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THE UNTANGLING THEOREM

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We show a tangle (B, t_1, t_2) is trivial if and only if $\pi_1(B-(t_1\cup t_2))\cong \mathbb{Z}*\mathbb{Z}$, $\pi_1(B-t_1)\cong\mathbb{Z}$ and $\pi_1(B-t_2)\cong\mathbb{Z}$. We use mainly the methods of the combinatorial group theory.

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handlebody of genus 2 | canonical longitude
characteristic loop | tangle

Introduction

A tangle is a triple $(B; t_1, t_2)$ where B is a 3-ball and t_i's are mutually disjoint properly embedded PL-arcs in B with $t_i\cap\partial B=\partial t_i$, $i=1$, 2, and such a tangle is trivial if there is a homeomorphism from a triple $(B; t_1, t_2)$ onto a triple $(D\times I; x\times I, y\times I)$, where D is a disc, x and y are distinct points in the interior of D, and I is the interval.

The purpose of this paper is to show:

Untangling Theorem. *A tangle $(B; t_1, t_2)$ is trivial if and only if the following three conditions hold:*

(1) $\pi_1(B-(t_1\cup t_2))\cong\mathbb{Z}*\mathbb{Z}$,
(2) $\pi_1(B-t_1)\cong\mathbb{Z}$,
(3) $\pi_1(B-t_2)\cong\mathbb{Z}$.

Figures 1 and 2 are two examples who show that those conditions (1), (2) and (3) in the theorem above are mutally independent.

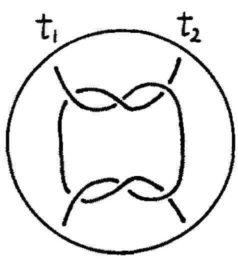

Fig. 1.

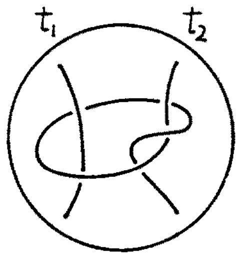

Fig. 2.

J. Simon announced this theorem in a more general situation (abstracts of Amer. Math. Soc. 7(5)(1986)310, ref. 828-57-67). He solved the "unknotting conjecture for planer graphs". Also Boileau and Costa have an alternative proof by using orbifolds and branched coverings. We have given the different proof by studying the handle body of genus 2.

1. Preliminaries

In this paper, we shall work in the PL category.

Let $(B; t_1, t_2)$ be a tangle. A second tangle $(B'; t_1', t_2')$ is *equivalent* to $(B; t_1, t_2)$, if there is a homeomorphism $h: B \to B'$ such that $h(t_1 \cup t_2) = t_1' \cup t_2'$. Recall that $(B; t_1, t_2)$ is trivial or untangled if it is equivalent to the trivial tangle $(D \times I; x \times I, y \times I)$. A *characteristic loop* of a tangle $(B; t_1, t_2)$ is a simple closed curve C on ∂V which is a meridian loop of either t_1 or t_2, where $V = B - \mathring{N}(t_1 \cup t_2)$ and N is the regular neighborhood in B. Thus there are only two distinct characteristic loops a, b of V up to isotopy (see Fig. 3). An unordered pair $\{a, b\}$ will be called the *characteristic loop system* of $(B; t_1, t_2)$.

We can easily observe that $(B; t_1, t_2)$ and $(B'; t_1', t_2')$ are equivalent if and only if there exists a homeomorphism from $B - \mathring{N}(t_1 \cup t_2)$ to $B' - \mathring{N}(t_1' \cup t_2')$ which preserves their characteristic loop systems.

