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The Finite Topology and Variational Calculus on Nontopological Vector Spaces

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Certain aspects of the calculus of variations are presented in the setting of nontopological vector spaces, and the results are shown to have certain advantages in the investigation of various optimization problems of economics that seem more directly accessible by these techniques than by the maximum principle of optimal control theory.

1. INTRODUCTION

The purposes of this paper are to present certain aspects of the calculus of variations in an essentially topology-free setting and to point out the utility of these techniques in the investigation of various important dynamic optimization problems of economics.

The usual necessary and sufficiency conditions of the calculus of variations are typically presented within the framework of a *topological* space, often taken to be a normed vector space (cf. Smith [8]). However, most optimization problems are stated originally without any natural topology appearing in the statements of the problems, and the introduction of any particular norm is often somewhat artificial in practice for problems on infinite dimensional spaces, as mentioned already in [8]. In the present paper we give a vector space presentation of certain aspects of the variational calculus without requiring any topology on the given vector space. We use only the Euclidean topology induced by the underlying real number field on suitable finite dimensional linear manifolds in the given vector space. Hence

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the presentation given here is broader than the customary presentation such as that of [8].

In recent years, economists have confronted, primarily in capital and growth theory, various optimization problems which have seemed more directly accessible by techniques of the calculus of variations rather than by the maximum principle of optimal control theory. We note in this regard the work of Kamien and Muller [5, 6] on capital replacement theory, the work of Black [2] on the optimal depletion of an exhaustible resource with putty–clay production, and the earlier work of Arrow [1], Nerlove and Arrow [7], and Solow *et al.* [10].

The primary reason for this preference for the calculus of variations is that the state variables in an optimal control approach to these problems are given by certain types of integral functions that need not be continuous, and this leads to certain technical difficulties in the attempt to solve these problems with the maximum principle of optimal control theory. Kamien and Muller [5, 6] have used a variational approach to obtain a maximum principle that may be applied in certain cases when such integral functions are required to be continuous, though not differentiable. (Reference [5] gives a more complete version of [6].) Black [2] has used a variational approach to solve a more complex problem in which even this latter continuity property must be relaxed. We shall briefly discuss this problem of Black below in our final section so as to illustrate our results of Sections 2 and 3.

In Section 2 we give a statement of the Euler–Lagrange multiplier theorem of the calculus of variations in a nontopological vector space setting, and in Section 3 we introduce a concept of *local directional extreme points*, again independent of any topology on the given vector space. This concept permits the development of a certain sufficiency result which is considerably less difficult to apply than the usual sufficiency theorems. Finally, in Section 4 we discuss the applications of these concepts to the problem of Black [2] on the optimal depletion of an exhaustible resource with putty–clay production. We point out why this problem seems most directly accessible by the present variational methods.

2. A Finite Topology Form of Euler–Lagrange Multiplier Theorem

We state here a form of the Euler–Lagrange multiplier theorem taken from Smith [9]. Certain notations used in the statement of the theorem will be explained below following the statement of the theorem itself.

Euler–Lagrange Multiplier Theorem. *Let $K_0, K_1, K_2, ..., K_m$ be real-valued functions which are defined on a subset D of a real vector space Σ, and let x be a maximum or minimum vector in $D|K_i = k_i$ for*

$i = 1, 2, \ldots, m]$ for K_0, where $k_1, k_2, \ldots, k_m$ are any given fixed numbers for which the set $D[K_i = k_i$ for $i = 1, 2, \ldots, m]$ is nonempty. Let D be finitely full at x, and let $K_0, K_1, \ldots, K_m$ have first variations which are finitely continuous at x. Then there are real numbers $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$\sum_{i=0}^{m} \lambda_i \delta K_i(x, z) = 0 \qquad \textit{for all vectors } z \quad \textit{in} \quad \Sigma, \tag{2.1}$$

holds, where λ_0 can be taken (up to an inessential constant multiple) as $\lambda_0 = |\det(\delta K_i(x, z_j))_{i,j=1,2,\ldots,m})|$ for any fixed vectors $z_1, z_2, \ldots, z_m$ in Σ, with $\lambda_0 \geqslant 0$, and where there are similar expressions for $\lambda_1, \lambda_1, \ldots, \lambda_m$ (cf. [8, pp. 78–79]). (These expressions for the multipliers $\lambda_0, \lambda_1, \ldots, \lambda_m$ typically play no role in the applications of the theorem.)

