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Euler class group of a Laurent polynomial ring: Local case

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Abstract

Let R be a Noetherian commutative ring of dimension $n > 2$ and let $A = R[T, T^{-1}]$. Assume that the height of the Jacobson radical of R is at least 2. Let P be a projective A-module of rank $n = \dim A - 1$ with trivial determinant. We define an abelian group called the "Euler class group of A," denoted by $E(A)$. Let χ be an isomorphism from A to $\det(P)$. To the pair (P, χ), we associate an element of $E(A)$, called the Euler class of P, denoted by $e(P, \chi)$. Then we prove that a necessary and sufficient condition for P to have a unimodular element is the vanishing of $e(P, \chi)$ in $E(A)$.

Earlier, Bhatwadekar and Raja Sridharan have defined the Euler class group of R, denoted by $E(R)$, and have proved similar results for projective R-module of rank n. Later, M.K. Das defined the Euler class group of the polynomial ring $R[T]$, denoted by $E(R[T])$, and proved similar results for projective $R[T]$-modules of rank n with trivial determinant.
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1. Introduction

Let A be a commutative Noetherian ring of dimension d. A classical result of Serre [18] asserts that if P is a projective A-module of rank $> d$, then P has a unimodular element. It is well known that this result is not true in general if rank $P = d = \dim A$. Therefore, it is interesting to know the obstruction for projective A-modules of rank $= \dim A$ to have a unimodular element.

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Let A be a commutative Noetherian ring of dimension n containing $\mathbb{Q}$ and let P be a projective A-module of rank n. In [8], an abelian group $E(A)$, called the *Euler class group* of A is defined and it is shown that P has a unimodular element if and only if the Euler class of P in $E(A)$ vanishes (see [8] for the definition of Euler class of P).

In view of the above result [8], we can ask the following:

Question 1.1. *Let A be a commutative Noetherian ring containing $\mathbb{Q}$. Let P be a projective A-module of* $\operatorname{rank} r < \dim A$ *having trivial determinant. What is the obstruction for P to have a unimodular element*?

Let R be a commutative Noetherian ring of dimension n containing $\mathbb{Q}$. In [10], an abelian group $E(R[T])$, called the Euler class group of $R[T]$ is defined and it is shown that if P is a projective $R[T]$-module of $\operatorname{rank} n = \dim R[T] - 1$ with trivial determinant, then P has a unimodular element if and only if the Euler class of P in $E(R[T])$ vanishes, thus answering the above question in the case $r = \dim A - 1$ and $A = R[T]$.

In this paper, we prove results similar to [10] for the ring $R[T, T^{-1}]$ under the assumption that height of the Jacobson radical of R is $\geqslant 2$. More precisely, we define the Euler class group of $R[T, T^{-1}]$ and prove that if $\widetilde{P}$ is a projective $R[T, T^{-1}]$-module of $\operatorname{rank} n = \dim R$ with trivial determinant, then $\widetilde{P}$ has a unimodular element if and only if the Euler class of $\widetilde{P}$ in $E(R[T, T^{-1}])$ vanishes (Corollary 4.8).

In Appendix A, we prove the following "Symplectic" cancellation theorem (Theorem A.7) (it is used in Section 7) which is a generalization of [3, Theorem 4.8], where it is proved in the polynomial ring case.

Theorem 1.2. *Let B be a ring of dimension d and $A = B[Y_1, \dots, Y_s, X_1^{\pm 1}, \dots, X_r^{\pm 1}]$. Let $(P, \langle\, , \rangle)$ be a symplectic A-module of* $\operatorname{rank} 2n > 0$. *If* $2n \geqslant d$, *then $ESp(A^2 \perp P, \langle\, , \rangle)$ acts transitively on* $\operatorname{Um}(A^2 \oplus P)$.

As an application, we get the following result (Theorem A.8), which gives a partial answer to a question of Weibel [22, Introduction].

Theorem 1.3. *Let R be a ring of dimension* 2 *and $A = R[X_1, \dots, X_r, Y_1^{\pm 1}, \dots, Y_s^{\pm 1}]$. Assume A^2 is cancellative. Then every projective A-module of* $\operatorname{rank} 2$ *with trivial determinant is cancellative.*

2. Preliminaries

All the rings considered in this paper are assumed to be commutative Noetherian and all the modules are finitely generated. We denote the Jacobson radical of A by $\mathcal{J}(A)$.

Let B be a ring and let P be a projective B-module. Recall that $p \in P$ is called a *unimodular element* if there exists an element $\psi \in P^* = \operatorname{Hom}_B(P, B)$ such that $\psi(p) = 1$. We denote by $\operatorname{Um}(P)$, the set of all unimodular elements of P.

Given an element $\varphi \in P^*$ and an element $p \in P$, we define an endomorphism φ_p as the composite $P \xrightarrow{\varphi} B \xrightarrow{p} P$. If $\varphi(p) = 0$, then ${\varphi_p}^2 = 0$ and hence $1 + \varphi_p$ is a unipotent automorphism of P.

By a *transvection*, we mean an automorphism of P of the form $1 + \varphi_p$, where $\varphi(p) = 0$ and either φ is unimodular in P^* or p is unimodular in P. We denote by $E(P)$ the subgroup of $\operatorname{Aut}(P)$ generated by all transvections of P. Note that $E(P)$ is a normal subgroup of $\operatorname{Aut}(P)$.

An existence of a transvection of P pre-supposes that P has a unimodular element. Now, let $P = B \oplus Q$, $q \in Q, \alpha \in Q^*$. Then $\Delta_q(b, q') = (b, q' + bq)$ and $\Gamma_\alpha(b, q') = (b + \alpha(q'), q')$ are transvections of P. Conversely, any transvection Θ of P gives rise to a decomposition $P = B \oplus Q$ in such a way that $\Theta = \Delta_q$ or $\Theta = \Gamma_\alpha$.

We begin by stating two classical results of Serre [18] and Bass [1], respectively.

Theorem 2.1. *Let A be a ring of dimension d. Then any projective A-module P of* rank $> d$ *has a unimodular element. In particular, if* $\dim A = 1$*, then any projective A-module of trivial determinant is free.*

Theorem 2.2. *Let A be a ring of dimension d and let P be a projective A-module of* rank $> d$. *Then $E(A \oplus P)$ acts transitively on* $\mathrm{Um}(A \oplus P)$. *In particular, P is cancellative.*

The following result is due to Lindel [11, Theorem 2.6].

Theorem 2.3. *Let A be a ring of dimension d and $R = A[T_1, \dots, T_n, Y_1^{\pm 1}, \dots, Y_r^{\pm 1}]$. Let P be a projective R-module of* rank $\geqslant \max(2, d+1)$. *Then $E(P \oplus R)$ acts transitively on* $\mathrm{Um}(P \oplus R)$. *In particular, projective R-modules of* rank $> d$ *are cancellative.*

The following result is due to Bhatwadekar and Roy [5, Proposition 4.1] and is about lifting an automorphism of a projective module.

Proposition 2.4. *Let A be a ring and $J \subset A$ an ideal. Let P be a projective A-module of* rank n. *Then any transvection $\widetilde{\Theta}$ of P/JP, i.e. $\widetilde{\Theta} \in E(P/JP)$, can be lifted to a (unipotent) automorphism Θ of P. In particular, if P/JP is free of* rank n*, then any element $\overline{\Psi}$ of $E((A/J)^n)$ can be lifted to $\Psi \in$* $\mathrm{Aut}(P)$. *If, in addition, the natural map* $\mathrm{Um}(P) \to \mathrm{Um}(P/JP)$ *is surjective, then the natural map $E(P) \to E(P/JP)$ is surjective.*

The following result is a consequence of a theorem of Eisenbud–Evans as stated in [17, p. 1420].

Lemma 2.5. *Let R be a ring and let P be a projective R-module of* rank r. *Let $(\alpha, a) \in (P^* \oplus R)$. Then there exists an element $\beta \in P^*$ such that* $\operatorname{ht} I_a \geqslant r$*, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geqslant r$, then* $\operatorname{ht} I \geqslant r$. *Further, if $(\alpha(P), a)$ is an ideal of height $\geqslant r$ and I is a proper ideal of R, then* $\operatorname{ht} I = r$.

The following result is due to Bhatwadekar and Keshari [4, Lemma 4.4].

Lemma 2.6. *Let C be a ring with* $\dim C/\mathcal{J}(C) = r$ *and let P be a projective C-module of* rank $m \geqslant r+1$. *Let I and L be ideals of C such that $L \subset I^2$. Let $\phi : P \twoheadrightarrow I/L$ be a surjection. Then ϕ can be lifted to a surjection $\Psi : P \twoheadrightarrow I$.*

The following result is due to Mandal and Raja Sridharan [16, Theorem 2.3].

Theorem 2.7. *Let A be a ring and let I_1, I_2 be two comaximal ideals of $A[T]$ such that I_1 contains a monic polynomial and $I_2 = I_2(0)A[T]$ is an extended ideal. Let $I = I_1 \cap I_2$. Suppose P is a projective A-module of* rank $n \geqslant \dim(A[T]/I_1) + 2$. *Let $\alpha : P \twoheadrightarrow I(0)$ and*

$\phi: P[T]/I_1P[T] \twoheadrightarrow I_1/I_1^2$ be two surjections such that $\phi(0) = \alpha \otimes A/I_1(0)$. Then there exists a surjective map $\Psi: P[T] \twoheadrightarrow I$ such that $\Psi(0) = \alpha$.

