar{W}_{iF}|_{F,B,X} = J^{-1} \bar{W}_{0F}|_{F,B,X}. \)

**Remark.** It is interesting to note that if instead of the magnetic induction vectors \( \mathbf{h} \) and \( \mathbf{B} \), we use the magnetic field vectors \( \mathbf{h} \) and \( \mathbf{H} \) as main variables, then the Legendre transformation (see Bustamante et al. (2006) for a similar formulation):
\[ \bar{U}_{0F}(\mathbf{F}, \mathbf{H}; X) = \bar{U}_{0F}(\mathbf{F}, \mathbf{B}; X) - \mathbf{h} \cdot \mathbf{B} \]  

(106)

gives us:
\[ \mathbf{B} = -\partial_{\mathbf{h}} \bar{U}_{0F}; \quad \bar{\mathbf{P}} = \partial_{\mathbf{F}} \bar{U}_{0F} \]  

(107.1-3)

and:
\[ \mathbf{h} = -J^{-1} F \cdot \partial_{\mathbf{h}} \bar{U}_{0F}; \quad \bar{\sigma} = J^{-1} \left[ \partial_{\mathbf{F}} \bar{U}_{0F} \right] \cdot F^t \]  

(108.1-2)

Furthermore, if we define some magnetic scalar potentials \( \lambda \) and \( \Lambda \) such that:
\[ \mathbf{h} = -\nabla X \lambda \quad \text{and} \quad \mathbf{H} = -\nabla X \Lambda \]  

(109.1-2)

then the magneto-elastostatic problem can be treated exactly the same way as the electro-elastostatic one. However, these scalar potentials can only be used in magnetostatics. In order to pave the way for the study of magneto-elastodynamics, we continue to use the magnetic induction vectors \( \mathbf{h} \) and \( \mathbf{B} \) as main variables together with the corresponding vector potentials \( \mathbf{a} \) and \( \mathbf{A} \).

### 5.3. Material motion problem in nonlinear magneto-elastostatics

Similar to nonlinear electro-elastostatics, by considering the total potential energy functional:
\[ \bar{T}_i := \int_{B_i} \bar{U}_{iF} \, d\mathbf{v} \]  

(110)

where \( \bar{U}_{iF} = \bar{W}_{iF} + V_{iF}, \) \( \bar{W}_{iF} = \bar{W}_{iF}(f, \mathbf{h}, \Phi), \) \( \bar{W}_{iF}|_{f,\Phi} = \bar{W}_{iF}|_{F,B,X}, \) one gets:
\[ \nabla_{\mathbf{x}} \cdot \bar{p} + \bar{B}_i = -\bar{B}_i^d \quad \text{and} \quad \mathbf{h} \times \mathbf{h} = 0 \quad \text{in} \ B_i \]
\[ \bar{p} \cdot \mathbf{n} = : \bar{T}_i^d \quad \text{and} \quad \mathbf{n} \times \mathbf{h} = 0 \quad \text{on} \ \partial B_i \]  

(111)

where \( \bar{B}_i^d \) and \( \bar{T}_i^d \) are the material body forces applied in \( B_i \) and the material tractions applied on \( \partial B_i \) and:
\[ \bar{p} := \partial_{\mathbf{F}} \bar{U}_{iF}; \quad \bar{T}_i := -\partial_{\Phi} \bar{U}_{iF} \]  

(112.1-2)

From the definitions of the stress tensors \( \bar{\sigma} \) and \( \bar{p}, \) one has the relationships:
\[ \bar{p} = \bar{U}_{iF} f^{-1} - f^{-1} \cdot \bar{\sigma} - [\mathbf{h} \cdot \mathbf{h}] f^{-1} + f^{-1} \cdot \mathbf{h} \otimes \mathbf{h} \]
\[ \bar{\sigma} = \bar{U}_{iF} I - f^{-1} \cdot \bar{p} - [\mathbf{h} \cdot \mathbf{h}] I + \mathbf{h} \otimes \mathbf{h} \]  

(113.1-2)

which lead to the following formulation of the material tractions \( \bar{T}_i^d \) in the presence of the boundary conditions \( \bar{\sigma} \cdot \mathbf{n} = 0 \) and \( \mathbf{n} \times \mathbf{h} = 0: \)
\[ \bar{T}_i^f = \bar{U}_i f^{-1} \cdot n \]  

(114)

With this formulation one can show that for a defect free body \((\bar{B}_i^f = 0)\) if \(\bar{p}, \bar{B}_r,\) and \(\bar{T}_i^f\) are defined by (112.1), (112.2) and (114), respectively, then the two systems (99) and (111) are equivalent.

