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Rank-width and tree-width of H-minor-free graphs

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ABSTRACT

We prove that for any fixed $r \geq 2$, the tree-width of graphs not containing K_r as a topological minor (resp. as a subgraph) is bounded by a linear (resp. polynomial) function of their rank-width. We also present refinements of our bounds for other graph classes such as K_r-minor free graphs and graphs of bounded genus.

1. Introduction

Tree-width and rank-width are width parameters of graphs, which are, roughly speaking, measures of their decomposability. These parameters play very important roles in Structural and Algorithmic Graph Theory. For example, if we restrict the input to graphs of bounded tree-width or rank-width, then many problems that are NP-hard in general can be solved in polynomial time.

It is natural to ask about the relations between various width parameters of graphs. Let us write $\mathbf{tw}(G)$ and $\mathbf{rw}(G)$ for tree-width and rank-width of a graph G, respectively. As was shown by Oum [22], for any graph G

$$\mathbf{rw}(G) \leq \mathbf{tw}(G) + 1. \tag{1}$$

On the other hand, there is no function f such that $\mathbf{tw}(G) \leq f(\mathbf{rw}(G))$. For instance, the complete graph K_n on n vertices has tree-width $n-1$ and rank-width 1. However, the situation changes when we impose some conditions on the structure of a graph G. Courcelle and Olariu [4] proved that such functions f exist under various conditions. Actually, their paper is about the clique-width of graphs, which has been defined earlier than the rank-width. In fact, the rank-width was defined by Oum and Seymour [23] so that graphs have bounded rank-width if and only if they have bounded clique-width. More precisely, they proved that

$$\mathbf{rw}(G) \leq \mathbf{cw}(G) \leq 2^{\mathbf{rw}(G)+1} - 1, \tag{2}$$

where $\mathbf{cw}(G)$ denotes the clique-width of a graph G.

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In particular, Courcelle and Olariu [4, Theorem 5.9] have shown that for every positive integer r, there exists a function f_r such that if a graph G has no subgraph isomorphic to the complete bipartite graph $K_{r,r}$ on $2r$ vertices, then $\mathbf{tw}(G) \leq f_r(\mathbf{cw}(G))$. The proof by Courcelle and Olariu is highly non-constructive. Later, Gurski and Wanke [11] proved that if a graph G has no subgraph isomorphic to $K_{r,r}$, then

$$\mathbf{tw}(G) + 1 \leq 3(r-1)\mathbf{cw}(G). \tag{3}$$

By combining (3) with (2), we can directly deduce that for every graph G having no $K_{r,r}$ as a subgraph,

$$\mathbf{tw}(G) + 1 \leq 3(r-1)(2^{\mathbf{rw}(G)+1} - 1). \tag{4}$$

In this paper, we show that the exponential bound (4) can be improved to a *polynomial* bound for graphs not containing $K_{r,r}$ as a subgraph and to a *linear* bound for graphs not containing K_r as a minor or a topological minor. We will apply our proof techniques to various classes of graphs while still obtaining linear bounds.

Let us summarize our theorems as follows. The results are ordered with respect to the generality of the corresponding class. In what follows G is a graph with at least one edge. We refer to Section 2 for the definitions of graph classes.

- Theorem 12: If G is planar, then
$$\mathbf{tw}(G) < 72\mathbf{rw}(G) - 1.$$
- Theorem 12: If the Euler genus of G is at most g, then
$$\mathbf{tw}(G) < 3\left(2 + \sqrt{2g}\right)(6\mathbf{rw}(G) + 5g) - 1.$$
- Theorem 10: If G contains no K_r as a minor, $r > 2$, then
$$\mathbf{tw}(G) = 2^{O(r \log\log r)}\mathbf{rw}(G).$$
- Theorem 16: If G contains no K_r as a topological minor for $r > 2$, then
$$\mathbf{tw}(G) = 2^{O(r \log r)}\mathbf{rw}(G).$$
- Theorem 18: If $\nabla_1(G) \leq r$, then
$$\mathbf{tw}(G) < 12 \cdot r \cdot 4^r\mathbf{rw}(G) - 1.$$
Here, ∇_1 is the greatest reduced average density with rank 1.
- Theorem 21: If G has no subgraph isomorphic to $K_{r,r}$ for $r \geq 2$, then
$$\mathbf{tw}(G) < 3(r-1)\left(\frac{2(r-2)}{r+1}\binom{\mathbf{rw}(G)}{r} + 2\sum_{i=0}^{r}\binom{\mathbf{rw}(G)}{i}\right) - 1.$$

2. Definitions

In this paper all graphs are simple undirected graphs without loops and parallel edges. For a vertex $v \in V(G)$ of graph G, we denote by $N_G(v)$ the set of vertices in G that are adjacent to v and we write $\mathbf{deg}_G(v) = |N_G(v)|$ to denote the *degree* of a vertex v in G. The *union* $G \cup H$ of two graphs G and H is the graph such that $V(G \cup H) = V(G) \cup V(H)$, and $E(G \cup H) = E(G) \cup E(H)$. Two distinct vertices x, y of G are *twins* if there are no vertices in $V(G) \setminus \{x, y\}$ that are adjacent to exactly one of x and y, or equivalently $N_G(x) \setminus \{x, y\} = N_G(y) \setminus \{x, y\}$. A *clique* of a graph is a set of pairwise adjacent vertices. Note that the empty set is a clique.

Subgraphs, minors, topological minors and star-minors. Let G be a graph on the vertex set $V(G)$ and with the edge set $E(G)$. For $v \in V(G)$ and $e \in E(G)$, we denote by $G - v$ the graph obtained from G by removal of v and all edges incident with v and by $G \setminus e$ the graph obtained by removal of e from G. For $\mathbf{deg}_G(v) = 2$, we call by the *dissolution of v in G* the graph obtained from G by adding an edge connecting the neighbors $N_G(v)$ of v (if there is no such an edge in G) and then by removing v. The result of the *contraction* of $e = \{x, y\}$ from G is the graph G/e obtained from $G - x - y$ by adding a new vertex $v_{x,y}$ and making it adjacent to all vertices of $(N_G(x) \cup N_G(y)) \setminus \{x, y\}$.