Now, we state the Addition and Subtraction principles, respectively, for arbitrary ring B ([4], Theorem 5.6 and Theorem 3.7, respectively). Note that the following results are valid in the case $d = n = 2$ also ([8], Theorem 3.2 and Theorem 3.3, respectively).

Proposition 2.8. *Let B be a ring of dimension d and let $I_1, I_2 \subset B$ be two comaximal ideals of height n, where $2n \geqslant d + 3$. Let $P = P_1 \oplus B$ be a projective B-module of rank n. Let $\Phi: P \twoheadrightarrow I_1$ and $\Psi: P \twoheadrightarrow I_2$ be two surjections. Then there exists a surjection $\Delta: P \twoheadrightarrow I_1 \cap I_2$ with $\Delta \otimes B/I_1 = \Phi \otimes B/I_1$ and $\Delta \otimes B/I_2 = \Psi \otimes B/I_2$.*

Proposition 2.9. *Let B be a ring of dimension d and let $I_1, I_2 \subset B$ be two comaximal ideals of height n, where $2n \geqslant d + 3$. Let $P = P_1 \oplus B$ be a projective B-module of rank n. Let $\Phi: P \twoheadrightarrow I_1$ and $\Psi: P \twoheadrightarrow I_1 \cap I_2$ be two surjections such that $\Phi \otimes B/I_1 = \Psi \otimes B/I_1$. Then there exists a surjection $\Delta: P \twoheadrightarrow I_2$ such that $\Delta \otimes B/I_2 = \Psi \otimes B/I_2$.*

We end this section by recalling some results from [8, 4.2, 4.3, 4.4] for later use.

Theorem 2.10. *Let B be a ring of dimension $n \geqslant 2$ containing $\mathbb{Q}$. Let J be an ideal of B of height n such that J/J^2 is generated by n elements. Let $w_J: (B/J)^n \twoheadrightarrow J/J^2$ be a surjection. Let P be a projective B-module of rank n with trivial determinant and $\chi: B \xrightarrow{\sim} \bigwedge^n P$. Then the following hold*:

1. *If $(J, w_J) = 0$ in $E(B)$, then w_J can be lifted to a surjection from B^n to J.*
2. *Suppose $e(P, \chi) = (J, w_J)$ in $E(B)$. Then there exists a surjection $\alpha: P \twoheadrightarrow J$ such that (J, w_J) is obtained from (α, χ).*
3. *$e(P, \chi) = 0$ in $E(B)$ if and only if P has a unimodular element.*

3. Some addition and subtraction principles

We begin with the following result which is proved in [6, Lemma 3.6]: in the case A is an affine algebra over a field, $f = T$ and $R = A[T]$. Since the same proof works in our case also, we omit the proof.

Lemma 3.1. *Let A be a ring of dimension d and $R = A[T, T^{-1}]$. Let $\widetilde{P}$ be a projective R-module of rank n, where $2n \geqslant d + 3$. Let $I \subset R$ be an ideal of height n. Let $J \subset I \cap A$ be any ideal of height $\geqslant d - n + 2$ and let $f \in R$ be any element. Assume that we are given a surjection $\phi: \widetilde{P} \twoheadrightarrow I/(I^2 f)$. Then ϕ has a lift $\widetilde{\phi}: \widetilde{P} \to I$ such that $\widetilde{\phi}(\widetilde{P}) = I''$ satisfies the following properties*:

1. *$I'' + (J^2 f) = I$,*
2. *$I'' = I \cap I'$, where $\operatorname{ht} I' \geqslant n$, and*
3. *$I' + (J^2 f) = R$.*

Notation 3.2. Let A be a ring and $R = A[T, T^{-1}]$. We say $f(T) \in A[T]$ is a *special monic polynomial* if $f(T)$ is a monic polynomial with $f(0) = 1$. By $\mathcal{R}$ we denote the ring obtained from R by inverting all the special monic polynomials of $A[T]$. It is easy to see that $\dim \mathcal{R} = \dim A$.

The following result is an analogue of [4, Lemma 4.5] for $A[T,T^{-1}]$.

Lemma 3.3. *Let A be a ring with $\dim A/\mathcal{J}(A)=r$ and $R=A[T,T^{-1}]$. Let I and L be ideals of R such that $L\subset I^2$ and L contains a special monic polynomial. Let Q be a projective R-module of $\operatorname{rank} m\geqslant r+1$. Let $\phi: Q\oplus R\twoheadrightarrow I/L$ be a surjection. Then we can lift ϕ to a surjection $\Phi: Q\oplus R\twoheadrightarrow I$ with $\Phi(0,1)$ a special monic polynomial.*

Proof. Let $\Phi'=(\Theta,g)$ be a lift of ϕ. Let $f\in L$ be a special monic polynomial. By adding some multiple of f to g, we can assume that the lift $\Phi'=(\Theta,g)$ of ϕ is such that g is a special monic polynomial. Let $C=R/(g)$. Since $A\hookrightarrow C$ is an integral extension, we have $\mathcal{J}(A)=\mathcal{J}(C)\cap A$ and, hence, $A/\mathcal{J}(A)\hookrightarrow C/\mathcal{J}(C)$ is also an integral extension. Therefore, $\dim C/\mathcal{J}(C)=r$.

Let "bar" denote reduction modulo (g). Then Θ induces a surjection $\alpha:\overline{Q}\twoheadrightarrow \overline{I}/\overline{L}$, which by 2.6 can be lifted to a surjection from $\overline{Q}$ to $\overline{I}$. Therefore, there exists a map $\Gamma: Q\to I$ such that $\Gamma(Q)+(g)=I$ and $(\Theta-\Gamma)(Q)=K\subset L+(g)$. Hence $\Theta-\Gamma\in KQ^*$. This shows that $\Theta-\Gamma=\Theta_1+g\Gamma_1$, where $\Theta_1\in LQ^*$ and $\Gamma_1\in Q^*$.

Let $\Phi_1=\Gamma+g\Gamma_1$ and let $\Phi=(\Phi_1,g)$. Then $\Phi(Q\oplus R)=\Phi_1(Q)+(g)=\Gamma(Q)+(g)=I$. Thus, $\Phi: Q\oplus R\twoheadrightarrow I$ is a surjection. Moreover, $\Phi(0,1)=g$ is a special monic polynomial. Since $\Phi-\Phi'=(\Phi_1-\Theta,0)$, $\Phi_1-\Theta\in LQ^*$ and Φ' is a lift of ϕ, we see that Φ is a (surjective) lift of ϕ. This proves the result. $\square$

The proof of the following result is the same as of [4, Lemma 4.6] using 2.3 and 3.3. Hence, we omit the proof.

Lemma 3.4. *Let A be a ring of dimension d and $R=A[T,T^{-1}]$. Let n be an integer such that $2n\geqslant d+3$. Let I be an ideal of R of height n such that $I+\mathcal{J}(A)R=R$. Assume that $\operatorname{ht}\mathcal{J}(A)\geqslant d-n+2$. Let $P=Q\oplus R^2$ be a projective R-module of $\operatorname{rank} n$ and let $\phi: P\twoheadrightarrow I/I^2$ be a surjection. If the surjection $\phi\otimes\mathcal{R}: P\otimes\mathcal{R}\twoheadrightarrow I\mathcal{R}/I^2\mathcal{R}$ can be lifted to a surjection from $P\otimes\mathcal{R}$ to $I\mathcal{R}$, then ϕ can be lifted to a surjection $\Phi: P\twoheadrightarrow I$.*

Proposition 3.5 *(Addition Principle). Let A be a ring of dimension d and $R=A[T,T^{-1}]$. Let $I_1,I_2\subset R$ be two comaximal ideals of height n, where $2n\geqslant d+3$. Let $P=P'\oplus R^2$ be a projective R-module of $\operatorname{rank} n$. Assume that $\operatorname{ht}\mathcal{J}(A)\geqslant d-n+2$. Let $\Phi: P\twoheadrightarrow I_1$ and $\Psi: P\twoheadrightarrow I_2$ be two surjections. Then there exists a surjection $\Delta: P\twoheadrightarrow I_1\cap I_2$ with $\Delta\otimes R/I_1=\Phi\otimes R/I_1$ and $\Delta\otimes R/I_2=\Psi\otimes R/I_2$.*

Remark 3.6. Since $\dim R=d+1$, if $2n\geqslant d+4$, then we can appeal to 2.8 for the proof (without the assumption $\operatorname{ht}\mathcal{J}(A)\geqslant d-n+2$). So, we need to prove the result only in the case $2n=d+3$. However, the proof given below works equally well for $2n>d+3$ and, hence, allows us to give a unified treatment. The same remark is also applicable to 3.7.

Proof. Step 1. Write $I=I_1\cap I_2$. Let $J=(I\cap A)\cap\mathcal{J}(A)$. Since $\operatorname{ht}(I\cap A)\geqslant n-1\geqslant(d-n+2)$, we have $\operatorname{ht}J\geqslant d-n+2$. The surjections Φ and Ψ induce a surjection $\Gamma: P\twoheadrightarrow I/I^2$ with $\Gamma\otimes R/I_1=\Phi\otimes R/I_1$ and $\Gamma\otimes R/I_2=\Psi\otimes R/I_2$. It is enough to show that Γ has a surjective lift from P to I.

