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On the intersections of polynomials and the Cayley–Bacharach theorem

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Abstract

Let $R = K[x_1, \ldots, x_n]$ and let $f_1, \ldots, f_n$ be products of linear forms with f_i of degree d_i. Assume that the f_i have $d_1, \ldots, d_n$ common zeros. Then we determine the maximum number of those zeros that a form of degree k can go through without going through all of them. This is a version of a conjecture of Eisenbud, Green, and Harris. We suggest a possible method for using this to explore the case where the f_i are arbitrary forms of degree d_i with the right number of common zeros.

MSC: 14N05; 14M05

1. Introduction

While the properties of intersections of curves in space have been studied throughout history, many recent advances have been described in the more advanced language of algebra. In a 1996 paper of David Eisenbud, Mark Green, and Joe Harris, these developments were concluded in a series of conjectures suggesting further extensions to the work of Cayley and Bacharach. See [3] for additional background on the conjectures. One of these Cayley–Bacharach conjectures, which we reproduce below, suggested at least a partial susceptibility to a more geometric attack.

Throughout, R denotes a polynomial ring in n variables over a field K.

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Cayley–Bacharach Conjecture. *Let Γ be a complete intersection of n quadratics in $\mathbb{P}^n$. If $X \subset \mathbb{P}^n$ is any hypersurface of degree k containing a subscheme Γ_0 of degree strictly greater than $2^n - 2^{n-k}$, then X contains Γ.*

In Section 2, we introduce a few special cases of the Eisenbud–Green–Harris conjecture and prove some results. In Section 3, we discuss areas for further generalization and study.

2. Results

We simplify the Cayley–Bacharach conjecture to a question about the common zeros of polynomials. We ask how many of these common zeros can also be roots of another polynomial without that polynomial vanishing at all of them. In the spirit of the conjecture, we begin by considering a set of quadratics with the maximum number of common zeros. To illustrate, we describe a few simple cases.

In dimension 2, the intersections of the zeros of the quadratics $x_1(x_1 - 1)$ and $x_2(x_2 - 1)$ are the vertices of a unit square embedded in the coordinate plane: $(0,0)$, $(0,1), (1,0), (1,1)$. It is possible to find a quadratic that vanishes on any three of these points without vanishing on the fourth (x_1x_2 is one example). This is clearly maximal because only 1 point is missed.

Similarly in dimension 3, the cubic $x_1x_2x_3$ is zero on seven of the eight vertices of the unit cube formed by the common zeros of the quadratics $x_1(x_1 - 1)$, $x_2(x_2 - 1)$, and $x_3(x_3 - 1)$. However, when the dimension of the system of quadratics exceeds the degree of the polynomial, it is no longer possible for a polynomial to vanish on all but one of the common zeros. We prove a special case of a conjecture of Eisenbud, Green, and Harris determining the minimum number of points that must be missed by a degree k polynomial in dimension n below.

Theorem 1. *Given the n quadratics in n variables $x_1(x_1 - 1), \ldots, x_n(x_n - 1)$ with 2^n common zeros, the maximum number of those common zeros a polynomial P of degree k can go through without going through them all is $2^n - 2^{n-k}$.*

Proof. We induct on the dimension n and show that the minimum number of missed points is 2^{n-k}. For our base case, we look at each degree k polynomial P in dimension $k + 1$ and show that at least two points must be missed. When $k = 0$ this is obvious.

When $k = j$ we have $j + 1$ variables. If P misses only one point, then we can say by symmetry that P is zero on all of the points in the face $x_{j+1} = 0$. We want to conclude that by replacing P mod the $x_i(x_i - 1)$ if necessary P is divisible by x_{j+1}. This is true because $x_1(x_1 - 1), \ldots, x_j(x_j - 1), x_{j+1}$ have the maximum number of common zeros that are allowed: namely the product of the degrees. Thus, by Bezout's theorem each zero is multiplicity one and R mod the ideal generated by $x_1(x_1 - 1), \ldots, x_j(x_j - 1), x_{j+1}$ is a direct sum of copies of the field K. Since P is zero at every common zero, it must be in the ideal. Thus P is congruent to $p_{j+1}x_{j+1}$ with $p_{j+1} \in R$. Thus, by changing P mod the ideal generated by the $x_i(x_i - 1)$, we can assume that P is divisible by x_{j+1}

(see [1] for details in the projective case using the n dimensional analogue of Max Noether's $Af + Bg$ theorem). We consider P divided by x_{j+1} to be a degree $j-1$ polynomial in dimension j by restricting it to the $x_{j+1} = 1$ face. By hypothesis this polynomial must miss at least two common zeros. Thus, any degree k polynomial P in dimension $k+1$ shares this property.