For graphs G and H, we say that H is an *induced subgraph of* G, and denote it by $H \subseteq_{is} G$, if H can be obtained from G after a sequence of vertex removals. Also, for $S \subseteq V(G)$, we call H the subgraph of G *induced by* S, and write $H = G[S]$, if the vertex set required to be removed from G in order to transform G to H is $V(G) \setminus S$.

We say that H is a *subgraph* of G, if H can be obtained from G after applying a sequence of vertex and edge removals. We say that H is a *topological minor* of G, if H can be obtained from G by applying a sequence of vertex/edge removals and dissolutions. Finally, we say that H is a *minor* of G if H can be obtained from G after applying a sequence of vertex removals or edge removals/contractions.

The *greatest reduced average density with rank p* of a graph G is

$$\nabla_p(G) = \max \frac{|E(H)|}{|V(H)|},$$

where maximum is taken over all the minors H of G obtained by contracting a set of vertex-disjoint subgraphs with radius at most p and then deleting any number of vertices and edges [18–20]. In this work, we consider only graphs with $p = 1$. We say that a graph H is a *star-minor* of G if H is obtained from a subgraph of G by contracting edges of vertex-disjoint subgraphs of radius 1 (or equivalently, vertex-disjoint stars). Thus $\nabla_1(G)$ is the maximum density among all star-minors of G. We also say that a graph G is *d-degenerate* if each of its subgraphs (including G itself) has a vertex of degree at most d. It is easy to observe that every graph G is $2 \cdot \nabla_p(G)$-degenerate for every $p \geq 0$.

Hypergraphs. A *hypergraph* H is a pair $(V(H), E(H))$ of a finite set $V(H)$, called the vertex set, and a set $E(H)$ of subsets of $V(H)$, called the hyperedge set. The *incidence graph* of a hypergraph H is the bipartite graph $I(H)$ on the vertex set $V(H) \cup E(H)$ such that $v \in V(H)$ is adjacent to $e \in E(H)$ in $I(H)$ if and only if v is incident with e in H (in other words, $v \in e$).

Bipartite graphs. For a graph G and a subset $X \subseteq V(G)$, we use notation $\overline{X}$ for $V(G) \setminus X$. For a bipartite graph G with bipartition X and $\overline{X}$, its *bipartite adjacency matrix* is an $|X| \times |\overline{X}|$ matrix

$$\boldsymbol{B}_G = (b_{i,j})_{i \in X, j \in \overline{X}},$$

over the binary field GF(2) such that $b_{i,j} = 1$ if and only if $\{i, j\} \in E(G)$.

For a nonempty subset X of the vertex set of G, we define the subgraph $G\langle X\rangle$ with vertex set $V(G)$ and edge set

$$\{\{x, x'\} \in E(G) \mid x \in X, x' \in \overline{X}\}.$$

Hence $G\langle X\rangle$ is the bipartite subgraph of G that contains only the edges with endpoints in X and $\overline{X}$.

Rank-width. For a graph G and $X \subseteq V(G)$, the *cut-rank* function is defined to be

$$\rho_G(X) = \mathbf{rank}(\boldsymbol{B}_{G\langle X\rangle}).$$

If $X = \emptyset$ or $X = V(G)$, then $\rho_G(X) = 0$. Let us note that $\boldsymbol{B}_{G\langle X\rangle}$ is a matrix over the binary field when we consider rank function of this matrix.

A tree is *ternary* if all its vertices are of degree 1 or 3. We denote by $L(T)$ the set of leaves of a tree T. A *rank-decomposition* of a graph G is a pair (T, μ) consisting of a ternary tree T and a bijection $\mu : V(G) \to L(T)$. Each edge e of T defines a partition (X_e, Y_e) of $L(T)$. The *width* of an edge e of T is $\rho_G(\mu^{-1}(X_e))$. The *width* of a rank-decomposition (T, μ) is the maximum width of all edges of T. The *rank-width* of a graph G, denoted by $\mathbf{rw}(G)$, is the minimum width of all rank-decompositions of G. If $|V(G)| \leq 1$, then G admits no rank-decompositions from the above definition. If that is the case, we define the rank-width of G to be 0.

Tree-width. A *tree-decomposition* of a graph G is a pair (T, X), where T is a tree, and $X = (\{X_v \mid v \in V(T)\})$ is a collection of subsets of $V(G)$ such that

(T1) For each edge e of G, the endpoints of e are contained in X_v for some $v \in V(T)$.
(T2) If $a, b, c \in V(T)$ and the path from a to c in T contains b, then $X_a \cap X_c \subseteq X_b$.
(T3) $\cup_{v \in V(T)} X_v = V(G)$.

The *width* of a tree-decomposition $(T, (X_v)_{v\in V(T)})$ is $\max_{v\in V(T)} |X_v| - 1$. The *tree-width* of a graph is the minimum width of all tree-decompositions of the graph.

Clique-width. For a positive integer k, a *k-graph* is a pair (G, lab) of a graph G and a labeling function

$$\mathrm{lab} : V(G) \to \{1, 2, \dots, k\}.$$

If $\mathrm{lab}(v) = i$, then we call i the *label* of v. From now on, we define *k-expressions*, which are algebraic expressions with the following four operations to describe how to construct k-graphs.