Applying 3.1 with $f=1$, we get a lift $\Gamma_1\in\operatorname{Hom}_R(P,I)$ of Γ such that the ideal $\Gamma_1(P)=I''$ satisfies the following properties: (1) $I=I''+J^2$, (2) $I''=I\cap K$, where $\operatorname{ht}K\geqslant n$, and (3) $K+J=R$.

Since $\dim \mathcal{R} = d$, applying 2.8 in the ring $\mathcal{R}$ for the surjections $\Phi \otimes \mathcal{R}: P \otimes \mathcal{R} \twoheadrightarrow I_1\mathcal{R}$ and $\Psi \otimes \mathcal{R}: P \otimes \mathcal{R} \twoheadrightarrow I_2\mathcal{R}$, we get a surjective map $\Delta: P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$ such that $\Delta \otimes \mathcal{R}/I_1\mathcal{R} = \Phi \otimes \mathcal{R}/I_1\mathcal{R}$ and $\Delta \otimes \mathcal{R}/I_2\mathcal{R} = \Psi \otimes \mathcal{R}/I_2\mathcal{R}$. It is easy to see, from the very construction of Γ, that Δ is a lift of $\Gamma \otimes \mathcal{R}$.

We have two surjections, $\Gamma_1 : P \twoheadrightarrow I \cap K$ and $\Delta: P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$. Since Γ_1 is a lift of Γ, we have $\Gamma_1 \otimes \mathcal{R}/I\mathcal{R} = \Delta \otimes \mathcal{R}/I\mathcal{R}$. Applying 2.9 in the ring $\mathcal{R}$ for the surjections $\Gamma_1 \otimes \mathcal{R}$ and Δ, we get a surjection $\Delta_1 : P \otimes \mathcal{R} \twoheadrightarrow K\mathcal{R}$ with $\Delta_1 \otimes \mathcal{R}/K\mathcal{R} = \Gamma_1 \otimes \mathcal{R}/K\mathcal{R}$. Since K is comaximal with J and hence with $\mathcal{J}(A)$, applying 3.4, we get a surjection $\Delta_2 : P \twoheadrightarrow K$ which is a lift of $\Gamma_1 \otimes R/K : P \twoheadrightarrow K/K^2$.

Step 2. We have two surjections, $\Gamma_1 : P \twoheadrightarrow I \cap K$ and $\Delta_2 : P \twoheadrightarrow K$, with $\Gamma_1 \otimes R/K = \Delta_2 \otimes R/K$. Recall that $P = P' \oplus R^2$, $J = (I \cap A) \cap \mathcal{J}(A)$, K is comaximal with J and $\operatorname{ht} J \geqslant d - n + 2$. Write $P_1 = P' \oplus R$ and $P = P_1 \oplus R$.

Let "bar" denote reduction modulo J^2. Then $\overline{R} = A/J^2[T, T^{-1}]$ and $\dim A/J \leqslant d - (d - n + 2) = n - 2$. Hence applying 2.3 and 2.4, we can assume that after performing some automorphism of $P_1 \oplus R$, $\Delta_2(P_1) = R$ modulo J^2 and $\Delta_2((0, 1)) \in J^2$. Assume that $\Delta_2((0, 1)) = \lambda \in J^2$. Replacing Δ_2 by $\Delta_2 + \lambda\Delta_3$ for some $\Delta_3 \in {P_1}^*$, we can assume, by 2.5, that $\operatorname{ht} \Delta_2(P_1) = n - 1$. Let $\Delta_2(p_1) = 1$ modulo J^2 for some $p_1 \in P_1$. Further, replacing λ by $\lambda + \Delta_2(p_1)$, we can assume that $\lambda = 1$ modulo J^2.

Let K_1 and K_2 be two ideals of $R[Y]$ defined by $K_1 = (\Delta_2(P_1), Y + \lambda)$ and $K_2 = IR[Y]$. Then $K_1 + K_2 = R[Y]$ since $\Delta_2(P_1) + J = R$ and $J \subset I$. Let $K_3 = K_1 \cap K_2$. Then we have two surjections, $\Gamma_1 : P \twoheadrightarrow K_3(0) = I \cap K$ and $\Lambda_1 : P[Y] \twoheadrightarrow K_1$, defined by $\Lambda_1 = \Delta_2$ on P_1 and $\Lambda_1((0, 1)) = Y + \lambda$. Then $\Lambda_1(0) = \Gamma_1 \mod K_1(0)^2$, as $\Delta_2 \otimes R/K = \Gamma_1 \otimes R/K$. Also, note that, since $\operatorname{ht} \Delta_2(P_1) = n - 1$ and $\Delta_2(P_1) + \mathcal{J}(A) = R$, $\dim R[Y]/K_1 = \dim R/\Delta_2(P_1) \leqslant d - n + 1 \leqslant n - 2$. Hence applying 2.7, we get a surjection $\Lambda_2 : P[Y] \twoheadrightarrow K_3$ with $\Lambda_2(0) = \Gamma_1$. Putting $Y = 1 - \lambda$, we get a surjection $\widetilde{\Delta} = \Lambda_2(1 - \lambda) : P \twoheadrightarrow I$ with $\widetilde{\Delta} \otimes R/I = \Gamma_1 \otimes R/I$.

Since Γ_1 is a lift of $\Gamma : P \twoheadrightarrow I/I^2$, we have $\widetilde{\Delta} \otimes R/I = \Gamma \otimes R/I$. This proves the result. □

Proposition 3.7 *(Subtraction Principle). Let A be a ring of dimension d and $R = A[T, T^{-1}]$. Let $I_1, I_2 \subset R$ be two comaximal ideals of height n, where $2n \geqslant d + 3$. Let $P = P' \oplus R^2$ be a projective R-module of* rank n. *Assume that* $\operatorname{ht} \mathcal{J}(A) \geqslant d - n + 2$. *Let $\Phi : P \twoheadrightarrow I_1 \cap I_2$ and $\Psi : P \twoheadrightarrow I_1$ be two surjections with $\Phi \otimes R/I_1 = \Psi \otimes R/I_1$. Then there exists a surjection $\Delta : P \twoheadrightarrow I_2$ with $\Phi \otimes R/I_2 = \Delta \otimes R/I_2$.*

Proof. Let $J = (I_2 \cap A) \cap \mathcal{J}(A)$. Since $\operatorname{ht}(I_2 \cap A) \geqslant n - 1$ and $n - 1 \geqslant d - n + 2$, we have $\operatorname{ht} J \geqslant d - n + 2$. We have a surjection $\phi : P \twoheadrightarrow I_2/I_2^2$ induced by Φ. Applying 3.1 with $f = 1$, we get a lift $\widetilde{\phi} \in \operatorname{Hom}(P, I_2)$ of ϕ such that $\widetilde{\phi}(P) = I''$ satisfies the following properties: (1) $I_2 = I'' + J^2$, (2) $I'' = I_2 \cap K$, where $\operatorname{ht} K \geqslant n$, and (3) $K + J^2 = R$.

We have two surjections, $\Phi : P \twoheadrightarrow I_1 \cap I_2$ and $\Psi : P \twoheadrightarrow I_1$, with $\Phi \otimes R/I_1 = \Psi \otimes R/I_1$. Since $\dim \mathcal{R} = d$, applying 2.9 in the ring $\mathcal{R}$ for the surjections $\Phi \otimes \mathcal{R}$ and $\Psi \otimes \mathcal{R}$, we get a surjection $\Gamma : P \otimes \mathcal{R} \twoheadrightarrow I_2\mathcal{R}$ with $\Gamma \otimes \mathcal{R}/I_2\mathcal{R} = \Phi \otimes \mathcal{R}/I_2\mathcal{R} = \widetilde{\phi} \otimes \mathcal{R}/I_2\mathcal{R}$.

Again applying 2.9 for the surjections Γ and $\widetilde{\phi} \otimes \mathcal{R}$, we get a surjection $\Gamma_1 : P \otimes \mathcal{R} \twoheadrightarrow K\mathcal{R}$ with $\Gamma_1 \otimes \mathcal{R}/K\mathcal{R} = \widetilde{\phi} \otimes \mathcal{R}/K\mathcal{R}$. Since $K + \mathcal{J}(A) = R$, applying 3.4, we get a surjection $\Gamma_2 : P \twoheadrightarrow K$ with $\Gamma_2 \otimes R/K = \widetilde{\phi} \otimes R/K$.

We have two surjections, $\widetilde{\phi} : P \twoheadrightarrow I_2 \cap K$ and $\Gamma_2 : P \twoheadrightarrow K$, with $\Gamma_2 \otimes R/K = \widetilde{\phi} \otimes R/K$. Recall that $K + \mathcal{J}(A) = R$. Following the proof of 3.5 Step 2, we get a surjection $\Delta : P \twoheadrightarrow I_2$ with $\Delta \otimes R/I_2 = \widetilde{\phi} \otimes R/I_2 = \Phi \otimes R/I_2$. This proves the result. □

Theorem 3.8. *Let A be a ring of dimension d and $R = A[T, T^{-1}]$. Let n be an integer such that $2n \geqslant d + 3$. Let I be an ideal of R of height n. Assume that $\operatorname{ht} \mathcal{J}(A) \geqslant d - n + 2$. Let $P = P' \oplus R^2$ be a projective R-module of $\operatorname{rank} n$ and let $\phi : P \twoheadrightarrow I/I^2$ be a surjection. Assume that $\phi \otimes \mathcal{R} : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}/I^2\mathcal{R}$ can be lifted to a surjection $\Phi : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$. Then ϕ can be lifted to a surjection $\Delta : P \twoheadrightarrow I$.*

Proof. Let $J = (I \cap A) \cap \mathcal{J}(A)$. Note that $\operatorname{ht} J \geqslant d - n + 2$. Applying 3.1 with $f = 1$, we get a lift $\Phi_1 \in \operatorname{Hom}(P, I)$ of ϕ such that the ideal $\Phi_1(P) = I''$ satisfies the following properties: (1) $I = I'' + J^2$, (2) $I'' = I \cap K$, where $\operatorname{ht} K \geqslant n$, and (3) $K + J^2 = R$.