Let M be an oriented 3-manifold, L a closed oriented 1-submanifold properly embedded in M, and S an oriented properly embedded 2-submanifold of M. We

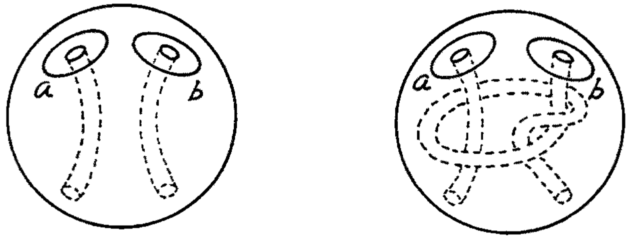

Fig. 3.

suppose L and S to be in general position in M. So $L \cap S$ are points. We denote the number of the components of $L \cap S$ by $|L \cap S|$, and the algebraic intersection number of L and S by $\langle [L], [S] \rangle$, where [ ] means the homology class in $H_i(M, \partial M)$.

Let H be a handlebody of genus 2. Let A and B be disjoint, properly embedded discs in H. We call the disc system $\{A, B\}$ a *canonical meridian disc system* of H if $H - \mathring{N}(A \cup B)$ is a 3-ball. Suppose that we are given an embedding of H in S^3. We call H *standard* if $S^3 - \mathring{H}$ is a handlebody of genus 2. Let H be the standard handlebody of genus 2. We call an ordered pair $\{a, b\}$ of simple closed curves on ∂H a *canonical longitude system* of H if there exists a canonical meridian disc system $\{A, B\}$ of H such that $|a \cap A| = |b \cap B| = 1$, $|a \cap B| = |b \cap A| = 0$, and there exists a canonical meridian disc system $\{A', B'\}$ of $S^3 - \mathring{H}$ such that $\partial A' = a$ and $\partial B' = b$. Note that a canonical longitude system represents a generating system of $\pi_1(H)$ and that a tangle $(B; t_1, t_2)$ is trivial if and only if $V = B - \mathring{N}(t_1 \cup t_2)$ is homeomorphic to the standard handlebody of genus 2 and the characteristic loop system of $(B; t_1, t_2)$ becomes a canonical longitude system of V.

Let H_0 be the 3-cell which is obtained by cutting H open along A and B, and A^+, A^- and B^+, B^- be the parallel copies of A and B in H_0. To draw the standard handlebody of genus 2, we shall use Fig. 4.

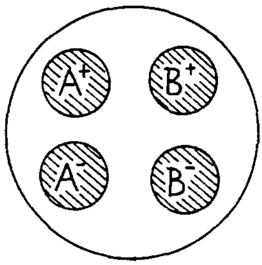

Fig. 4.

Lemma 1. *Let H be a standard handlebody of genus* 2, $\{A, B\}$ *a canonical meridian disc system of H, and l an arc on ∂H with $l \cap A = \partial l$ and $l \cap B = \emptyset$. Then there exists a homeomorphism $h: H \to H$ such that $h(l)$ is one of* 1)–5) *in Fig.* 5.

Proof. We may assume $\partial l \subset \partial A^+$ in H_0. Take a simple closed curve C on $\partial H_0 - \operatorname{Int}(A^+ \cup A^- \cup B^+ \cup B^-)$. Let R be a rectangle on ∂H_0 whose boundary consists of four arcs r_1, r_2, r_3, r_4 with $r_1 \cap r_3 = \emptyset$, $r_2 \cap r_4 = \emptyset$, $R \cap A^+ = r_1$, $R \cap A^- = R \cap B^+ = R \cap B^- = \emptyset$, $R \cap C = r_3$, and $\partial r_1 = \partial l$. Then $r_2 \cup (C - \mathring{r}_3) \cup r_4$ is an arc whose end points coincide those of l. We call such an arc a band sum from ∂l to C. The equivalence class of l by ambient isotopies of $\partial H_0 - \operatorname{Int}(A^+ \cup A^- \cup B^+ \cup B^-)$ is represented by a band sum from ∂l to C. At first we classify the simple closed curve C on ∂H_0 up to ambient isotopies of $\partial H_0 - \operatorname{Int}(A^+ \cup A^- \cup B^+ \cup B^-)$. Note that there is a 1-1-correspondence between such an ambient isotopy class and the free