- For $i \in \{1, 2, \dots, k\}$, $\cdot_i$ is a k-graph consisting of a single vertex of label i.
- For distinct $i, j \in \{1, 2, \dots, k\}$, $\rho_{i\to j}(G, \mathrm{lab}) = (G, \mathrm{lab}')$ in which $\mathrm{lab}'(v) = \mathrm{lab}(v)$ if $\mathrm{lab}(v) \neq i$ and $\mathrm{lab}'(v) = j$ if $\mathrm{lab}(v) = i$.
- For distinct $i, j \in \{1, 2, \dots, k\}$, $\eta_{i,j}(G, \mathrm{lab}) = (G', \mathrm{lab})$ in which $V(G') = V(G)$ and $E(G') = E(G) \cup \{vw : \mathrm{lab}(v) = i, \mathrm{lab}(w) = j\}$.
- $\oplus$ is the disjoint union of two k-graphs. In other words, $(G_1, \mathrm{lab}_1) \oplus (G_2, \mathrm{lab}_2) = (G, \mathrm{lab})$ in which G is the disjoint union of G_1 and G_2, and $\mathrm{lab}(v) = \mathrm{lab}_1(v)$ if $v \in V(G_1)$ and $\mathrm{lab}(v) = \mathrm{lab}_2(v)$ if $v \in V(G_2)$.

The *clique-width* of a graph G is the minimum k such that there exists a k-expression with value (G, lab) for some labeling function lab.

3. Rank-width and clique-width

For a graph G and a set $X \subseteq V(G)$, we define

$$\lambda_G(X) = |\{N_{G\langle X\rangle}(v) \mid v \in \overline{X}\}|,$$

which is the number of distinct neighborhoods of the vertices in $\overline{X}$ in the graph $G\langle X\rangle$. By the definition of $G\langle X\rangle$, each such a neighborhood is a subset of X. For a positive integer k, we also define

$$\lambda_G(k) = \max\{\lambda_G(X) \mid X \subseteq V(G), |X| \le k\}.$$

Clearly, in general, $\lambda_G(k) \le 2^k$. As we will see in the following sections, better bounds can be obtained when G belongs to certain graph classes.

Lemma 1. *Let G be a graph and let X be a subset of $V(G)$ such that $\rho_G(X) \le k$. Then the bipartite adjacency matrix of $G\langle X\rangle$ has at most $\lambda_G(k)$ distinct rows.*

Proof. Let M be the bipartite adjacency matrix of $G\langle X\rangle$. We may assume that M has exactly $\rho_G(X)$ columns, because there exist $\rho_G(X)$ columns whose linear combination spans all other column vectors. □

The following lemma is implicit in [23]. For a set X of vertices of a graph G, let $c_G(X)$ be the number of distinct *nonzero* rows in the bipartite adjacency matrix of $G\langle X\rangle$. For a rank-decomposition (T, μ) of G, we define $\beta_G(T, \mu) = \max\{\max\{c_G(X_e), c_G(Y_e)\} \mid e \in E(T)\}$.

Lemma 2. *Let (T, μ) be a rank-decomposition of a graph G. Then the clique-width of G is at most $2\beta_G(T, \mu) + 1$.*

Proof. We set $C = \beta_G(T, \mu)$. We may assume that $|V(G)| \ge 3$. We may assume that T has a vertex of degree 2 by subdividing an edge. We turn T into a rooted directed tree by choosing an internal vertex r of degree 2 as a root and by directing all edges from the root.

For a vertex v in T, let $D_v = \{x \in V(G) : \mu(x)$ is a descendant of v in $T\}$. Let G_v be the subgraph of G induced on D_v.

We claim that for each vertex v of T, there is a $(2C + 1)$-expression t_v with value (G_v, lab_v) for some map $\mathrm{lab}_v : V(G_v) \to \{1, 2, \dots, C, 2C + 1\}$ satisfying the following two conditions:

- If $\mathrm{lab}_v(x) = \mathrm{lab}_v(y)$, then every vertex in $V(G) \setminus D_v$ is either adjacent to both x and y, or nonadjacent to both x and y.
- If x in D_v has no neighbor in $V(G) \setminus D_v$, then $\mathrm{lab}_v(x) = 2C + 1$.

We proceed by induction on the number of descendants of v of T. If v is a leaf, then we let $t_v = \cdot_{2C+1}$. Now let us assume that v has two children v_1 and v_2. By the induction hypothesis, we have $(2C+1)$-expressions t_{v_1} and t_{v_2} with values $(G_{v_1}, \mathrm{lab}_{v_1})$, $(G_{v_2}, \mathrm{lab}_{v_2})$, respectively. We glue t_{v_1} and t_{v_2} to obtain a $(2C+1)$-expression t_v for G_v. Let F be the set of pairs (i, j) such that there exist a vertex $x \in D_{v_1}$ and a vertex $y \in D_{v_2}$ such that $\mathrm{lab}_{v_1}(x) = i$, $\mathrm{lab}_{v_2}(y) = j$, and x is adjacent to y in G. Let N be the set of integers $i \in \{1, 2, \ldots, 2C\}$ such that there exists a vertex v of label i in D_{v_1} or a vertex v of label $(i - C)$ in D_{v_2} such that v has no neighbors in $V(G) \setminus D_v$. Then let

$$t^* = (\underset{i\in N}{\circ} \rho_{i\to 2C+1})((\underset{(i,j)\in F}{\circ} \eta_{i,j+C})(t_{v_1} \oplus \rho_{1\to C+1}(\rho_{2\to C+2}(\cdots(\rho_{C\to 2C}(t_{v_2}))\cdots)))).$$

Then t^* is a $(2C+1)$-expression with value (G_v, lab^*) say. So far, lab^* satisfies the condition that if two vertices in D_v have the same lab^* value, then they have the identical set of neighbors out of D_v.

Since $c_G(D_v) \le C$, there are at most C distinct nonempty subsets of $V(G) \setminus D_v$ that are sets of the neighbors of a vertex of D_v. We obtain a $(2C+1)$-expression t' from t^* by applying $\rho_{i\to j}$ to merge two labels i, j whenever vertices of i and j have the same nonempty set of neighbors in $V(G) \setminus D_v$. Let (G_v, lab') be the value of t'. Then lab' has at most $C+1$ distinct values.