This statement of the Euler–Lagrange multiplier theorem is broader than that of [8] (but not that of [9]). However, it is easy to check that the same proof as given in [8] suffices also to prove the present form of the theorem, and we omit the details.

The notation used in the above statement of the theorem is as follows. The symbol $D[K_i = k_i$ for $i = 1, 2, \ldots, m]$ denotes the subset of D consisting of all vectors x in D which simultaneously satisfy all the constraints

$$K_i(x) = k_i \qquad \text{for } i = 1, 2, \ldots, m,$$

where $k_1, k_2, \ldots, k_m$ may be any given real numbers.

For any vectors $x, x_1, x_2, \ldots, x_n$ in a vector space Σ, the subset μ of Σ defined as

$$\mu = \left\{ z \in \Sigma \mid z = x + \sum_{i=1}^{n} \alpha_i x_i \text{ for arbitrary real numbers } \alpha_1, \alpha_2, \ldots, \alpha_n \right\} \tag{2.2}$$

is called the *linear manifold through x generated by $x_1, x_2, \ldots, x_n$*. A subset D of Σ is said to be *full at x along the linear manifold μ* of (2.2) if D contains all vectors $z = x + \sum_{i=1}^{n} \alpha_i x_i$ for all sufficiently small numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. The set D is said to be *finitely full at x* if D is full at x along *every* linear manifold μ of the form (2.2) (i.e., for every positive integer n and for every choice of the vectors $x_1, x_2, \ldots, x_n$ in Σ). Hence the set D is *finitely* full at x if the intersection of D with any *finite* dimensional linear manifold through x contains a full neighborhood of x in the relative topology of the linear manifold. This notion is related to that of the *finite topology* of Kakutani and Klee [4]. (The second author is indebted to H. Halkin for this reference to the finite topology of a vector space; cf. [9].)

It is customary to refer to the realvalued functions $K_0, K_1, K_2, \ldots, K_m$ as *functionals* on D. If D is finitely full at x, then any functional L on D is said to have a (first) *variation* at x whenever there is a functional $\delta L(x)$, with values $\delta L(x, z)$, defined for all vectors z in Σ and such that

$$\delta L(x, z) = \underset{(\varepsilon \text{ real})}{\lim_{\varepsilon \to 0}} \frac{L(x + \varepsilon z) - L(x)}{\varepsilon} = \frac{d}{d\varepsilon} L(x + \varepsilon z) \Big|_{\varepsilon = 0} \tag{2.3}$$

holds for every vector z in Σ. The functional $\delta L(x)$ is called the (Gateaux) *variation of L at x*. The variation of L is said to be ***finitely continuous*** at x whenever the expression

$$\delta L \left(x + \sum_{i=1}^{n} \alpha_i x_i, z \right) \tag{2.4}$$

exists and is a continuous function of the real n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ near the origin in Euclidean n-space, for every choice of the vectors $x_1, x_2, \ldots, x_n$ in Σ, and for each $z = x_i$ (for $i = 1, 2, \ldots, n$).

The variation of L need not in general be a linear functional with respect to z (cf. the example given in Exercise 6 on p. 40 of [8]). However, if the variation is finitely continuous at x, then the vatiation at x is easily seen to be linear in z. The finite continuity of the variation seems to be just what one requires in practice, and in typical applications it is an easy matter to check whether or not the functionals of interest are finitely continuous (cf. [2]).