We induct on the dimension n to show that a polynomial P of degree k can miss no fewer than 2^{n-k} common zeros. Assume that P misses exactly p of the 2^n points with $p < 2^{n-k}$ and consider the set of p missed points. We note that our base case shows it is impossible to miss fewer than two points in dimension $k+1$, so when $n \geqslant k+1$, $p \geqslant 2$. Because these points are distinct, there exists at least one variable, say x_i, in which some points differ. This partitions the p points into two non-empty subsets: one contained in the plane $x_i=0$, and the other in the plane $x_i=1$. However, one of these subsets contains fewer than 2^{n-k-1} points, and by our inductive hypothesis, it is impossible for a form of degree $k-1$ to be nonzero on fewer than 2^{n-k-1} points without vanishing on all of the points in that subset. Thus, we conclude that there does not exist a degree k polynomial that misses fewer than 2^{n-k} of the 2^n common solutions to the quadratics $x_1(x_1-1), \ldots, x_n(x_n-1)$.

Now we need only show the existence of a polynomial of degree k that is zero on $2^n - 2^{n-k}$ of the intersections. We consider the polynomial $x_1 x_2, \ldots, x_k = 0$. When at least one of the x_i is zero, the polynomial is also zero. This occurs on exactly $2^n - 2^{n-k}$ of the common zeros. On the remaining 2^{n-k} points, when $x_1 = x_2 = \cdots = x_k = 1$, the polynomial is not zero. Clearly, this polynomial satisfies our requirements for all positive k. Thus, 2^{n-k} is the minimum number of vertices that can be missed by any polynomial that does not contain all 2^n points. □

While it is easy to think of the set of common zeros as the vertices of a hypercube, the proof is equivalent in the case where we consider the common zeros of n quadratics in n variables with the requirement that the quadratics intersect in 2^n distinct points and each quadratic can be factored into linear terms. This result is stated below.

Corollary 1. *Given n quadratics in n variables $f_1, \ldots, f_n$ with each quadratic $f_i = l_{i1} l_{i2}$ the product of linear forms and 2^n common zeros, the maximum number of common zeros a polynomial P of degree k can go through without going through them all is $2^n - 2^{n-k}$.*

To generalize this result, we consider the set of common zeros of polynomials of any degree. This can be thought of as a hypercube with multiple points on each side. In the spirit of our last theorem, we choose polynomials f_i that are products of linear forms l_{ij}, where d_i is the degree of the polynomial. Before we can determine the minimum number of common zeros a polynomial of degree k can miss, we must consider a seemingly unrelated problem.

Lemma 1. *Let $d_1, \ldots, d_n$, $k \in \mathbb{Z}^+$ so that $1 < d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. Let $d_i = d_i' + a_i$ where d_i', $a_i \in \mathbb{Z}$ with $d_i' \geqslant 1$ and $a_i \geqslant 0$. If $\Sigma a_i = k$ is fixed, the product $d_1' \cdots d_n'$ is minimized by maximizing a_1, then a_2, then a_3, and so on.*

As an illustration, let $d_1=4$, $d_2=5$, $d_3=5$, and $d_4=6$, and $k=8$. We subtract as much as possible from the smallest d_i so $d'_1=1$, $d'_2=1$, $d'_3=4$, and $d'_4=6$. This gives the product $d'_1\cdots d'_4=24$ which is minimal.

Proof. For convenience, we call the algorithm described above the k reduction algorithm and define it rigorously as follows. Let i be such that $k \geqslant d_1+\cdots+d_i$ but $k < d_1+\cdots+d_i+d_{i+1}$. Then if d'_i represents the remains of d_i at the algorithm's completion, $d'_i=\cdots=d'_i=1$, $d'_{i+1}=d_{i+1}-(k-(d_1+\cdots+d_i-i))$, and $d'_j=d_j$ for all $i+1<j\leqslant n$.

We begin by proving that this algorithm minimizes the product in the case where $n=2$. With $d_1,d_2,k\in\mathbb{Z}^+$ so that $d_1\leqslant d_2$ and $k\leqslant d_1+d+2-2$, we let $a_i=d'_i$ for the k reduction algorithm and $b_i=d'_i$ for some other reduction. Clearly $b_i>a_1$ and $a_2>b_2$. Also, $b_1-a_1=a_2-b_2$ because $a_1+a_2=b_1+b_2=d_1+d_2-k$.