If $\operatorname{ht} K > n$, then $K = R$ and Φ_1 is a lift of ϕ. Hence, we assume that $\operatorname{ht} K = n$. We have two surjections, $\Phi : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$ and $\Phi_1 : P \twoheadrightarrow I \cap K$, with $\Phi \otimes \mathcal{R}/I\mathcal{R} = \Phi_1 \otimes \mathcal{R}/I\mathcal{R}$. Applying 2.9 in the ring $\mathcal{R}$ for the surjections Φ and $\Phi_1 \otimes \mathcal{R}$, we get a surjection $\Psi : P \otimes \mathcal{R} \twoheadrightarrow K\mathcal{R}$ such that $\Psi \otimes \mathcal{R}/K\mathcal{R} = \Phi_1 \otimes \mathcal{R}/K\mathcal{R}$. Since $K + \mathcal{J}(A) = R$, applying 3.4, we get a surjection $\Delta_1 : P \twoheadrightarrow K$ which is a lift of $\Phi_1 \otimes R/K$.

We have two surjections, $\Phi_1 : P \twoheadrightarrow I \cap K$ and $\Delta_1 : P \twoheadrightarrow K$, with $\Phi_1 \otimes R/K = \Delta_1 \otimes R/K$. Applying 3.7, we get a surjection $\Delta : P \twoheadrightarrow I$ such that $\Delta \otimes R/I = \Phi_1 \otimes R/I = \phi$. This proves the result. □

As a consequence of the above result, we have the following:

Corollary 3.9. *Let A be a ring of dimension $n \geqslant 3$ with $\operatorname{ht} \mathcal{J}(A) \geqslant 2$ and $R = A[T, T^{-1}]$. Let I be an ideal of R of height n. Let $\phi : (R/I)^n \twoheadrightarrow I/I^2$ be a surjection. Assume that $\phi \otimes \mathcal{R}$ can be lifted to a surjection from $\mathcal{R}^n$ to $I\mathcal{R}$. Then ϕ can be lifted to a surjection $\Phi : R^n \twoheadrightarrow I$.*

4. Euler class group of $A[T, T^{-1}]$

Notation 4.1. We will denote the following hypothesis by $(*)$: Let A be a ring containing $\mathbb{Q}$ of dimension $n \geqslant 3$ with $\operatorname{ht} \mathcal{J}(A) \geqslant 2$ and $R = A[T, T^{-1}]$.

Assume $(*)$. We proceed to define the *nth Euler class group* of R. The results of this section are similar to [10, Section 4], where it is proved for the ring $A[T]$ (without the assumption $\operatorname{ht} \mathcal{J}(A) \geqslant 2$).

Let $I \subset R$ be an ideal of height n such that I/I^2 is generated by n elements. Let α and β be two surjections from $(R/I)^n$ to I/I^2. We say that α and β are *related* if there exists $\sigma \in \mathrm{SL}_n(R/I)$ such that $\alpha\sigma = \beta$. It is easy to see that this is an equivalence relation on the set of surjections from $(R/I)^n$ to I/I^2. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$ *a local orientation* of I.

If a surjection α from $(R/I)^n$ to I/I^2 can be lifted to a surjection $\Theta : R^n \twoheadrightarrow I$, then so can any β equivalent to α. For, let $\beta = \alpha\sigma$ for some $\sigma \in \mathrm{SL}_n(R/I)$. If $I\mathcal{R} = \mathcal{R}$, then $\beta \otimes \mathcal{R}$ can be lifted to a surjection from $\mathcal{R}^n \twoheadrightarrow I\mathcal{R}$ and hence we can appeal to 3.9. We assume that $I\mathcal{R}$ is a proper ideal of $\mathcal{R}$. Since $\dim \mathcal{R} = n$, we have $\dim \mathcal{R}/I\mathcal{R} = 0$. Hence, $\mathrm{SL}_n(\mathcal{R}/I\mathcal{R}) = E_n(\mathcal{R}/I\mathcal{R})$. Therefore, by 2.4, $\sigma \otimes \mathcal{R}$ can be lifted to an element of $\mathrm{SL}_n(\mathcal{R})$. Thus $\beta \otimes \mathcal{R}$ can be lifted to a surjection from $\mathcal{R}^n \twoheadrightarrow I\mathcal{R}$. By 3.9, β can be lifted to a surjection from $R^n \twoheadrightarrow I$. Therefore, from now on, we shall identify a surjection α with the equivalence class $[\alpha]$ to which it belongs.

We call a local orientation $[\alpha]$ of I *a global orientation* of I, if the surjection $\alpha : (R/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\Theta : R^n \twoheadrightarrow I$.

Let G be the free abelian group on the set of pairs (I, w_I), where $I \subset R$ is an ideal of height n having the property that $\mathrm{Spec}(R/I)$ is connected, I/I^2 is generated by n elements and $w_I : R^n \twoheadrightarrow I/I^2$ is a local orientation of I.

Let $I \subset R$ be an ideal of height n such that I/I^2 is generated by n elements. Then I can be decomposed as $I = I_1 \cap \cdots \cap I_r$, where I_k's are pairwise comaximal ideals of R of height n and $\mathrm{Spec}(R/I_k)$ is connected. From [10, Lemma 4.4], it follows that such a decomposition is unique. We say that I_k's are the connected components of I. Let $w_I : (R/I)^n \twoheadrightarrow I/I^2$ be a surjection. Then w_I induces surjections $w_{I_k} : (R/I_k)^n \twoheadrightarrow I_k/I_k^2$. By (I, w_I), we denote the element $\sum(I_k, w_{I_k})$ of G.

Let H be the subgroup of G generated by the set of pairs (I, w_I), where $I \subset R$ is an ideal of height n and w_I is a global orientation of I. We define the *nth Euler class group* of R, denoted by $E^n(R)$, to be G/H. By abuse of notation, we will write $E(R)$ for $E^n(R)$ throughout this paper.

Let P be a projective R-module of rank n having trivial determinant. Let $\chi : R \xrightarrow{\sim} \bigwedge^n P$ be an isomorphism. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(R)$ as follows:

Let $\lambda : P \twoheadrightarrow I$ be a surjection, where $I \subset R$ is an ideal of height n (such a surjection exists by 2.5). Let "bar" denote reduction modulo I. We obtain an induced surjection $\bar{\lambda} : P/IP \twoheadrightarrow I/I^2$. Since P has trivial determinant and $\dim R/I \leqslant 1$, by 2.1, P/IP is a free R/I-module of rank n. We choose an isomorphism $\bar{\gamma} : (R/I)^n \xrightarrow{\sim} P/IP$ such that $\bigwedge^n(\bar{\gamma}) = \bar{\chi}$. Let w_I be the surjection $\bar{\lambda}\bar{\gamma} : (R/I)^n \twoheadrightarrow I/I^2$. Let $e(P, \chi)$ be the image of (I, w_I) in $E(R)$. We say that (I, w_I) is *obtained* from the pair (λ, χ).

Lemma 4.2. *The assignment sending the pair (P, χ) to the element $e(P, \chi)$, as described above, is well defined.*

Proof. Let $\mu : P \twoheadrightarrow I_1$ be another surjection, where $I_1 \subset R$ is an ideal of height n. Let (I_1, w_{I_1}) be obtained from the pair (μ, χ). Let $J = (I \cap I_1) \cap A$. Recall that $w_I : (R/I)^n \twoheadrightarrow I/I^2$ is a surjection. By 3.1, w_I can be lifted to $\Phi : R^n \twoheadrightarrow I \cap K$, where $\mathrm{ht}\, K = n$ and $K + J = R$.

Since K and I are comaximal, Φ induces a local orientation w_K of K. Clearly, $(I, w_I) + (K, w_K) = 0$ in $E(R)$. Let $L = K \cap I_1$. Since $K + I_1 = R$, w_K and w_{I_1} together induce a local orientation w_L of L. It is enough to show that $(L, w_L) = 0$ in $E(R)$. (Since $(L, w_L) = (K, w_K) + (I_1, w_{I_1})$ in $E(R)$ and $(L, w_L) = 0$ implies $(I, w_I) = (I_1, w_{I_1})$ in $E(R)$.)

Since $\dim \mathcal{R} = n = \mathrm{rank}\, P$, $e(P \otimes \mathcal{R}, \chi \otimes \mathcal{R})$ is well defined in $E(\mathcal{R})$ [8, Section 4]. Hence, it follows that $w_L \otimes \mathcal{R}$ is a global orientation of $L\mathcal{R}$. Therefore, by 3.9, w_L is a global orientation of L, i.e. $(L, w_L) = 0$ in $E(R)$. This proves the lemma. □

Notation 4.3. We define the *Euler class* of (P, χ) to be $e(P, \chi)$.