Let t_v be a $(2C+1)$-expression obtain from t' by applying $\rho_{i\to j}$ operations whenever $2C \ge i > C \ge j$ and there are no vertices of label j. Then t_v is what we wanted. This proves the induction claim.

Now t_r is a $(2C+1)$-expression of G and therefore the clique-width of G is at most $2C+1$. □

Lemma 3. *Let G be a graph with at least one edge. Then*

$$\mathbf{rw}(G) \le \mathbf{cw}(G) \le 2\lambda_G(\mathbf{rw}(G)) - 1.$$

Proof. The first inequality $\mathbf{rw}(G) \le \mathbf{cw}(G)$ was shown by Oum and Seymour [23].

Let $\mathbf{rw}(G) \le k$ and let (T, τ) be a rank-decomposition of G of width at most k. Since $|E(G)| > 0$, we have that $k > 0$. For every $e \in E(T)$, the rank of the bipartite adjacency matrix M_e of $G\langle\tau^{-1}(X_e)\rangle$ is at most k. If $\mathbf{rank}(M_e) = 0$, then $c_G(X_e) = 0$. Now let us assume that $\mathbf{rank}(M_e) > 0$. Let $M'_e = \binom{M_e}{0}$ be the matrix obtained by adding a zero row to M_e. By Lemma 1, M'_e has at most $\lambda_G(k)$ distinct rows. Then M_e has at most $\lambda_G(k) - 1 \ge 0$ nonzero distinct rows. In any case, we deduce that $c_G(X_e) \le \lambda_G(k) - 1$ and thus $\beta_G(T, \tau) \le \lambda_G(k) - 1$. By Lemma 2, we deduce that $\mathbf{cw}(G) \le 2 \cdot \lambda_G(\mathbf{rw}(G)) - 1$. □

Lemma 3 along with the fact that $\lambda_G(k) \le 2^k$ yields the exponential upper bound in (2). In general, such a bound is unavoidable because Corneil and Rotics [3] showed that, for each k, there is a graph G_k such that $\mathbf{cw}(G_k) \ge 2^{\lfloor k/2\rfloor - 1}$ and $\mathbf{tw}(G_k) = k$, which implies $\mathbf{rw}(G_k) \le k+1$ by (1). In the following sections we refine the bound in (2) for certain graph classes. Our main tool is to derive better estimations of the function λ_G.

4. Graphs with no complete graph minor

Our goal in this section is to prove that, for a fixed $r > 2$, the tree-width, the rank-width and the clique-width of a graph with no K_r-minor are within a constant factor, where the constant only depends on r. We also aim to make this section as a reference to be used later for other graph classes.

Let us consider the following problems for a fixed positive integer r.

P1: Does there exist a constant c_1 such that, for all $n > 0$, every n-vertex graph has at most $c_1 n$ edges if it has no K_r-minor?

P2: Does there exist a constant c_2 such that, for all $n > 0$, every n-vertex graph has at most $c_2 n$ cliques if it has no K_r-minor?

P3: Does there exist a constant c_3 such that, for all $n > 0$, every n-vertex hypergraph has at most $c_3 n$ hyperedges if its incidence graph has no K_r-minor?

P4: Does there exist a constant c_4 such that, for all $n > 0$, every binary matrix of rank n has at most $c_4 n$ distinct rows if the bipartite graph having the matrix as a bipartite adjacency matrix has no K_r-minor?

P5: Does there exist a constant c_5 such that, for all $n > 0$, the tree-width of every graph of rank-width n is at most $c_5 n$ if the graph has no K_r-minor?

Note that these problems are trivial if $r \le 2$ and therefore we will assume that $r > 2$.

The problem P1 was answered by Kostochka [15,16] and Thomason [28] independently. Later Thomason determined the exact constant as follows.

Proposition 4 (*P1; Thomason [29]*)**.** *There is a constant α such that every n-vertex graph with no K_r-minor has at most $(\alpha r\sqrt{\log r})n$ edges. Moreover, this result is tight up to the value of $\alpha = 0.319\cdots + o(1)$.*

This proposition implies that $c_1 = \alpha r\sqrt{\log r}$ satisfies c_1. Now we will explain that any upper bound of c_i will give upper bounds for c_{i+1}. Moreover, our proof technique can be applied to classes of graphs more general than graphs with no K_r-minor which we will discuss later.

To answer P2, we claim that every n-vertex graph with no K_r-minor will have at most $2^{O(r\sqrt{\log r})}n$ cliques. To see this, we use a simple induction argument by counting cliques containing a vertex v of the minimum degree to show that every n-vertex graph with no K_r-minor has at most $2^{2c_1}n$ cliques if $c_1 \geq 1/2$. More precisely, one can prove that if an n-vertex graph is d-degenerate and $n \geq d$, then it has at most $2^d(n-d+1)$ cliques; see [30]. We now aim to show that the above bound on the number of cliques can be improved to $2^{O(r\log\log r)}n$.

Lemma 5. *There is a constant α such that, for $r \geq 2$, every n-vertex graph with no K_r-minor has at most $\frac{1}{r+1}\binom{r+1}{k}(2\alpha\sqrt{\log r})^{k-1}n$ cliques of size k for $1 \leq k \leq r-1$.*

Proof. Let G be an n-vertex graph with no K_r-minor. We take α from Proposition 4. We apply induction on r. If $r = 2$ or $k = 1$, then it is trivial. So we may assume that $r > 2$ and $k > 1$. For a vertex v, the subgraph induced on the neighbors of v contains at most $\frac{1}{r}\binom{r}{k-1}(2\alpha\sqrt{\log(r-1)})^{k-2}\mathbf{deg}(v)$ cliques of size $k-1$ because it has no K_{r-1}-minor. Since each clique of size k is counted k times, G has at most $\frac{1}{k}\sum_{v\in V(G)}\frac{1}{r}\binom{r}{k-1}(2\alpha\sqrt{\log(r-1)})^{k-2}\mathbf{deg}(v)$ cliques of size k. The conclusion follows because $\sum_{v\in V(G)}\mathbf{deg}(v) \leq (2\alpha r\sqrt{\log r})n$ by Proposition 4 and $\binom{r+1}{k} = \frac{r+1}{k}\binom{r}{k-1}$. □