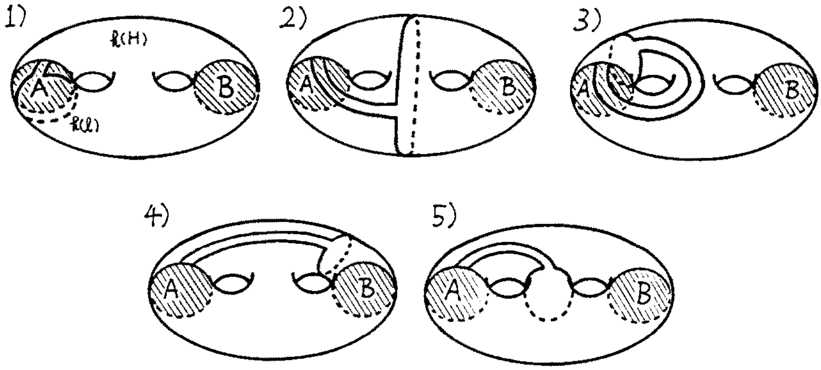

Fig. 5.

homotopy classes of simple closed curves on $\partial H_0 - \mathrm{Int}(A^+ \cup A^- \cup B^+ \cup B^-)$. We can see that there are seven representatives of such C as in Fig. 6 by observing primitive elements of π_1 (disc with 3 holes). Note that the positions of B^+ and B^- are interchanged in Fig. 6.2), 3), 6), 7).

There are five types of C up to ambient isotopies of $\partial H_0 - \mathrm{Int}(A^+ \cup A^- \cup B^+ \cup B^-)$, and there is a unique band from ∂l to C up to ambient isotopies of $\partial H_0 - \mathrm{Int}(A^+ \cup A^- \cup B^+ \cup B^-)$ as in Fig. 7.

Let I be the ambient isotopy. We can extend $I(1)$ to a homeomorphism of the 2-sphere and further to a homeomorphism of the 3-ball. Since the homeomorphism preserves A^+, A^-, B^+, and B^-, by pasting A^+ to A^-, and B^+ to B^-, we can get a standard handlebody of genus 2 again. Thus we can extend the homeomorphism of the 3-ball to a homeomorphism of H. We can take the homeomorphism as the desired h. □

Lemma 2. *Let H be the standard handlebody of genus 2, C a simple closed curve on ∂H, and $\{a, b\}$ a canonical longitude system of H. If C is freely homotopic in H to*

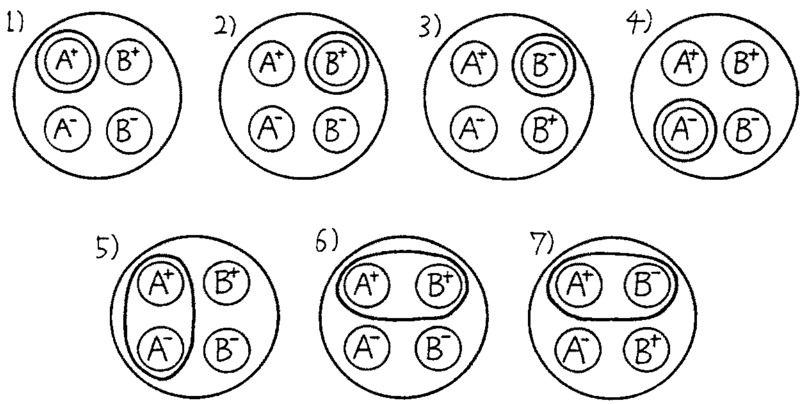

Fig. 6.