Theorem 4.4. *Assume $(*)$. Let $I \subset R$ be an ideal of height n such that I/I^2 is generated by n elements and let $w_I : R^n \twoheadrightarrow I/I^2$ be a local orientation of I. Suppose that the image of (I, w_I) in $E(R)$ is zero. Then w_I is a global orientation of I.*

Proof. Since $(I, w_I) = 0$ in $E(R)$, $(I\mathcal{R}, w_I \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$. Therefore, by 2.10, $w_I \otimes \mathcal{R}$ can be lifted to a surjection from $\mathcal{R}^n \twoheadrightarrow I\mathcal{R}$ (as $\dim \mathcal{R} = n$). By 3.9, w_I can be lifted to a surjection from $R^n \twoheadrightarrow I$ and hence is a global orientation of I. □

Theorem 4.5. *Assume $(*)$. Let P be a projective R-module of* rank n *with trivial determinant and let $I \subset R$ be an ideal of height n. Assume that we are given a surjection $\psi : P \twoheadrightarrow I/I^2$. Assume* *further that $\psi \otimes \mathcal{R}$ can be lifted to a surjection $\Psi : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$. Then there exists a surjection $\widetilde{\Psi} : P \twoheadrightarrow I$, which is a lift of ψ.*

Proof. Let $J = I \cap \mathcal{J}(A)$. Then $\operatorname{ht} J \geqslant 2$. By 3.1, ψ can be lifted to $\Phi : P \twoheadrightarrow I \cap I'$, where $\operatorname{ht} I' = n$ and $I' + J^2 = R$.

Fix $\chi : R \xrightarrow{\sim} \bigwedge^n P$. Let $\lambda : (R/(I \cap I'))^n \xrightarrow{\sim} P/(I \cap I')P$ such that $\bigwedge^n \lambda = \chi \otimes R/(I \cap I')$. Then $e(P, \chi) = (I \cap I', w_{I \cap I'})$ in $E(R)$, where $w_{I\cap I'} = (\Phi \otimes R/(I \cap I'))\lambda$. Therefore, $e(P, \chi) = (I, w_I) + (I', w_{I'})$, where w_I and $w_{I'}$ are local orientations of I and I', respectively, induced from $w_{I\cap I'}$.

Since $e(P \otimes \mathcal{R}, \chi \otimes \mathcal{R}) = (I\mathcal{R}, w_I \otimes \mathcal{R})$ (using Ψ), $(I'\mathcal{R}, w_{I'} \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$, i.e. $w_{I'} \otimes \mathcal{R}$ can be lifted to a surjection from $\mathcal{R}^n$ to $I'\mathcal{R}$. By 3.9, $w_{I'}$ can be lifted to a set of n generators of I', say $I' = (f_1, \ldots, f_n)$. Since $I' + \mathcal{J}(A) = R$ and $\operatorname{ht} I' = n$, $\dim R/I' = 0$. Hence, applying 2.3, 2.4 and 2.5, after performing some elementary transformation on the generators of I', we can assume that

(1) $\operatorname{ht}(f_1, \ldots, f_{n-1}) = n - 1$,
(2) $\dim R/(f_1, \ldots, f_{n-1}) \leqslant 1$, and
(3) $f_n = 1$ modulo J^2.

Write $C = R[Y]$, $K_1 = (f_1, \ldots, f_{n-1}, Y + f_n)$, $K_2 = IC$ and $K_3 = K_1 \cap K_2$.

Claim. *There exists a surjection $\Delta(Y) : P[Y] \twoheadrightarrow K_3$ such that $\Delta(0) = \Phi$.*

First we show that the theorem follows from the claim. Specializing $\Delta(Y)$ at $Y = 1 - f_n$, we obtain a surjection $\Delta_1 : P \twoheadrightarrow I$. Since $1 - f_n \in J^2 \subset I^2$, $\Delta_1 = \Phi$ modulo I^2. Therefore, Δ_1 is a lift of ψ. This proves the result.

Proof of the claim. λ induces an isomorphism $\delta : (R/I')^n \xrightarrow{\sim} P/I'P$ such that $\bigwedge^n \delta = \chi \otimes R/I'$. Also, $(\Phi \otimes R/I')\delta = w_{I'}$. Since $\dim C/K_1 = \dim R/(f_1, \ldots, f_{n-1}) \leqslant 1$, and P has trivial determinant, by 2.1, $P[Y]/K_1P[Y]$ is free of rank n. Choose an isomorphism $\Gamma(Y) : (C/K_1)^n \xrightarrow{\sim} P[Y]/K_1P[Y]$ such that $\bigwedge^n(\Gamma(Y)) = \chi \otimes C/K_1$.

Since $\bigwedge^n \delta = \chi \otimes R/I'$, $\Gamma(0)$ and δ differ by an element of $\mathrm{SL}_n(R/I')$. Since $\dim R/I' = 0$, $\mathrm{SL}_n(R/I') = E_n(R/I')$. Therefore, we can alter $\Gamma(Y)$ by an element of $\mathrm{SL}_n(C/K_1)$ and assume that $\Gamma(0) = \delta$.

Let $\Lambda(Y) : (C/K_1)^n \twoheadrightarrow K_1/K_1^2$ be the surjection induced by the set of generators $(f_1, \ldots, f_{n-1}, Y + f_n)$ of K_1. Thus, we get a surjection

$$\Delta(Y) = \Lambda(Y)\Gamma(Y)^{-1} : P[Y]/K_1P[Y] \twoheadrightarrow K_1/K_1^2.$$

Since $\Gamma(0) = \delta$, $\Phi \otimes R/I' = w_{I'}\delta^{-1}$ and $\Lambda(0) = w_{I'}$, we have $\Delta(0) = \Phi \otimes R/I'$. By 2.7, we get a surjection $\widetilde{\Delta} : P[Y] \twoheadrightarrow K_3$ such that $\widetilde{\Delta}(0) = \Phi$. This proves the claim. □

Lemma 4.6. *Assume (∗). Let P be a projective R-module of* rank n *having trivial determinant and $\chi : R \xrightarrow{\sim} \bigwedge^n P$. Let $e(P, \chi) = (I, w_I)$ in $E(R)$, where $I \subset R$ is an ideal of height n. Then there exists a surjection $\Delta : P \twoheadrightarrow I$ such that (I, w_I) is obtained from (Δ, χ).*

Proof. Since $\dim R/I \leqslant 1$ and P has trivial determinant, by 2.1, P/IP is a free R/I-module of rank n. Choose $\lambda : (R/I)^n \xrightarrow{\sim} P/IP$ such that $\bigwedge^n \lambda = \chi \otimes R/I$. Let $\gamma = w_I \lambda^{-1} : P/IP \twoheadrightarrow I/I^2$.

Since $e(P \otimes \mathcal{R}, \chi \otimes \mathcal{R}) = (I\mathcal{R}, w_I \otimes \mathcal{R})$ in $E(\mathcal{R})$, by 2.10, there exists a surjection $\Gamma : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$ such that $(I\mathcal{R}, w_I \otimes \mathcal{R})$ is obtained from the pair $(\Gamma, \chi \otimes \mathcal{R})$, i.e. Γ is a lift of $\gamma \otimes \mathcal{R}$. Applying 4.5, there exists a surjection $\Delta : P \twoheadrightarrow I$ such that Δ is a lift of γ. Since $(\Delta \otimes R/I)\lambda = w_I$ and $\bigwedge^n(\lambda) = \chi \otimes R/I$, (I, w_I) is obtained from the pair (Δ, χ). □

The following result is essentially 3.1.

Lemma 4.7. *Assume* $(*)$. *Let* $(I, w_I) \in E(R)$. *Then there exist an ideal* $I_1 \subset R$ *of height* n *and a local orientation* w_{I_1} *of* I_1 *such that* $(I, w_I) + (I_1, w_{I_1}) = 0$ *in* $E(R)$. *Further,* I_1 *can be chosen to be comaximal with any ideal* $K \subset R$ *of height* $\geqslant 2$.

Corollary 4.8. *Assume* $(*)$. *Let* P *be a projective* R*-module of* rank n *with trivial determinant and* $\chi : R \xrightarrow{\sim} \bigwedge^n(P)$. *Then* $e(P, \chi) = 0$ *if and only if* P *has a unimodular element. In particular, if* P *has a unimodular element, then*

(1) *P maps onto any ideal of height n generated by n elements* (*see Lemma* 4.6).
(2) *Let $\beta : P \twoheadrightarrow I$ be a surjection, where I is an ideal of R of height n. Then I is generated by n elements.*

Proof. Let $\alpha : P \twoheadrightarrow I$ be a surjection, where $I \subset R$ is an ideal of height n. Let $e(P, \chi) = (I, w_I)$ in $E(R)$, where (I, w_I) is obtained from the pair (α, χ).

Assume that $e(P, \chi) = 0$ in $E(R)$. Then $(I, w_I) = 0$ in $E(R)$. By 4.7, there exist an ideal I' of height n such that $I' + \mathcal{J}(A) = R$ and a local orientation $w_{I'}$ of I' such that $(I, w_I) + (I', w_{I'}) = 0$ in $E(R)$. Since $(I, w_I) = 0$, $(I', w_{I'}) = 0$ in $E(R)$. Hence, without loss of generality, we can assume that $I + \mathcal{J}(A)R = R$.

By 4.4, I is generated by n elements, say $I = (f_1, \dots, f_n)$. Since $I + \mathcal{J}(A)R = R$, $\dim R/I = 0$. Hence, applying 2.3 and 2.4, after performing some elementary transformations on the generators of I, we can assume that $\dim R/(f_1, \dots, f_{n-1}) \leqslant 1$.