Proposition 6 (*P2*)**.** *There is a constant μ such that, for $r > 2$, every n-vertex graph with no K_r-minor has at most $n2^{\mu r\log\log r}$ cliques.*

Proof. Let α be the constant in Proposition 4. We may assume that $\alpha \geq 0.5$ by taking a larger value if necessary. (It is likely that α is bigger than 0.5 if we want it to be satisfied by all graphs, not just large graphs.) Since $\log r \geq 1$, we have that the number of cliques of size i is at most $n\frac{1}{r+1}\binom{r+1}{i}(2\alpha)^{i-1}(\sqrt{\log r})^{r-1} \leq n\binom{r}{i-1}(2\alpha)^{i-1}(\sqrt{\log r})^r$ when $1 \leq i \leq r-1$ by Lemma 5. Let C be the number of cliques in the graph. Then we obtain the following.

$$\begin{aligned}
C &\leq 1 + n\left(\sqrt{\log r}\right)^r\sum_{i=1}^{r-1}\binom{r}{i-1}(2\alpha)^{i-1} \\
&\leq n\left(\sqrt{\log r}\right)^r\sum_{i=0}^{r}\binom{r}{i}(2\alpha)^i \quad \text{because } (2\alpha)^r \geq 1 \\
&= n\left(\sqrt{\log r}\right)^r(1+2\alpha)^r.
\end{aligned}$$

Let $c = \frac{\log(1+2\alpha)}{\log\log 3}$. Then $\log(1+2\alpha) \leq c\log\log r$ and so $C \leq n2^{(\frac{1}{2}+c)\frac{1}{\log 2}r\log\log r}$. □

Proposition 7 (*P3*)**.** *Let c_2 be a constant satisfying* P2. *Then every n-vertex hypergraph has at most c_2n hyperedges if its incidence graph has no K_r-minor; therefore $c_3 = c_2$ satisfies* P3.

Proof. Let H be a hypergraph with n vertices whose incidence graph $I(H)$ has no K_r-minor. We may assume that every subset of a hyperedge e of H is a hyperedge of H, because otherwise we may replace e by its proper subset. Let G be a graph on $V(H)$ obtained from H by deleting all hyperedges of arity other than 2. It is easy to observe that G is a minor of $I(H)$ (actually, G is a topological minor or a star-minor of $I(H)$) and therefore G has no K_r-minor. Moreover for each hyperedge e of H, G has a corresponding clique on the same vertex set. Thus, the number of hyperedges of H is at most c_2n. □

Proposition 8 (*P4*)**.** *Let c_3 be a constant satisfying* P3. *Then every binary matrix of rank n has at most c_3n distinct rows if the bipartite graph having the matrix as a bipartite adjacency matrix has no K_r-minor; this implies that $c_4 = c_3$ satisfies* P4.

Proof. Let M be a binary matrix of rank n. Let G be the bipartite graph having M as a bipartite adjacency matrix. We claim that M has at most c_3n distinct rows. We may assume that M has n columns by deleting linearly dependent columns. We may also assume that M has no identical rows. Then let H be a hypergraph such that its incidence graph is G and the vertices of H correspond to vertices of G representing the columns of M. (Note that in this paper, a hypergraph has no parallel hyperedges.) Since G has no K_r-minor, H has at most c_3n hyperedges and therefore M has at most c_3n rows. □

Proposition 9 (*P5*)**.** *Let c_4 be a constant satisfying* P4. *If G is a graph with no K_r-minor, then $\mathbf{cw}(G) \le 2c_4\mathbf{rw}(G) - 1$ and $\mathbf{tw}(G) + 1 \le 3(r-2)(2c_4\mathbf{rw}(G) - 1)$.*

Therefore $c_5 = 6(r-2)c_4$ satisfies P5.

Proof. Let G be a graph of rank-width at most n with no K_r-minor. We will only need the following two facts:

- G has no $K_{r-1,r-1}$ as a subgraph.
- Every bipartite subgraph of G has no K_r-minor.

First we claim that the clique-width is at most $2c_4n - 1$. By Lemma 3, it is enough to prove that $\lambda_G(n) \le c_4n$. Let X be a subset of at most n vertices of G. (Here, n is the rank-width of G.) Let M be the bipartite adjacency matrix of G whose rows and columns are indexed by $V(G) \setminus X$ and X, respectively. Then obviously $\mathbf{rank}(M) \le n$. Moreover the bipartite graph having M as a bipartite adjacency matrix has no K_r-minor. Thus M has at most c_4n distinct rows and therefore $\lambda_G(n) \le c_4n$. This proves the claim.

Since G has no K_r-minor, G does not contain $K_{r-1,r-1}$ as a subgraph. By (3), the tree-width of G is at most $3(r-2)(2c_4n-1) - 1$. □

Let us summarize what we have for graphs with no K_r-minor.

Theorem 10. *There is a constant μ such that for each integer $r > 2$, if G is a graph with no K_r-minor, then*

$$\mathbf{cw}(G) < 2 \cdot 2^{\mu r \log\log r}\mathbf{rw}(G),$$
$$\mathbf{tw}(G) + 1 < 6(r-2)2^{\mu r \log\log r}\mathbf{rw}(G).$$

5. Graphs of bounded genus

If Σ is a surface which can be obtained from the sphere by adding k crosscaps and h handles, then *Euler genus* $\varepsilon(\Sigma)$ of the surface Σ is $k + 2h$. We refer to the book of [17] for more details on graph embeddings. *Euler genus* $\varepsilon(G)$ of a graph G is the minimum r such that the graph can be embedded into a surface of Euler genus r.