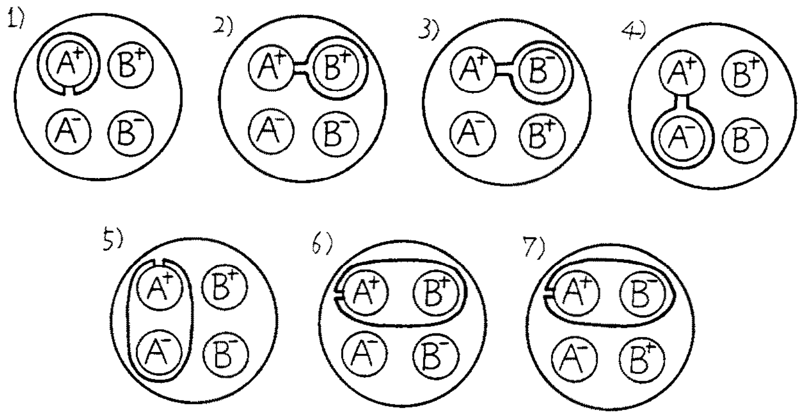

Fig. 7.

$a^{\pm 1}$, then there exist a homeomorphism $h: H \to H$ and a meridian disc system M_1, M_2 of $h(H)$, such that $|h(C) \cap M_1| = 1$ and $\langle [h(C)], [\partial M_2] \rangle = 0$.

Proof. Take a canonical meridian disc system A, B of H so that C intersects $A \cup B$ transversally, and $|a \cap A| = |b \cap B| = 1$ and $|a \cap B| = |b \cap A| = 0$. We read C as a word $W(a, b)$ in symbols $a^{\pm 1}$ and $b^{\pm 1}$ according to the order and directions in which C meets the discs A, B (cf. Fig. 8).

Let n be the number of occurrences of a in the word $W(a, b)$. We prove Lemma 2 by induction on n.

If $n = 1$, then we can take $M_1 = A$, $M_2 = B$ and $h = \mathrm{id}$. Suppose that $n > 1$. Since $W(a, b)$ can be reduced to a by trivial reductions, there is a trivial part aa^{-1} (or $a^{-1}a$) in $W(a, b)$ or there is a part axa^{-1} (or $a^{-1}xa$) in $W(a, b)$, where x is a word in b which is nonempty and reducible to the empty word by trivial reductions. At first, we consider the case that there is a trivial part aa^{-1} (or $a^{-1}a$). There must be a subarc l of C such that $\mathrm{Int}(l)$ intersects neither A nor B and $\partial l \subset \partial A$. By Lemma 1, we may assume that l is one of 1)-5) in Fig. 5. There are two subarcs l_1 and l_2 in ∂A such that $l_1 \cap l_2 = \partial l_1 = \partial l_2 = \partial l$. We characterize l_i, $i = 1, 2$, by the homology class $[l_i \cup l] \in H_1(\partial H)$ in 1)-5) of fig. 5, as in Table 1.

We can find discs D, D' properly embedded in H such that $\partial D = l_1 \cup l$ and $\partial D' = l_2 \cup l$ (cf. Fig. 9).

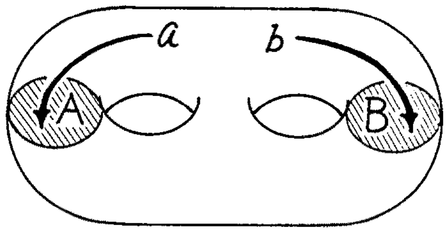

Fig. 8.

Table 1

	1)	2)	3)	4)	5)
$[l_1 \cup l]$	$[\partial A]$	0	$[\partial A]$	$[\partial B]$	$[\partial A]+[\partial B]$
$[l_2 \cup l]$	0	$[\partial A]$	0	$[\partial A]+[\partial B]$	$[\partial B]$

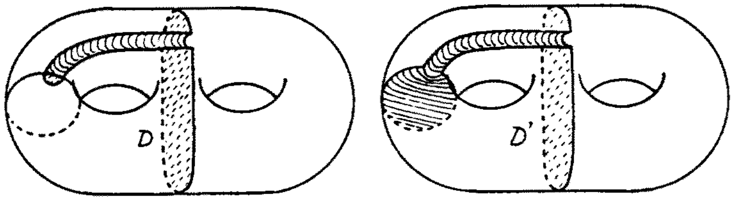

Fig. 9.