Let $C = R[Y]$ and $K = (f_1, \dots, f_{n-1}, Y + f_n)$ be an ideal of C. We have two surjections, $\alpha : P \twoheadrightarrow K(0) (= I)$ and $\phi : P[Y]/KP[Y] \twoheadrightarrow K/K^2$, such that $\phi(0) = \alpha \bmod K(0)^2$, where ϕ is the composition of two maps, $\phi_1 : P[Y]/KP[Y] \xrightarrow{\sim} (C/K)^n$ with $\bigwedge^n \phi_1 = \chi^{-1} \otimes C/K$ and $\phi_2 : (C/K)^n \twoheadrightarrow K/K^2$ defined by $(f_1, \dots, f_{n-1}, Y + f_n)$. Applying 2.7 with $I_1 = K$ and $I_2 = C$, we get a surjection $\Phi : P[Y] \twoheadrightarrow K$. Since $\Phi(1 - f_n) : P \twoheadrightarrow R$, P has a unimodular element.

Conversely, we assume that P has a unimodular element. Applying 2.10, we have $(I\mathcal{R}, w_I \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$. By 3.9, $(I, w_I) = 0 = e(P, \chi)$ in $E(R)$. This proves the result. □

The following result is a direct consequence of 3.9.

Theorem 4.9. *Assume* $(*)$. *Then the canonical map* $E(R) \to E(\mathcal{R})$ *is injective.*

Assume $(*)$. We have a canonical map $\Phi : E(A) \to E(R)$. It is easy to see that Φ is injective. It is natural to ask, when is Φ surjective? First, we prove an analogue of [4, Theorem 4.13] for $A[T, T^{-1}]$.

Theorem 4.10. *Let A be a regular domain of dimension d essentially of finite type over an infinite perfect field k and $R = A[T, T^{-1}]$. Let n be an integer such that $2n \geqslant d + 3$. Let $I \subset R$ be an ideal of height n and let P be a projective A-module of* rank n*. Assume that I contains some $f \in A[T]$ such that either f is a monic polynomial or $f(0) = 1$. Then any surjection $\phi : P \otimes R \twoheadrightarrow I/I^2$ can be lifted to a surjection $\Phi : P \otimes R \twoheadrightarrow I$.*

Proof. First we assume that $f(0) = 1$. Let $J = I \cap A[T]$. Let $\psi : P \otimes R \to I$ be a lift of ϕ. Since $(P \otimes R)^* = P^* \otimes R$, there exists $\widetilde{\psi} \in P[T]^*$ such that $\psi = \widetilde{\psi}/T^r$ for some positive integer r. It follows that $\widetilde{\psi} : P[T] \to J$. Let $\Psi : P[T] \to J/J^2$ be the map induced by $\widetilde{\psi}$. Since $\Psi_T = \phi$ and $(J/J^2)_f = 0$, we get that Ψ is a surjection. Since $f \in I$, by [4, Lemma 3.5], Ψ can be lifted to a surjection $\Delta : P[T] \twoheadrightarrow J/J^2(f-1)$. Since $f - 1 \in (T)$, Δ induces a surjection $\widetilde{\Delta} : P[T] \twoheadrightarrow J/J^2T$. Applying [4, Theorem 4.13], we get a surjection $\Phi : P[T] \twoheadrightarrow J$ which lifts $\widetilde{\Delta}$ and hence Ψ. Now, $T^r(\Phi \otimes R) : P \otimes R \twoheadrightarrow I$ is a lift of ϕ. This proves the result in the case $f(0) = 1$.

Now, we assume that $f(T)$ is a monic polynomial. Let $J = I \cap A[X]$, where $X = T^{-1}$. Then J contains an element $g(X) = T^{-r}f(T)$, where $r = \deg f$. Note that $g(0) = 1$. Now, we are reduced to the previous case. □

As a consequence of 4.10, we have the following result.

Theorem 4.11. *Let A be a regular domain of dimension $n \geqslant 3$ essentially of finite type over an infinite perfect field k with $\operatorname{ht} \mathcal{J}(A) \geqslant 2$. Let $(I, w_I) \in E(A[T, T^{-1}])$. Assume that I contains some $f(T) \in A[T]$ such that either f is a monic polynomial or $f(0) = 1$. Then $(I, w_I) = 0$.*

Remark 4.12. In [15], 4.10 is proved for an arbitrary ring under the assumption that I contains a special monic polynomial. Hence 4.11 is valid for an arbitrary ring if I contains a special monic polynomial.

Let A be a ring of dimension n containing an infinite field and let P be a projective $A[T]$-module of rank n. In [9], it is proved that if $P_{f(T)}$ has a unimodular element for some monic polynomial $f(T) \in A[T]$, then P has a unimodular element. We will prove the analogous result for $A[T, T^{-1}]$.

Theorem 4.13. *Assume $(*)$. Let P be a projective R-module of* rank n *with trivial determinant. If $P_{f(T)}$ has a unimodular element for some special monic polynomial $f(T) \in A[T]$, then P has a unimodular element.*

Proof. Fix $\chi : R \xrightarrow{\sim} \bigwedge^n(P)$. Since P_f has a unimodular element, $e(P \otimes \mathcal{R}, \chi \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$. By 4.9, $e(P, \chi) = 0$ in $E(R)$. Hence P has a unimodular element, by 4.8. □

5. Weak Euler class group of $A[T, T^{-1}]$

Results in this section are similar to [10, Section 5]. Assume $(*)$. We define the *nth weak Euler class group* $E_0^n(R)$ of R in the following way:

Let G be the free abelian group on (I), where $I \subset R$ is an ideal of height n with the property that I/I^2 is generated by n elements and $\operatorname{Spec}(R/I)$ is connected. Let $I \subset R$ be an ideal of height n such that I/I^2 is generated by n elements. Then I can be decomposed as $I =$

$I_1 \cap \cdots \cap I_r$, where I_i's are pairwise comaximal ideals of height n and $\operatorname{Spec}(R/I_i)$ is connected for each i. In the previous section, we have seen that such a decomposition of I is unique. By (I), we denote the element $\sum_i (I_i)$ of G.

Let H be the subgroup of G generated by elements of the type (I), where $I \subset R$ is an ideal of height n such that I is generated by n elements.

We define $E_0^n(R) = G/H$. By abuse of notation, we will write $E_0(R)$ for $E_0^n(R)$ in what follows. Note that there is a canonical surjective homomorphism from $E(R)$ to $E_0(R)$ obtained by forgetting the orientations.

Remark 5.1. Assume $(*)$. Let $I \subset R$ be an ideal of height n and let $w_I : (R/I)^n \twoheadrightarrow I/I^2$ be a local orientation of I. Let $\theta \in \mathrm{GL}_n(R/I)$ be such that $\det\theta = \overline{f}$. Then $w_I\theta$ is another orientation of I, which we denote by $\overline{f} w_I$. On the other hand, if w_I and $\widetilde{w}_I$ are two local orientations of I, then by [8, Lemma 2.2], it is easy to see that $\widetilde{w}_I = \overline{f} w_I$ for some unit $\overline{f} \in R/I$.

The proof of the following lemma is contained in [8, 2.7, 2.8 and 5.1] and, hence, we omit the proof.

Lemma 5.2. *Assume $(*)$. Let P be a projective R-module of* rank n *having trivial determinant and $\chi : R \xrightarrow{\sim} \bigwedge^n P$. Let $\alpha : P \twoheadrightarrow I$ be a surjection, where $I \subset R$ is an ideal of height n. Let (I, w_I) be obtained from (α, χ). Let $f \in R$ be a unit mod I. Then there exist a projective R-module P_1 of* rank n *such that $[P] = [P_1]$ in $K_0(R)$, $\chi_1 : R \xrightarrow{\sim} \bigwedge^n P_1$, and a surjection $\beta : P_1 \twoheadrightarrow I$ such that $(I, \overline{f^{n-1}} w_I)$ is obtained from (β, χ_1).*

The following lemma can be proved using [8, Lemmas 5.3, 5.4] and 3.9.

Lemma 5.3. *Assume $(*)$. Let $(I, w_I) \in E(R)$. Let $\overline{f} \in R/I$ be a unit. Then $(I, w_I) = (I, \overline{f^2} w_I)$ in $E(R)$.*

Adapting the proof of [7, Lemma 3.7] and using 2.5 in place of Swan's Bertini theorem, the proof of the following lemma follows.

Lemma 5.4. *Assume $(*)$ with n even. Let P be a stably free R-module of* rank n *and $\chi : R \xrightarrow{\sim} \bigwedge^n P$. Suppose that $e(P, \chi) = (I, w_I)$ in $E(R)$. Then $(I, w_I) = (I_1, w_{I_1})$ in $E(R)$ for some ideal $I_1 \subset R$ of height n generated by n elements. Moreover, I_1 can be chosen to be comaximal with any ideal of R of height $\geqslant 2$.*

The following result can be proved by adapting the proofs of [7, 3.8, 3.9, 3.10, 3.11].

Proposition 5.5. *Assume $(*)$ with n even. Then we have the following*:

(1) *Let $I_1, I_2 \subset R$ be two comaximal ideals of height n and $I_3 = I_1 \cap I_2$. If any two of I_1, I_2 and I_3 are surjective images of stably free R-modules of* rank n, *then so is the third.*
(2) *Let $(I, w_I) \in E(R)$. Then $(I) = 0$ in $E_0(R)$ if and only if I is a surjective image of a stably free projective R-module of* rank n.
(3) *Let P be a projective R-module of* rank n *with trivial determinant. Then $e(P) = 0$ in $E_0(R)$ if and only if $[P] = [Q \oplus R]$ in $K_0(R)$ for some projective R-module Q of* rank $n - 1$.