A hypergraph is *planar* if its incidence graph is planar; see [32]. Also a hypergraph is *embeddable* on a surface of Euler genus r if so is its incidence graph. For formal definitions of hypergraph embeddings on surfaces (called "paintings") see [26]. *Euler genus* $\varepsilon(H)$ of a hypergraph H is the minimum r such that H can be embedded into a surface of Euler genus r.

For graphs of Euler genus at most r, Euler's formula allows us to answer P1 easily; every n-vertex graph of Euler genus r has at most $3n - 6 + 3r$ edges if $n \ge 3$. We may obtain easy answers to P2 and P3 by using the fact that such graphs have vertices of small degree. However, that approach will give us the following: the number of hyperedges of a hypergraph of Euler genus at most r is at most $64n + f(r)$ for some function f. In the next lemma, we improve 64 to 6 for P3.

We remark that Wood [30] showed that an n-vertex planar graph has at most $8(n-2)$ cliques if $n > 2$; This answers P2 for planar graphs. However, for P3, we can improve $8(n-2)$ to $6n - 9$ by the following proposition. As a generalization of [30], Dujmović et al. [6] showed that an n-vertex graph embedded on a surface has at most $8n + \frac{3}{2}2^{\omega} + o(2^{\omega})$ cliques, where ω is the maximum integer such that the complete graph K_{ω} can be embedded on the same surface. Notice that their bound can also be used to answer P3 but our bound for P3 improves their $8n + O(1)$ to $6n + O(1)$ for a fixed surface.

Proposition 11 (*P3*)**.** *Let H be an n-vertex hypergraph embeddable on a surface of Euler genus r where $r > 0$ or $n > 2$. Then H has at most $(6n - 9 + 5r)$ hyperedges.*

Proof. If $n = 1$, then $E(H) \leq 2^1 \leq 6 - 9 + 5r$ because $r \geq 1$. If $n = 2$, then $r \geq 1$ and $E(H) \leq 2^2 \leq 12 - 9 + 5r$. Now we may assume that $n \geq 3$.

We assume that the incidence graph of H is embedded on a surface Σ of the Euler genus $r = \varepsilon(H)$. There is at most 1 hyperedge of arity 0 because $\emptyset$ is the only possible hyperedge of arity 0. It is also trivial that there are at most n hyperedges of arity 1.

We now count hyperedges of arity at least 2. We define sub-hypergraphs H_2 and $H_{\geq 3}$ of H such that

$$V(H_2) = V(H) \quad \text{and} \quad E(H_2) = \{e \in E(H) \mid |e| = 2\},$$

and

$$V(H_{\geq 3}) = V(H) \quad \text{and} \quad E(H_{\geq 3}) = \{e \in E(H) \mid |e| \geq 3\}.$$

In other words, H_2 contains the hyperedges of H of arity 2 and $H_{\geq 3}$ contains the hyperedges of arity greater than 2. Clearly, both H_2 an $H_{\geq 3}$ are hypergraphs embeddable on Σ. Because H_2 has no parallel edges or loops, by Euler's formula, we have $|E(H_2)| \leq 3n - 6 + 3r$. To bound the number of hyperedges in $H_{\geq 3}$, we construct a graph F as follows. For each hyperedge $e = \{v_1, v_2, \ldots, v_l\}$ whose endpoints are cyclically ordered as $v_1, v_2, \ldots, v_l, v_1$ in the surface, we remove e and add edges $\{v_1, v_2\}, \ldots, \{v_{l-1}, v_l\}, \{v_l, v_1\}$. We will not create parallel edges or loops. Then each hyperedge of $H_{\geq 3}$ corresponds to a face of the embedding of F in Σ and no two hyperedges are mapped to the same face in F. The graph F has n vertices, and, again by Euler's formula, we derive that $|E(H_{\geq 3})| \leq 2n - 4 + 2r$. So we conclude that $|E(H)| \leq 1 + n + (3n - 6 + 3r) + (2n - 4 + 2r) = 6n - 9 + 5r$. □

Proposition 11 is tight; Given any plane triangulation, we attach a hyperedge of arity 3 for each triangle. Then we obtain $6n - 9$ hyperedges in the planar hypergraph.

To answer P4 for graphs of Euler genus at most r, it is fairly straightforward to apply the same argument of Proposition 8 to deduce that every binary matrix of rank n has at most $(6n - 9 + 5r)$ distinct rows if the matrix induces a bipartite graph whose Euler genus is at most r.

Finally let us consider the problem P5 for graphs of Euler genus at most r. To mimic the argument of Proposition 9, we need to determine the largest complete bipartite graphs with Euler genus at most r. Ringel [24,25] showed that if $m, n \geq 2$, then the orientable genus of $K_{m,n}$ is $\lceil (m-2)(n-2)/4 \rceil$ and if $m, n \geq 3$, then the nonorientable genus of $K_{m,n}$ is $\lceil (m-2)(n-2)/2 \rceil$. It follows that if $t > 2 + \sqrt{2r}$, then $K_{t,t}$ is not embeddable on a surface of Euler genus r.

Theorem 12. *Let G be a graph embeddable on a surface of Euler genus r. Then*

$$\mathbf{cw}(G) < 12\mathbf{rw}(G) + 10r,$$
$$\mathbf{tw}(G) + 1 < 3(2 + \sqrt{2r})(6\mathbf{rw}(G) + 5r).$$

Proof. Let t be a minimum integer such that $t > 2 + \sqrt{2r}$. Then $K_{t,t}$ is not embeddable on the surface of Euler genus r and therefore G has no $K_{t,t}$ subgraph. By Gurski and Wanke's inequality (3), we have

$$\mathbf{tw}(G) + 1 \leq 3(t - 1)\mathbf{cw}(G).$$

From Proposition 11, we have $\lambda_G(n) \leq 6n - 9 + 5r$ unless $r = 0$ and $n \leq 2$. We use a relaxed inequality $\lambda_G(n) < 6n + 5r$, true for all $r \geq 0$ and $n \geq 1$. Then $\mathbf{cw}(G) < 12\mathbf{rw}(G) + 10r$ and $\mathbf{tw}(G) + 1 < 3(2 + \sqrt{2r})(6\mathbf{rw}(G) + 5r)$. □

6. Graphs excluding topological minors

We now relax our problems to graphs with no K_r topological minor. As we did in Section 4, we begin by answering P1; how many edges can a graph have if it has no K_r topological minor?