3. Local Directional Extreme Points: A Sufficiency Condition

Let D be a nonempty subset of a vector space Σ, and let K be a functional defined on D. A vector x in D is said to be a *local directional maximum point* in D for K if, for every fixed nonzero vector z in Σ,

$$K(x + \varepsilon z) < K(x) \tag{3.1}$$

holds for all sufficiently small (depending on z) $\varepsilon \neq 0$ such that $x + \varepsilon z$ is in D. That is, given any nonzero vector z in Σ, there is a positive number ε_0 (depending possibly on z) such that (3.1) holds for all vectors $x + \varepsilon z$ in D with $0 < |\varepsilon| \leqslant \varepsilon_0$. A local directional *minimum* point in D for K is defined similarly, with the inequality reversed in (3.1). We say that x is a local directional *extreme* point in D for K if x is either a local directional maximum point or a local directional minimum point, and in this case we say that K has a local directional *extremum* at x.

THEOREM 3.1. *Let K be a functional defined on a subset D of a vector space Σ, and let x be a fixed vector in D with D finitely full at x. Suppose that $K(x+\varepsilon z)$ is twice continuously differentiable with respect to ε for every fixed nonzero vector z in Σ and for all sufficiently small ε (depending on z), and suppose that $\delta K(x,z)=0$ and $\delta^2 K(x,z)<0$ hold for all nonzero vectors z in Σ. Then x is a local directional maximum point in D for K.*

The proof of Theorem 3.1 is an easy consequence of the above definitions and the identity

$$K(x+\varepsilon z)=K(x)+\varepsilon\delta K(x,z)+\int_0^{\varepsilon}(\varepsilon-t)\frac{d^2}{dt^2}K(x+tz)\,dt, \tag{3.2}$$

where identity (3.2) is obtained by integrating by parts the integral on the right side here. In the statement of Theorem 3.1 the notation $\delta^2 K$ denotes the usual *second* variation of K. We omit the details of the proof.

Of course, Theorem 3.1 amounts to nothing more than the usual local sufficiency theorem of one-variable real calculus which the above definitions allow us to carry over to a general vector space.

There is an obvious, corresponding theorem for a local directional *minimum* point which we omit.

EXAMPLE (Exercise 1 on p. 351 of [8]). Let Σ be the vector space consisting of all continuously differentiable functions $x=x(t)$ on the interval $0\leqslant t\leqslant 1$ satisfying the fixed boundary conditions $x(0)=x(1)=0$, with the usual vector space structure. Let the functional K be defined by the formula

$$K(x)=-\int_0^1[x(t)^2-x'(t)^4]\,dt \qquad \text{for any } x \quad \text{in} \quad \Sigma.$$

One directly finds the results $\delta K(x^*,z)=0$ for all z in Σ and $\delta^2 K(x^*,z)<0$ for all nonzero z in Σ, where here x^* denotes the zero vector $x^*=0$ with $x^*(t)=0$ for all t. One sees easily that all of the conditions of Theorem 3.1 are satisfied (with $D=\Sigma$), and so we conclude that *the zero vector $x^*=0$ is a local directional maximum point in Σ for K.* However, if we take $z_n=z_n(t)=[\sin 2\pi nt]/n^{1/2}$ for $n=1,2,\ldots$, we find

$$K(x^*+\varepsilon z_n)=-\varepsilon^2\left[\frac{1}{2n}-6(n\pi\varepsilon)^2\right],$$

so that there holds

$$K(x^*+\varepsilon z_n)>K(x^*) \qquad \text{for each fixed } \varepsilon\neq 0 \text{ and for all sufficiently large } n.$$

Note that we have introduced no topology on Σ, and we should *not* think of the vectors z_n as somehow tending towards the zero vector $x^* = 0$ with increasing n. Note also that inequality (3.1) does indeed hold for any fixed z_n (with n fixed) for all sufficiently small $\varepsilon \neq 0$.

This example illustrates both the limitations and the advantages of Theorem 3.1. The theorem provides a positive, albeit weak, result which is sometimes quite useful in practice, particularly when stronger results are not readily available. An advantage of the theorem is its ease of use. The conditions of the theorem are usually relatively easy to check in practice.

For extremal problems that can be cast so as to involve only a finite number of equality constraints we have a similar result given by the following theorem taken from [2] in the case $m = 1$.