(4) *Let P be a projective R-module of* rank n *with trivial determinant. Suppose that $e(P) = (I)$ in $E_0(R)$, where $I \subset R$ is an ideal of height n. Then there exists a projective R-module Q of* rank n *such that $[Q] = [P]$ in $K_0(R)$ and I is a surjective image of Q.*

The proof of the following result is the same as of [8, Proposition 6.5] using the above results.

Theorem 5.6. *Assume $(*)$ with n even. Let $(I, \widetilde{w_I}) \in E(R)$ belong to the kernel of the canonical homomorphism $E(R) \twoheadrightarrow E_0(R)$. Then there exists a stably free R-module P_1 of* rank n *and $\chi_1 : R \xrightarrow{\sim} \bigwedge^n P_1$ such that $e(P_1, \chi_1) = (I, \widetilde{w_I})$ in $E(R)$.*

6. The case of dimension two

In this section, we briefly outline the results similar to those in the previous sections in the case when dimension of the base ring is two. The results of this section are similar to [10, Section 6], where they are proved for $A[T]$.

We begin by stating the following result of Mandal [14].

Lemma 6.1. *Let A be a ring and $R = A[T, T^{-1}]$. Let P be a projective R-module. Let $f \in R$ be a special monic polynomial. If P_f is free, then P is free.*

The proof of the following result is similar to [10, Theorem 7.1].

Theorem 6.2. *Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. Let $I \subset R$ be an ideal of height 2 such that $I = (f_1, f_2) + I^2$. Suppose that there exist $F_1, F_2 \in I\mathcal{R}$ such that $I\mathcal{R} = (F_1, F_2)$ and $F_i - f_i \in I^2\mathcal{R}$ for $i = 1, 2$. Then there exist $h_1, h_2 \in I$ and $\theta \in \mathrm{SL}_2(R/I)$ such that $I = (h_1, h_2)$ and $(\overline{f}_1, \overline{f}_2)\theta = (\bar{h}_1, \bar{h}_2)$, where "bar" denotes reduction modulo I.*

Proof. Since a unimodular row of length two is always completable to a matrix of determinant 1, it follows (using patching argument) that there is a projective R-module P of rank 2 with trivial determinant mapping onto I. Let $\alpha : P \twoheadrightarrow I$ be the surjection. Fix $\chi : R \xrightarrow{\sim} \bigwedge^2 P$. Since $\dim R/I \leqslant 1$, by 2.1, P/IP is free of rank 2. Hence α and χ induce a set of generators of I/I^2, say $I = (g_1, g_2) + I^2$.

It is easy to see that there exists a matrix $\bar{\sigma} \in \mathrm{GL}_2(R/I)$ with determinant $\overline{f}$ such that $(\overline{f}_1, \overline{f}_2) = (\bar{g}_1, \bar{g}_2)\bar{\sigma}$. Now, following [8, Lemmas 2.7 and 2.8], there exist a projective R-module P_1 of rank 2 having trivial determinant, $\chi_1 : R \xrightarrow{\sim} \bigwedge^2 P_1$, and a surjection $\beta : P_1 \twoheadrightarrow I$ such that if the set of generators of I/I^2 induced by β and χ_1 is $\bar{h}_1, \bar{h}_2$, then $(\bar{h}_1, \bar{h}_2) = (\bar{g}_1, \bar{g}_2)\bar{\delta}$, where $\bar{\delta} \in \mathrm{GL}_2(R/I)$ has determinant $\overline{f}$. Therefore, the two sets of generators, $(\overline{f}_1, \overline{f}_2)$ and $(\bar{h}_1, \bar{h}_2)$ of I/I^2, are connected by a matrix in $\mathrm{SL}_2(R/I)$.

From the above discussion, it is easy to see that $e(P_1 \otimes \mathcal{R}, \chi_1 \otimes \mathcal{R}) = (I\mathcal{R}, w_I \otimes \mathcal{R})$ in $E(\mathcal{R})$, where $w_I : (R/I)^2 \twoheadrightarrow I/I^2$ is the surjection corresponding to the generators $(\overline{f}_1, \overline{f}_2)$. Therefore, from the given condition of the theorem, it follows that $(I\mathcal{R}, w_I \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$. Hence, we have $e(P_1 \otimes \mathcal{R}, \chi_1 \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$. Since $\dim \mathcal{R} = 2$, by 2.10, $P_1 \otimes \mathcal{R}$ has a unimodular element and hence is free (as rank $P_1 = 2$ and determinant of P_1 is trivial). Therefore, by 6.1, P_1 is a free R-module.

Assume that the surjection β is given by h_1, h_2. Then $I = (h_1, h_2)$ and $(\overline{f}_1, \overline{f}_2)\theta = (\bar{h}_1, \bar{h}_2)$, for some $\theta \in \mathrm{SL}_2(R/I)$. This proves the result. □

As applications of the above theorem, we prove the following results.

Corollary 6.3 *(Addition Principle). Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. Let $I_1, I_2 \subset R$ be two comaximal ideals of height 2. Suppose that $I_1 = (f_1, f_2)$ and $I_2 = (g_1, g_2)$. Then there exist $h_1, h_2 \in I_1 \cap I_2$ and $\theta_i \in \mathrm{SL}_2(R/I_i)$, $i = 1, 2$, such that $I_1 \cap I_2 = (h_1, h_2)$ and $(f_1, f_2) \otimes R/I_1 = ((h_1, h_2) \otimes R/I_1)\theta_1$ and $(g_1, g_2) \otimes R/I_2 = ((h_1, h_2) \otimes R/I_2)\theta_2$.*

Proof. Write I for $I_1 \cap I_2$. The generators of I_1 and I_2 induce a set of generators of I/I^2, say $I = (H_1, H_2) + I^2$. Since $\dim \mathcal{R} = 2$, applying 2.8 in the ring $\mathcal{R}$, we get $I\mathcal{R} = (F_1, F_2)$ with $F_i - f_i \in {I_1}^2\mathcal{R}$ and $F_i - g_i \in {I_2}^2\mathcal{R}$. Hence, it is easy to see that $F_i - H_i \in I^2\mathcal{R}$, for $i = 1, 2$.

Applying 6.2, there exist $h_1, h_2 \in I$ and $\theta \in \mathrm{SL}_2(R/I)$ such that $I = (h_1, h_2)$ and $(H_1, H_2) \otimes R/I = ((h_1, h_2) \otimes R/I)\theta$. Let $\theta_i = \theta \otimes R/I_i$. Then $\theta_i \in \mathrm{SL}_2(R/I_i)$, $i = 1, 2$, and we have $(f_1, f_2) \otimes R/I_1 = ((h_1, h_2) \otimes R/I_1)\theta_1$ and $(g_1, g_2) \otimes R/I_2 = ((h_1, h_2) \otimes R/I_2)\theta_2$. □

Corollary 6.4 *(Subtraction Principle). Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. Let $I_1, I_2 \subset R$ be two comaximal ideals of height 2. Suppose that $I_1 = (f_1, f_2)$ and $I_1 \cap I_2 = (h_1, h_2)$ such that $f_i - h_i \in I_1^2$, for $i = 1, 2$. Then there exist $g_1, g_2 \in I_2$ and $\theta \in \mathrm{SL}_2(R/I_2)$ such that $I_2 = (g_1, g_2)$ and $(g_1, g_2) \otimes R/I_2 = ((h_1, h_2) \otimes R/I_2)\theta$.*

Proof. We have $I_2 = (h_1, h_2) + I_2^2$. Since $\dim \mathcal{R} = 2$, applying 2.9 in the ring $\mathcal{R}$, we get that $I_2\mathcal{R} = (G_1, G_2)$ with $G_i - h_i \in I_2^2\mathcal{R}$. Now, applying 6.2, we get the result. □

Remark 6.5. Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. We can define the *Euler class group* and the *weak Euler class group* of R in exactly the same way as we did in the previous sections. The only difference is that, for an ideal I of R of height 2, a local orientation $[\alpha]$ will be called a global orientation if there is a surjection $\theta : R^2 \twoheadrightarrow I$ and some $\sigma \in \mathrm{SL}_2(R/I)$ such that $\alpha\sigma = \theta \otimes R/I$. For a rank 2 projective R-module P having trivial determinant, the Euler class of P is defined as in the previous section.

The following result can be proved using 6.2 and 2.10 ((i) follows from 4.4, (ii)'s proof is similar to [10, Theorem 7.6] using A.7 and (iii), (iv) follow from 6.1).