Proposition 13 (*P1; Bollobás and Thomason [1]; Komlós and Szemerédi [14]*)**.** *There is a constant β such that for every r, every graph of average degree at least βr^2 contains K_r as a topological minor. Subsequently every n-vertex graph with more than $\frac{\beta}{2}r^2 n$ edges contains K_r as a topological minor.*

Thomas and Wollan's Theorem [27] can be used to obtain that $\beta = 10$ satisfies the above proposition; see [5, Theorem 7.2.1] with the corrected proof in the web site of Diestel.[1]

If we use the fact that every graph with no K_r topological minor has a vertex of degree at most βr^2, we can easily show that every n-vertex graph with no K_r topological minor can have at most $2^{\beta r^2} n$ cliques. We aim to improve $2^{O(r^2)} n$ to $2^{O(r \log r)} n$ as we did in Proposition 6.

Lemma 14. *Let $r \geq 2$. There is a constant β such that every n-vertex graph with no K_r topological minor has at most $\frac{1}{r+1}\binom{r+1}{k}(\beta r)^{k-1} n$ cliques of size k for $1 \leq k \leq r-1$.*

Proof. We take the same β of Proposition 13. We proceed by induction on r. Let G be an n-vertex graph with no K_r topological minor. We may assume that $k \geq 2$ and $r \geq 3$. For each vertex v, there are at most $\frac{1}{r}\binom{r}{k-1}(\beta(r-1))^{k-1}\mathbf{deg}(v)$ cliques of size k containing v. Since each clique of size k is counted k times, there are at most $\frac{1}{kr}\binom{r}{k-1}(\beta(r-1))^{k-1}(2|E(G)|)$ cliques of size k. By Proposition 13, $2|E(G)| \leq \beta r^2 n$. The conclusion follows because $\frac{1}{k}\binom{r}{k-1} = \frac{1}{r+1}\binom{r+1}{k}$. $\square$

Proposition 15 (*P2*)**.** *There is a constant τ such that, for $r > 2$, every n-vertex graph with no K_r topological minor has at most $2^{\tau r \log r} n$ cliques.*

Proof. Let G be an n-vertex graph with no K_r topological minor. Let β be the constant in Proposition 13. Since planar graphs have no K_5 topological minor, $\frac{25}{2}\beta \geq 3$ and so $\beta \geq \frac{6}{25}$. We may assume that $n \geq 3$ by choosing τ so that $2^{3\tau \log 3} \geq 2$. By Lemma 14, G has at most $C = 1 + \frac{1}{r+1}\sum_{k=1}^{r-1}\binom{r+1}{k}(\beta r)^{k-1} n$ cliques.

$$
\begin{aligned}
C &\leq \frac{4}{3(r+1)}\sum_{k=1}^{r-1}\binom{r+1}{k}(\beta r)^{k-1} n \quad \text{because } 1 + \frac{1}{n} \leq \frac{4}{3},\\
&\leq \frac{1}{3}\frac{1+\beta r}{\beta r}(1+\beta r)^r n \quad \text{because } r+1 \geq 4,\\
&\leq \frac{43}{54}\left(\left(\beta + \frac{1}{3}\right) r\right)^r n \quad \text{because } \beta r \geq \frac{18}{25} \text{ and } r \geq 3.
\end{aligned}
$$

Therefore if we let $\tau = \max(\frac{1}{3\log 3}, \frac{1}{\log 2} + \frac{\log(\beta+\frac{1}{3})}{\log 2 \log 3})$, then $2^{\tau r \log r} n \geq ((\beta + \frac{1}{3})r)^r n \geq C$. $\square$

When $\beta = 10$, $\max(\frac{1}{3\log 3}, \frac{1}{\log 2} + \frac{\log(\beta+\frac{1}{3})}{\log 2 \log 3}) < 4.51$ and therefore $\tau = 4.51$ satisfies Proposition 15.

We can deduce the following theorem from Proposition 15 by using almost identical proofs of Propositions 7–9.

1 http://diestel-graph-theory.com/corrections/3rd.edn.corrections.pdf.

Theorem 16. *There is a constant τ such that for every integer $r > 2$, if G is a graph with no K_r topological minor, then*

$$\mathbf{cw}(G) < 2 \cdot 2^{\tau r \log r}\mathbf{rw}(G),$$
$$\mathbf{tw}(G) + 1 < \frac{3}{4}(r^2 + 4r - 5)2^{\tau r \log r}\mathbf{rw}(G).$$

Proof. Let $t = \lceil r/2 \rceil + \binom{\lceil r/2 \rceil}{2}$. It is obvious that $K_{t,t}$ has a topological minor isomorphic to K_r. So if G is a graph with no K_r topological minor, then G has no $K_{t,t}$ subgraph and therefore

$$\mathbf{tw}(G) \leq 3(t-1)\mathbf{cw}(G)$$

by (3). From Proposition 15, we can deduce that there is a constant τ such that $\mathbf{cw}(G) < 2 \cdot 2^{\tau r \log r}\mathbf{rw}(G)$. Thus we deduce the desired inequality, as $t - 1 \leq \frac{1}{8}r^2 + \frac{r}{2} - \frac{5}{8}$. $\square$

7. Graphs of bounded ∇_1

As mentioned in [21], for every r there is a function f (resp. f') such that if G is a graph excluding G as a minor (resp. topological minor), then $\nabla_1(G) \leq f(r)$ (resp. $\nabla_1(G) \leq f'(r)$) (see also [18]). In that sense, the class of graphs with bounded ∇_1 is more general than all the classes we considered in the previous sections. However, the same line of arguments allows us to prove that when ∇_1 is bounded, then tree-width, rank-width, and clique-width are still linearly dependent. For this we first observe the following analogue of Proposition 7.