In any case, $|D \cap C|, |D' \cap C| < |A \cap C|$. By an easy homology argument, $\langle[\partial D'], [C]\rangle = \pm 1$ in 2) and 4) in Fig. 5, and $\langle[\partial D], [C]\rangle = \pm 1$ in 1), 3), and 5) in Fig. 5. We take D as the new A in 1), 3), and 5) in Fig. 5, D' in 2) and 4) in Fig. 5, B remains unaltered in all cases. For these new A, B, $\langle[C], [\partial A]\rangle = \pm 1$, $\langle[C], [\partial B]\rangle = 0$ and $|C \cap A| < n$. So the lemma follows by induction. Next, we consider the case that there are no trivial parts aa^{-1} (or $a^{-1}a$). In this case, there is a part axa^{-1} (or $a^{-1}xa$) in $W(a, b)$, where x is a word in b which is nonempty and reducible to the empty word by trivial reductions. So there must be a subarc l of C corresponding to x and a subarc l' of ∂A, such that $l \cup l'$ bounds a disc properly embedded in H. Let H_1 be the solid torus which is obtained by cutting H open along A. Since l does not intersect ∂A, we may assume that $l \cup l'$ is a simple closed curve on ∂H_1 which is contractible in H_1. Since $\mathrm{Ker}(\pi_1(\partial H_1) \to \pi_1(H_1)) = \langle[\partial B]\rangle$, we can isotope $l(\mathrm{rel}\, \partial l)$ on ∂H so that l does not intersect B. So we can reduce to the case where x is empty. □

Remark. We can find a homeomorphism $f: H \to H$ such that $f(C)$ is a canonical longitude of $f(H)$ by cutting $h(H)$ open along M_1 and sliding M_1^+ along $h(C)$ and pasting M_1^+ to M_1^- (see Fig. 10).

We can prove the following statement in almost the same way as lemma 2. In this case, since b does not intersect l, we may assume l is one of 1), 2) or 3) in Fig. 11.

Lemma 3. *Let H be the standard handlebody of genus 2, C a simple closed curve on ∂H, and $\{a, b\}$ a canonical longitude system of H. If C is freely homotopic in H to $a^{\pm 1}b^r$ for some integer r and is disjoint from b, then there exist a homeomorphism $h: H \to H$ and a meridian disc system M_1, M_2 of $h(H)$ such that*

$$|h(C) \cap M_1| = 1, \qquad |h(b) \cap M_1| = 0, \qquad |h(b) \cap M_2| = 1.$$

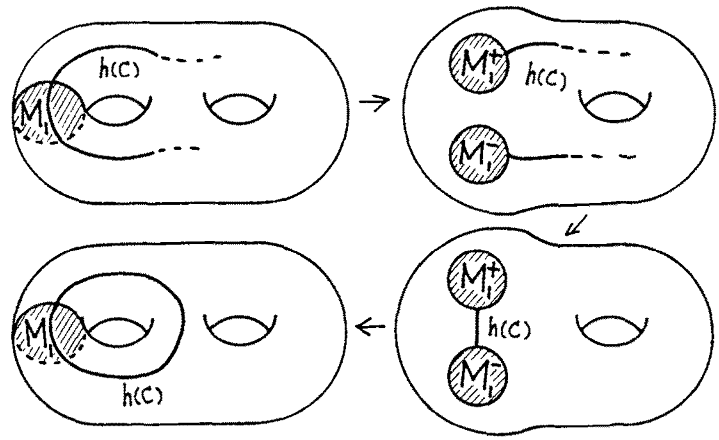

Fig. 10.