In reference to the material configuration, the consideration of the total potential energy functional:

\[ T_0^m := \int_{B_0} \bar{U}_o f dV \]  

(115)

where \(\bar{U}_o f = \bar{W}_o f + V_o f, \bar{W}_o f = \bar{W}_o f(f, b, \Phi), \bar{W}_o f|_{F,B,X},\) leads to:

\[ \nabla_X \cdot \bar{\Sigma} + \bar{B}_0 = -\bar{B}_0^d \quad \text{and} \quad \nabla_X \times \bar{H} = 0 \quad \text{in} \ B_0 \]

(116)

where the material motion stress \(\bar{\Sigma}\) and the material body force \(\bar{B}_0\) are defined as:

\[ \bar{\Sigma} := \bar{U}_o f I + \bar{\partial}_f \bar{U}_o f \cdot f^t \quad \text{and} \quad \bar{B}_0 := -\bar{\partial}_b \bar{U}_o f \]  

(117.1-2)

Note that the two systems (116) and (111) are equivalent. The definitions of the stress tensors \(\bar{\Sigma}\) and \(\bar{P}\) give us the relationship:

\[ \bar{\Sigma} = \bar{U}_o f I - F^t \cdot \bar{P} - [B \cdot \bar{H}] I + \bar{H} \otimes B \]  

(118)

or by using the Legendre transformation (106):

\[ \bar{\Sigma} = \bar{U}_o f I - F^t \cdot \bar{P} + \bar{H} \otimes B \]  

(119)

The relationship (118) has a similar format as (113.2) and leads to the following formulation of the material tractions \(\bar{T}_0^f\) in the presence of the boundary conditions \(\bar{P} \cdot N = 0\) and \(N \times \bar{H} = 0\):

\[ \bar{T}_0^f = \bar{U}_o f N \]  

(120)

This formulation enables us to prove that for a defect free body \((\bar{B}_i^f = 0)\) if \(\bar{\Sigma}, \bar{B}_0,\) and \(\bar{T}_0^f\) are defined by (117.1), (117.2) and (120), respectively, then the two systems (98) and (116) are equivalent.

Finally, by considering the formulations (94), (118) and with the help of (86), (100.1) and (104.3) one has:

\[ \bar{\sigma} = J^{-1} \bar{\partial}_f W_{0b} \cdot F^t + [\mu_0^{-1} [\bar{b} \otimes \bar{b} - \frac{1}{2} [\bar{b} \cdot \bar{b}] I] + [\bar{b} \cdot \bar{m}] - [\bar{m} \otimes \bar{b}]] \]  

(121)

and:

\[ \bar{\Sigma} = [W_{0b} + V_{0f}] I - F^t \cdot \bar{\partial}_f W_{0b} + [B \cdot \bar{M}] I - \bar{M} \otimes B \]  

(122)

According to these formulations, in vacuum the spatial motion stress \(\bar{\sigma}\) survives whereas the material motion stress \(\bar{\Sigma}\) vanishes, which means that material motion stresses are defined only in the body.

6. Conclusion

The governing equations and variational formulations of spatial and material motion problems in nonlinear electro- and magneto-elastostatics are considered in this work. It is observed that the three problems of elastostatics, electro-elastostatics and magneto-elastostatics are very similar. For the spatial motion problem, the balance equations of linear momentum can be cast in the form:

\[ \nabla_X \cdot \sigma^s + b_i = 0 \quad \text{and} \quad \nabla_X \cdot \bar{P}^s + b_0 = 0 \]

where \(\{\bullet\}^s = \{\bullet, (\bullet), (\bullet)\}\). For the material motion problem, the form of the corresponding balance equations are:

\[ \nabla_X \cdot \bar{p}^s + \bar{B}_i^s = -\bar{B}_i^d \quad \text{and} \quad \nabla_X \cdot \bar{\Sigma}^s + \bar{B}_0^s = -\bar{B}_0^d \]

The stresses can be computed through some potential energy densities as:

\[ \bar{P}^s = J \sigma^s \cdot f^{-1} = \bar{\partial}_f U_{0f}^* \quad \text{and} \quad \bar{p}^s = J^{-1} \bar{\Sigma}^s \cdot f^{-1} = \bar{\partial}_f U_{0f}^* \]
Furthermore, the spatial motion stresses $\sigma^*$ and the material motion stresses $\Sigma^*$ (Eshelby’s stresses) have a similar energy-momentum type:

$$\sigma = U^* \mathbf{I} - f^* \cdot \mathbf{p}$$
$$\Sigma = U^0 \mathbf{I} - F^* \cdot \mathbf{P}$$

for nonlinear elastostatics,

$$\dot{\sigma} = \dot{U}^* \mathbf{I} - f^* \cdot \dot{\mathbf{p}} + e \otimes \mathbf{d}$$
$$\dot{\Sigma} = \dot{U}^0 \mathbf{I} - F^* \cdot \dot{\mathbf{P}} + \mathbf{E} \otimes \mathbf{D}$$

for nonlinear electro-elastostatics, and

$$\sigma = U^* \mathbf{I} - f^* \cdot \mathbf{p} - [\mathbf{b} \cdot \mathbf{h}] \mathbf{I} + h \otimes \mathbf{b}$$
$$\Sigma = U^0 \mathbf{I} - F^* \cdot \mathbf{P} - [\mathbf{B} \cdot \mathbf{H}] \mathbf{I} + \mathbf{H} \otimes \mathbf{B}$$

for nonlinear magneto-elastostatics. The three problems of elastostatics, electro-elastostatics and magneto-elastostatics can therefore be treated in a very similar way. With the introduction of the material tractions $T^d_0$, $T^d_t$ and body forces $B^d_0$, $B^d_t$ associated with changes in material configuration and material motions of defects relative to the ambient material, it is shown that the two problems of material and spatial motions are only equivalent for defect free bodies. Despite the fact that in this work we only consider conservative systems wherein external body forces can be derived from a potential, and for simplicity we take into account only homogeneous boundary conditions, the results can be extended to more general cases.

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**References**