Proposition 17. *Let $r \geq 1$. Every n-vertex hypergraph H with $\nabla_1(I(H)) \leq r$ has at most $4^r \cdot n$ hyperedges.*

Proof. We consider the graph G as in the proof of Proposition 7 and recall that G is a star-minor of $I(H)$. Since $\nabla_1(I(H)) \leq r$, G is $2r$-degenerate. We conclude that G contains at most $4^r \cdot (n - 2r + 1) \leq 4^r \cdot n$ cliques (from [30]). The result follows, as for each hyperedge of H, there is a clique in G on the same vertex set. $\square$

It is now easy to produce an analogue of Proposition 9 by observing that (i) G cannot have $K_{2r+1,2r+1}$ as a subgraph (this graph has density more than r) and (ii) if G' is a bipartite subgraph of G, then $\nabla_1(G') \leq r$. We conclude the following.

Theorem 18. *If G is a graph with at least one edge where $\nabla_1(G) \leq r$, then*

$$\mathbf{cw}(G) < 2 \cdot 4^r\mathbf{rw}(G),$$
$$\mathbf{tw}(G) + 1 < 12 \cdot r \cdot 4^r\mathbf{rw}(G).$$

Proposition 17 does not hold any more if we replace ∇_1 with ∇_0: The complete graph K_n as a hypergraph has $\binom{n}{2}$ hyperedges and yet $I(K_n)$ is 2-degenerate. In fact we can go a little bit further as Theorem 18 also holds for the "grad"-variant $\nabla_{\frac{1}{2}}$. This parameter was defined in [31] as a variant of ∇_1 where we additionally ask that every two contracted stars, if adjacent in the star-minor, have centers of distance at most 2 in G. As this requirement is also satisfied in the proof of Proposition 7, this implies that Theorem 18 holds for $\nabla_{\frac{1}{2}}$ as well. As proved in [7], graphs of bounded $\nabla_{\frac{1}{2}}$ are those with bounded arrangeability, a natural parameter defined in [2] (see also [12,13]). Therefore, we may conclude that if G is a graph with bounded arrangeability, then $\mathbf{tw}(G) = O(\mathbf{rw}(G))$.

8. Bounds when excluding $K_{r,r}$ as a subgraph

In this section, we investigate graphs with no $K_{r,r}$ subgraph, motivated by the inequality (3) of Gurski and Wanke, which is

$$\mathbf{tw}(G) \leq 2(r-1)\mathbf{cw}(G) - 1.$$

One natural question we might ask is the relation between tree-width and rank-width for graphs with no $K_{r,r}$ subgraph. By our approach, it is enough to find an upper bound on the number of hyperedges in a hypergraph with no $K_{r,r}$ subgraph in its incidence graph. What are those hypergraphs? In fact, if $\mathcal{F}$ is a collection of hyperedges of such a hypergraph, then the intersection of r hyperedges can have at most $r-1$ elements. The problem of finding the maximum possible number of sets with k-wise restricted intersection was studied more generally by Füredi and Sudakov [10]. We cite their lemma here.

Lemma 19 (*Füredi and Sudakov [10, Lemma 2.1]*). *Let $k \geq 2$ and s be two positive integers. If $\mathcal{F}$ is a family of subsets of an n-element set such that $|A_1 \cap A_2 \cap \cdots \cap A_k| < s$ for all $A_1, A_2, \ldots, A_k \in \mathcal{F}$, then*

$$|\mathcal{F}| \leq \frac{k-2}{s+1}\binom{n}{s} + \sum_{i=0}^{s}\binom{n}{i}.$$

(In [10], k is assumed to be larger than 2. However, when $k = 2$, then the above lemma is implied by a theorem of [9, Theorem 11].) In our case, we let $k = s = r$. Then the above inequality answers P3; It provides an upper bound on the number of hyperedges in a hypergraph whose incidence graph has no subgraph isomorphic to $K_{r,r}$.

Proposition 20 (P3). *Let H be an n-vertex hypergraph. Let $r \geq 2$. If the incidence graph of H has no $K_{r,r}$ subgraph, then*

$$|E(H)| \leq \frac{r-2}{r+1}\binom{n}{r} + \sum_{i=0}^{r}\binom{n}{i}.$$

Theorem 21. *Let $r \geq 2$. Let G be an n-vertex graph with no subgraph isomorphic to $K_{r,r}$. Then*

$$\mathbf{cw}(G) < \frac{2(r-2)}{r+1}\binom{\mathbf{rw}(G)}{r} + 2\sum_{i=0}^{r}\binom{\mathbf{rw}(G)}{i},$$

$$\mathbf{tw}(G) + 1 < 3(r-1)\left(\frac{2(r-2)}{r+1}\binom{\mathbf{rw}(G)}{r} + 2\sum_{i=0}^{r}\binom{\mathbf{rw}(G)}{i}\right).$$

Proof. From Proposition 20, $\lambda_G(n) \leq \frac{r-2}{r+1}\binom{n}{r} + \sum_{i=0}^{r}\binom{n}{i}$. □

9. Conclusions

We observe that Theorem 10 has important algorithmic consequences for approximating rank-width. By Feige et al. [8], for every fixed r there exists a polynomial time constant factor approximating algorithm computing the tree-width of a graph excluding K_r as a minor. By combining this result with Theorem 10, we deduce that for every fixed r, there is a polynomial time algorithm approximating within constant factor the rank-width of a K_r-minor free graph.

As a side remark, we proved in Proposition 6 that every n-vertex graph with no K_r-minor has at most $2^{\mu r \log\log r} n$ cliques for a fixed μ. The previous best upper bound $2^{O(r\sqrt{\log r})}$ was observed by Wood [30]. He posed an open problem whether such a graph has at most $c^r n$ cliques for a constant c. It will be interesting to resolve this open problem.

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