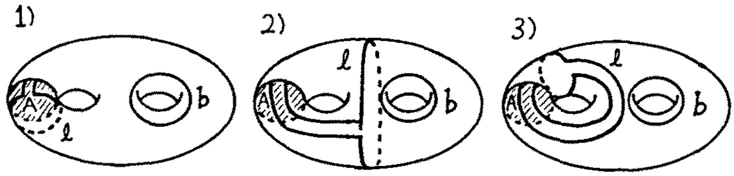

Fig. 11.

Lemma 3 follows as in the proof of Lemma 2, except that as C is disjoint from b, subcases 4) and 5) of Fig. 5 do not occur when considering aa^{-1} (see Fig. 11).

We can find a homeomorphism $f: H \to H$ such that $\{f(C), f(b)\}$ is a canonical longitude system of $f(H)$ as in the remark following Lemma 2 (see Fig. 10).

2. The proof of the main theorem

Untangling Theorem. *A tangle* $(B; t_1, t_2)$ *is trivial if and only if* $\pi_1(B-(t_1 \cup t_2)) \cong \mathbb{Z} * \mathbb{Z}$, $\pi_1(B-t_1) \cong \mathbb{Z}$, *and* $\pi_1(B-t_2) \cong \mathbb{Z}$.

Proof. The "only if" part is trivial. We show the "if" part. First we show that $B - \mathring{N}(t_1 \cup t_2)$ is homeomorphic to the handlebody of genus 2. By the hypothesis that $\pi_1(B-(t_1 \cup t_2)) \cong \mathbb{Z} * \mathbb{Z}$ and since $B - \mathring{N}(t_1 \cup t_2)$ is irreducible, we can show that there exists a disc which separates B into two 3-balls B_1 and B_2 such that each of those contains an arc t_1 and t_2, respectively, and hence that $\pi_1(B_i - t_i) \cong \mathbb{Z}$, $i=1$, 2. The fact that $\pi_1(B_i - t_i) \cong \mathbb{Z}$, $i=1$, 2 implies that t_i is a trivial arc in B_i. So $B_i - \mathring{N}(t_i)$ is homeomorphic to the solid torus. Since $B - \mathring{N}(t_i \cup t_2)$ is a boundary connected sum of $B_1 - \mathring{N}(t_1)$ and $B_2 - \mathring{N}(t_2)$, $B - \mathring{N}(t_1 \cup t_2)$ is a handlebody of genus 2.

Let V be the standard handlebody of genus 2, h a homeomorphism from $B - \mathring{N}(t_1 \cup t_2)$ to V, and m_1, m_2 a characteristic loop system of $B - \mathring{N}(t_1 \cup t_2)$ of $(B; t_1, t_2)$.

We have only to show that there exists a homeomorphism h_1 of V such that $\{h_1(h(m_1)),\ h_1(h(m_2))\}$ is a canonical longitude system of V. Since

$$\pi_1(B-t_2)\cong\pi_1(B-(t_1\cup t_2))/\langle[m_1]\rangle\cong\pi_1(V)/\langle[h(m_1)]\rangle,$$

$$\pi_1(B-t_2)\cong\mathbb{Z},$$

we have

$$\mathbb{Z}*\mathbb{Z}/\langle[h(m_1)]\rangle\cong\mathbb{Z}.$$

Thus, by [5, Proposition 5.10 of Chapter II], $[h(m_1)]$ is an element of a generating system of $\pi_1(V)\cong\mathbb{Z}*\mathbb{Z}$. So $h(m_1)$ is freely homotopic in V to a canonical longitude of V. Thus, by Zieschang [9] and Lemma 2, there exists a homeomorphism h_2 of V such that $h_2(h(m_1))$ is a canonical longitude of $h_2(V)$.