THEOREM 3.2. *Let $K_0, K_1, ..., K_m$ be functionals defined on a subset D of a real vector space Σ, let D be finitely full at x, and let $K_0, K_1, ..., K_m$ have first variations which are finitely continuous at x. Suppose also that $K_0, K_1, ..., K_m$ have second variations at x, and suppose that x is a candidate provided by the Euler–Lagrange multiplier theorem for a maximum vector in $D[K_i = k_i$ for $i = 1, ..., m]$ for K_0, where $k_1, ..., k_m$ are any given fixed numbers for which the set $D[K_i = k_i$ for $i = 1, ..., m]$ is nonempty. Let $\lambda_0, \lambda_1, ..., \lambda_m$ be the multipliers appearing in (2.1) of the multiplier theorem, with $\lambda_0 \geqslant 0$, and define the functional G on D as $G = \sum_{i=0}^{m} \lambda_i K_i$. Suppose that $G(x + \varepsilon z)$ is twice continuously differentiable with respect to ε for every fixed nonzero vector z in Σ and for all sufficiently small ε (depending on z). If $\delta^2 G(x, z) < 0$ now holds for all nonzero vectors z in Σ, then x is a local directional maximum point in $D[K_i = k_i$ for $i = 1, ..., m]$ for K_0.*

Proof. The first variation of G vanishes at x by (2.1), and so we may apply identity (3.2) to G and find

$$G(x + \varepsilon z) - G(x) = \int_0^{\varepsilon} (\varepsilon - t) \frac{d^2}{dt^2} G(x + tz)\, dt. \tag{3.3}$$

For any $x + \varepsilon z$ in the constraint set $D[K_i = k_i$ for $i = 1, ..., m]$ we have

$$G(x + \varepsilon z) - G(x) = \lambda_0 [K_0(x + \varepsilon z)) - K_0(x)],$$

so that (3.3) implies

$$\lambda_0 [K_0(x + \varepsilon z) - K_0(x)] = \int_0^{\varepsilon} (\varepsilon - t) \frac{d^2}{dt^2} G(x + tz)\, dt \tag{3.4}$$

for all $x + \varepsilon z$ in the given constraint set. The smoothness of $G(x + tz)$ with respect to t (near $t = 0$, for fixed z) along with the given strict negativeness of the second variation of G at x implies that the right side of (3.4) is strictly negative for all sufficiently small nonzero ε, and the stated result of Theorem 3.2 then follows directly from (3.4). (There also follows immediately the additional result $\lambda_0 > 0$, although we have no need of this result here.)

4. The Optimal Depletion of an Exhaustible Resource

Black [2] has proposed and solved a more general and realistic version of the resource depletion problem solved by Dasgupta and Heal [3]. Both [2] and [3] consider a fixed population in a one-good world in which the output of the consumption good is a result of the use of two factors: capital, which is reproducible, and resource, which exists in finite supply and is not reproducible. Dasgupta and Heal [3] assume also that capital is perfectly malleable so that the ratio of resource to capital can be freely altered for the entire stock of capital at any time. This latter assumption is called *putty–putty production*, and provides a useful model in certain situations but not in others. Black [2] considers a more general model based on putty–clay production, which takes into account the fact that capital equipment is typically designed so that the effective use of the equipment requires the use of a certain minimal amount of resource. In this case the use of the equipment leads to no output if the amount of resource used falls below this minimal amount. Thus, in Black's model, the stock of capital consists of a variety of different types, depending on its vintage, and the planner must decide how much, if any, of each type of capital to use at any time. Black also introduces capital depreciation, not included in the model of Dasgupta and Heal. We refer the reader to [2] for a more complete discussion of the model of Black; we now turn directly to a mathematical statement of the model.