We see that we can simplify our problem to one where $d_1{=}b_1, d_2{=}a_2$, and $k{=}b_1{-}a_1$. To do so we first demonstrate that $a_2\geqslant b_1$. We know that either $a_1=1$ or $a_1=d_1-k$. Because b_2 is no less than d_2-k or 1 if $k\geqslant d_2$ (in which case $a_1=1$), we conclude that $b_2\geqslant a_1$. From $b_1-a_1=a_2-b_2$ we see that $a_2\geqslant b_1$. Thus $d_1\leqslant d_2$ and we can proceed.

With this simplification, the k reduction algorithm subtracts all of k from d_1, while the other algorithm subtracts k from d_2. More concretely, the k reduction algorithm reduces the product by multiplying d_1d_2 by $(1-1/d_1)(1-1/(d_1-1))\cdots(1-1/(d_1-k))$. The other algorithm multiplies d_1d_2 by $(1-1/d_2)(1-1/(d_2-1))\cdots(1-1/(d_2-k))$. Because $d_1\leqslant d_2$ each factor $(1-1/(d_1-i))$ is no greater than $(1-1/(d_2-i))$ and the product $d'_1d'_2$ is minimized by the k reduction algorithm.

In the general case, we consider adjacent pairs $d'_id'_{i+1}$ in some reduced sequence and define a pair to be k reduced if the d_i follow the k reduction algorithm for $k=d_i-d'_i+d_{i+1}-d'_{i+1}$.

We start with $d'_1d'_2$ and find the first pair that is not k reduced. We then redistribute the k within that pair so that it matches the k reduction algorithm, minimizing the product of those d'_i (and reducing the overall product $d'_1\cdots d'_n$). We then return to $d'_1d'_2$ and repeat the process. When all adjacent pairs are k reduced, the algorithm matches the k reduction algorithm, and the product $d'_1\cdots d'_n$ is minimized. □

We apply this lemma to the question of common zeros of polynomials to achieve the following result.

Theorem 2. *Let $f_1,\ldots,f_n$ be polynomials of degree $d_1,\ldots d_n$ where $f_i=l_{i1}\cdots l_{id_i}$ with l_{ij} of degree 1 so that there are $d_1\cdots d_n$ common zeros. The minimum number of common zeros that a polynomial P of degree k can miss without vanishing on them all is the product achieved by subtracting k from the degrees of the polynomials d_i with the k reduction algorithm.*

Proof. We prove this result by induction and use the result from Corollary 1, in which each $d_i{=}2$, for our base case. To do so, we note that the k reduction algorithm subtracts 1 from the first kd_i, leaving 2^{n-k} as the desired product.

We induct on $d = d_1 \cdots d_n$ in the general case. Without loss of generality, we assume that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. We consider the hyperplanes $l_{11} = 0, \ldots, l_{1d_1} = 0$. Either the minimum number of missed points is achieved either when the polynomial P vanishes on all of the points in at least one of these planes or when P misses the minimum number of points in each plane.

In the first case if P passes through all of the points in one face, then it is divisible by the equation of that face by an argument identical to the one given earlier. This reduces the degree of f_1 by one and the degree of P to $k-1$. This reduction is in accordance with the k reduction algorithm and by the inductive hypothesis the minimum number of missed points will be found by its continuation.

Now, we need only show that it is not more efficient for a polynomial P to miss the minimum number of common zeros in each linear face. In this case, the minimum number of points missed in each face is the product achieved by subtracting a total of k from $d_2, \ldots, d_n$. Because these points are missed in each of the d_1 faces, the total number of points is d_1 times this product. However, by Lemma 1 this is greater than or equal to the product achieved by subtracting k from $d_1, \ldots, d_n$ so the k reduction algorithm minimizes the number of missed common zeros. $\square$

3. Further study

We see two directions to take this result. The first would be to remove the restriction that the f_i are products of linears. One could sufficiently find some deformation of the situation where f_1, the form of lowest degree, had just one linear factor. A related deformation occurs in the proof of Bezout's theorem found in Mumford [5]. Note that the deformation will be more delicate here. If we take two cubics in two variables that have nine common zeros but with no three of them in a line, then the Cayley–Bacharach theorem (see [4] for details) says that the maximum number of the 9 zeros a quadric can go through without going through all of them is 5. But if we deform the situation so that one of the f_i has a linear factor the three of the zeros lie on a line and the number of common zeros a quadric can go through is 6—namely the 6 not on that line. Thus we note that for general f_i the minimum will not always be attained.

The other direction to generalize is to have more than one additional form. We conjecture that if we have f_i of degree d_i and additional forms g_j of degree e_j that the number of common zeros of the f's and g's should be the length of R mod the lex plus powers ideal (see [2]) formed by the power of x_i being d_i and then fill in the ideal by taking the earliest thing in lex order of degree e_j.

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