Let a, b be a generating system of $\pi_1(h_2(V))$ which is represented by a canonical longitude system of $h_2(V)$. We may assume that $a=[h_2(h(m_1))]$. Since

$$\pi_1(B-t_1)\cong\pi_1(B-(t_1\cup t_2))/\langle[m_2]\rangle$$

$$\cong\pi_1(h_2(V))/\langle[h_2(h(m_2))]\rangle,$$

$$\pi_1(B-t_1)\cong\mathbb{Z},$$

we have

$$\mathbb{Z}*\mathbb{Z}/\langle h_2(h(m_2))]\rangle\cong\mathbb{Z}.$$

Thus $[h_2(h(m_2))]$ is an element of a generating system of $\pi_1(h_2(V))\cong\mathbb{Z}*\mathbb{Z}$. Thus, by Kaneto [3] or Cohen, Mitzler and Zimmermann [1], $[h_2(h(m_2))]$ is written up to cyclic permutations and inversions, in the following two forms:

(1) $ab^{\varepsilon_1}\ldots ab^{\varepsilon_k}$, or

(2) $a^{\varepsilon_1}b\ldots a^{\varepsilon_k}b$,

where $\varepsilon_i\in\mathbb{Z}$ and $|\varepsilon_i-\varepsilon_j|\leq 1$, $1\leq i, j\leq k$.

Since $\pi_1(B-(t_1\cup t_2))/\langle[m_1]\rangle\cong\pi_1(B-t_2)\cong\langle[m_2]\rangle$, $[h_2(h(m_2))]$ must become $b^{\pm1}$ when we put $[h_2(h(m_1))]=a=1$. Thus, $[h_2(h(m_2))]$ is $a^rb^{\pm1}$ for some $r\in\mathbb{Z}$. Thus, by Lemma 3, there exists a homeomorphism h_3 of $h_2(V)$ such that $h_3(h_2(h(m_1)))$, $h_3(h_2(h(m_2)))$ is the canonical longitude system of $h_3(h_2(V))$. We may take $h_1=h_3\circ h_2$. □

An application. Situate two arcs A, B in $\mathbb{R}^3_+$ as in Fig. 12.

Zeeman [8] shows that $G=\pi_1(\mathbb{R}^3_+-(A\cup B))$ is not free by using a lower central series. We give here a proof using our main theorem. The pair $(\mathbb{R}^3_+, A, B)$ is homeomorphic to a tangle (B, t_1, t_2) as in Fig. 1. Since t_1 and t_2 are trivial arcs and $B-\mathring{N}(t_1\cup t_2)$ is irreducible, if G is free, then $\pi_1(B-(t_1\cup t_2))\cong\mathbb{Z}*\mathbb{Z}$. Then the tangle (B, t_1, t_2) is the untangle by the Untangling Theorem. But this tangle is not the untangle because any sum of two untangles is a 2-bridge knot or link. The untangle can be added to this tangle to produce the square knot and the bridge index of the square knot is 3. So G is not free.

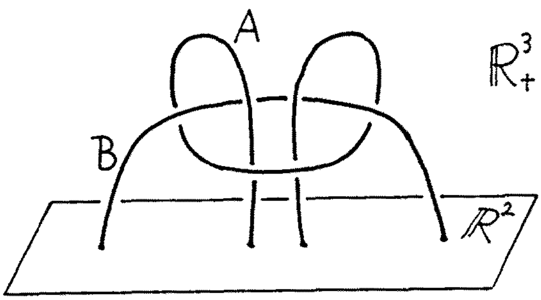

Fig. 12.

Finally we remark that the statement of the theorem is the best possible. As mentioned above, an example of a nontrivial tangle which satisfies $\pi_1(B-t_i)\cong\mathbb{Z}$, $i=1, 2$ and $\pi_1(B-(t_1\cup t_2))\cong\mathbb{Z}*\mathbb{Z}$ is shown in Fig. 1. An example of a nontrivial tangle which satisfies $\pi_1(B-(t_1\cup t_2))\cong\mathbb{Z}*\mathbb{Z}$ and $\pi_1(B-t_1)\cong\mathbb{Z}$ is in Fig. 2. As to the tangle of Fig. 2, note that $B-\mathring{N}(t_1\cup t_2)$ is homeomorphic to the standard handlebody (cf. Rolfsen [7]).

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