Let the output of the single good be $y = y(t)$, given by

$$y(t) = y_0 e^{-at} + \int_0^t v(s,t)\, I(s)\, f(r(s))\, e^{-a(t-s)}\, ds, \tag{4.1}$$

where $I = I(s)$ is the investment in new capital at time s, and a is the positive constant depreciation rate of capital. The function $f = f(r)$ is the intensive form of the production function which is assumed to be increasing, linearly homogeneous, strictly concave, and as smooth as necessary. The function $r = r(s)$ is the resource intensity of capital of vintage s, and its lack of dependence on the current time t embodies the putty–clay assumption. The utilization function $v = v(s, t)$ is the fraction of capital of vintage s used at the later time t. The planning interval is taken to begin at $t = 0$, and the given positive constant y_0 is assumed to account for all investment decisions made prior to that time.

The total amount of the resource used over an infinite planning horizon is given by

$$\int_0^{\infty}\int_0^{t} v(s,t)\, I(s)\, r(s)\, e^{-a(t-s)}\, ds\, dt = \int_0^{\infty} I(s)\, r(s) \int_s^{\infty} v(s,t)\, e^{-a(t-s)}\, dt\, ds,$$

and since the resource is exhaustible, we impose the constraint

$$\int_0^{\infty} I(s)\, r(s) \int_s^{\infty} v(s,t)\, e^{-a(t-s)}\, dt\, ds = R_0, \tag{4.2}$$

where R_0 is a given positive constant, representing the finite supply of the resource.

The objective functional of the problem in both Black [2] and Dasgupta and Heal [3] is the utility of consumption U given by

$$U = \int_0^{\infty} u(c(t))\, e^{-bt}\, dt, \tag{4.3}$$

where the consumption $c = c(t)$ is given by

$$c(t) = y(t) - I(t),$$

and where u is the utility function, assumed to be increasing, strictly concave, and as smooth as necessary. The positive constant b is the rate at which the utility of consumption is discounted.

The optimization problem is then to maximize the utility U of (4.3), with $c(t)$ replaced by $y(t) - I(t)$ and $y(t)$ given by (4.1), subject to the constraint (4.2), by suitable choices of the control functions I, r, and v. We must also impose the pointwise inequality constraints

$$I(s) \geqslant 0, \qquad r(s) \geqslant 0, \qquad 0 \leqslant v(s,t) \leqslant 1 \tag{4.4}$$

for all $s \geqslant 0$ and all $0 \leqslant s \leqslant t$.

The admissible controls are further restricted by the requirements that I must be integrable on $[0, \infty)$ and r must be bounded and locally integrable there. These restrictions are justifiable on economic grounds if the production function f satisfies $f(0) = 0$, which we assume. Finally, the utilization function v is required only to be locally integrable, and it is here that the use of the calculus of variations is indicated. Indeed, we wish to permit the planner the option of discontinuously shutting down the capital stock accumulated over an interval of s at any later time t, and inspection of

(4.1) shows that this leads to an output function y that need not satisfy the required smoothness properties for a state variable in an optimal control approach to the problem. For this reason Black [2] chose to employ a calculus of variations approach.

The use of a variational approach to problems of this sort involves the use of the necessary condition provided by the Euler–Lagrange multiplier theorem in a search for candidates for the extremal solution. Pointwise inequality constraints such as (4.4) are customarily handled through the use of suitable slack functions (following Valentine; cf. [8]). After finding such a candidate, or candidates, one is faced with the problem of verifying whether or not such a candidate is in fact a solution to the optimization problem. This verification can be done directly, but usually this method is tractable only for relatively simple problems. There are various sufficiency theorems, both global and local, but these are sometimes prohibitively difficult to use for complex problems in practice. Moreover the local theorems typically presuppose a topology on the underlying vector space of control functions, and indeed the resulting local sufficiency theorems are strongly topology dependent, whereas most optimization problems in practice do *not* come equipped with any *natural* topology.

Black [2] used slack variables to account for the pointwise inequality constraints (4.4) and thereby cast the above optimization problem into a form covered by the Euler–Lagrange multiplier theorem of Section 2 (with $m = 1$). Black then solved the resulting necessary condition (2.1) and obtained a candidate solution provided a certain economically meaningful parameter restriction is satisfied, and in this case Black used the above Theorem 3.2 to prove that this candidate solution is in fact a local directional maximum for the optimization problem. We refer the reader to [2] for the details along with the economic interpretation of Black's solution.

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