Theorem 6.6. *Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. Let $I \subset R$ be an ideal of height 2 such that I/I^2 is generated by 2 elements. Let $w_I : (R/I)^2 \twoheadrightarrow I/I^2$ be a local orientation of I. Let P be a projective R-module of* rank 2 *with trivial determinant and $\chi : R \xrightarrow{\sim} \bigwedge^2 P$. We have the following results*:

- (i) *Suppose that the image of (I, w_I) is zero in $E(R)$. Then I is generated by 2 elements and w_I is a global orientation of I.*
- (ii) *Suppose that $e(P, \chi) = (I, w_I)$ in $E(R)$. Then there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I, w_I) is obtained from (α, χ).*
- (iii) *$e(P, \chi) = 0$ in $E(R)$ if and only if P has a unimodular element and hence P is free.*
- (iv) *The canonical map $E(R) \to E(\mathcal{R})$ is injective.*

Remark 6.7. Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. Let $I \subset R$ be an ideal of height 2 such that I/I^2 is generated by 2 elements and let w_I be a local orientation of I. It is easy to see, as in 6.2, that there exist a projective R-module P of rank 2 together with an isomorphism $\chi : R \xrightarrow{\sim} \bigwedge^2 P$ and a surjection $\alpha : P \twoheadrightarrow I$ such that (I, w_I) is obtained from the pair (α, χ)

The theory of weak Euler class group described in the last section also follows in a like manner in the two-dimensional case.

7. Relations between $E(R)$ and $\widetilde{K_0}Sp(R)$

In this section, we prove results similar to [8, Section 7].

Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. Let $\widetilde{K}_0Sp(R)$ be the set of isometry classes of (P, s), where P is a projective R-module of rank 2 with trivial determinant and $s : P \times P \to R$ a non-degenerate alternating bilinear form. We note that there is (up to isometry) a unique non-degenerate alternating bilinear form on R^2, which we denote by h, namely $h((a, b), (c, d)) = ad - bc$. We write $H(R)$ for (R^2, h).

We define a binary operation $*$ on $\widetilde{K}_0Sp(R)$ as follows. Let (P_1, s_1) and (P_2, s_2) be two elements of $\widetilde{K}_0Sp(R)$. Since $\dim A = 2$, $R = A[T, T^{-1}]$ and $P_1 \oplus P_2$ has rank 4, hence by 2.1, $P_1 \oplus P_2$ has a unimodular element, say p. Then there exists $q \in P_1 \oplus P_2$ such that if $s = s_1 \perp s_2$, then $s(p, q) = 1$. Let $P_3 = \{\widetilde{p} \in P_1 \oplus P_2 \mid s(p, \widetilde{p}) = 0 = s(q, \widetilde{p})\}$. Then the restriction $s_3 : P_3 \times P_3 \to R$ of s to P_3 is non-degenerate (i.e. (P_3, s_3) is symplectic) and $P_1 \oplus P_2 = (Rp \oplus Rq) \oplus P_3$. Hence $(P_1, s_1) \perp (P_2, s_2)$ is isometric to $(P_3, s_3) \perp (R^2, h)$. We define $(P_1, s_1) * (P_2, s_2) = (P_3, s_3)$. By A.7, (P_3, s_3) is determined uniquely up to isometry. Hence $*$ is a well defined operation and for every symplectic R-module (P, s) of rank 2, $(P, s) * (R^2, h) = (P, s)$. Hence $\widetilde{K}_0Sp(R)$ is a commutative semigroup under $*$ with the isometry class of (R^2, h) as the identity element. We will briefly indicate that in fact $\widetilde{K}_0Sp(R)$ is an abelian group under $*$.

For a projective R-module P of rank 2 with trivial determinant, the alternating bilinear form s_P on $P \oplus P^*$ defined by

$$s_P\big((p, f), (q, g)\big) = g(p) - f(q), \quad p, q \in P,\ f, g \in P^*$$

is non-degenerate. We write $H(P)$ for the symplectic module $(P \oplus P^*, s_P)$. If (P, s) is a symplectic R-module of rank 2, then $(P, s) \perp (P, -s) \xrightarrow{\sim} H(P)$ [21, Lemma A.3]. By [12, Theorem 2.1], every projective R-module of rank $\geqslant 3$ has a unimodular element. Hence, by 2.2, there exists a projective R-module P_1 of rank 2 such that $P \oplus P_1 \xrightarrow{\sim} R^4$. Therefore,

$$H(P_1) \perp (P, -s) \perp (P, s) \xrightarrow{\sim} H(P_1 \oplus P) \xrightarrow{\sim} H\big(R^4\big) \xrightarrow{\sim} H\big(R^2\big) \perp H(R) \perp H(R).$$

Since the symplectic module $H(P_1) \perp (P, -s)$ has rank 6, $H(P_1) \perp (P, -s) \xrightarrow{\sim} H(R^2) \perp (\widetilde{P}, \widetilde{s})$ for some symplectic R-module $(\widetilde{P}, \widetilde{s})$ of rank 2. By Bass result [2],

$$(\widetilde{P}, \widetilde{s}) \perp (P, s) \xrightarrow{\sim} H(R) \perp H(R)$$

and therefore $(\widetilde{P}, \widetilde{s}) * (P, s) = H(R)$. Thus, $\widetilde{K}_0Sp(R)$ is an abelian group under $*$.

Let P be a projective R-module of rank 2 with trivial determinant. Then having a non-degenerate alternating bilinear form s on P is equivalent to giving an isomorphism $\lambda : \bigwedge^2 P \xrightarrow{\sim} A$. Thus, we can identify the pair (P, s) with (P, χ), where χ is the generator of $\bigwedge^2 P$ given by $\lambda^{-1}(1)$. It is easy to see that the isometry classes of (P, s) coincide with the isomorphism classes of (P, χ).

We will begin with the following result, the proof of which is the same as of [8, Theorem 7.2].

Theorem 7.1. *Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. Then the map from $\widetilde{K}_0Sp(R)$ to $E(R)$ sending (P, χ) to $e(P, \chi)$ is an isomorphism.*

Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. Let G be the set of isometry classes of non-degenerate alternating bilinear forms on R^4. Let $H(R^4) = (R^2, h) \perp (R^2, h)$. As before, we can define the group structure on G as follows: We set $(R^4, s_1) * (R^4, s_2) = (R^4, s_3)$, where s_3 is the unique (up to isometry) alternating bilinear form on R^4 satisfying the property that $(R^4, s_1) \perp (R^4, s_2)$ is isometric to $(R^4, s_3) \perp H(R^2)$. Then G is a group with $H(R^2)$ as the identity element. Let s be a non-degenerate alternating bilinear form on R^4. Since $\dim A = 2$ and $R = A[T, T^{-1}]$, by 2.2, we get $(R^4, s) \xrightarrow{\sim} (P, s') \perp (R^2, h)$. The assignment sending (R^4, s) to (P, s') gives rise to an injective homomorphism from G to $\widetilde{K}_0Sp(R)$.

In view of the above theorem, we have the following result, the proof of which is the same as [8, Theorem 7.3].

Theorem 7.2. *Let A be a ring of dimension 2 and $R = A[T, T^{-1}]$. Then we have the following exact sequence*

$$0 \to G \to \widetilde{K}_0Sp(R)\big(\xrightarrow{\sim} E(R)\big) \to E_0(R) \to 0.$$

Corollary 7.3. *Assume $(*)$. Let (I, w_I) be an element of $E(R)$ such that its image in $E_0(R)$ (which is independent of w_I) is zero. Then the element $(I, w_I) + (I, -w_I) = 0$ in $E(R)$.*

Proof. Let $(I, w_I) + (I, -w_I) = (J, w_J)$ in $E(R)$. Since $\dim \mathcal{R} = n$, applying [8, Corollary 7.9] in the ring $\mathcal{R}$, we get that $(J \otimes \mathcal{R}, w_J \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$. By 4.9, $(J, w_J) = 0$ in $E(R)$. This proves the result. □

As an application of 7.3, following the proof of [8, Corollary 7.10], we have the following result.

Corollary 7.4. *Assume $(*)$ with n odd. Let P be a projective R-module of* rank n *having trivial determinant. Assume that the kernel of the canonical surjection $E(R) \twoheadrightarrow E_0(R)$ has no non-trivial 2-torsion. If $e(P) = 0$ in $E_0(R)$, then P has a unimodular element.*

Following the proof of [8, Theorem 7.13] gives the following result.

Theorem 7.5. *Assume $(*)$ with n odd. Let P be a projective R-module of* rank n *having trivial determinant. Suppose that there exists a projective R-module Q of* rank $n-1$ *such that $[P] = [Q \oplus R]$ in $K_0(R)$. Then P has a unimodular element.*

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Appendix A

We will freely use results and notations from [3]. Let $(P, \langle\,,\rangle)$ be an A-module with an alternating bilinear form $\langle\,,\rangle$ (P need not be projective and $\langle\,,\rangle$ need not be non-degenerate). Let $E(A^2 \perp P, \langle\,,\rangle)$ denote the subgroup of $\mathrm{Aut}(A^2 \perp P, \langle\,,\rangle)$ generated by $\theta_{(c,q)}$ and $\sigma_{(d,q)}$ for $c, d \in A$ and $q \in P$, where $\theta_{(c,q)}$ and $\sigma_{(d,q)}$ are defined as

$$\theta_{(c,q)}(a,b,p) = \big(a, b + ca + \langle p, q\rangle, p + aq\big),$$
$$\sigma_{(d,q)}(a,b,p) = \big(a + bd + \langle q, p\rangle, b, p + bq\big)$$

for $(a, b, p) \in A^2 \oplus P$.

Remark A.6. It is easy to see that [3, Lemmas 4.3, 4.5, 4.7] holds for $(P, \langle\,,\rangle)$ replacing $ESp(A^2 \perp P, \langle\,,\rangle)$ with $E(A^2 \perp P, \langle\,,\rangle)$ with further assumption in 4.5 that $sP \subset F$.

The following result is a symplectic analogue of 2.3 and is a generalization of [2] and [3, Theorem 4.8], where it is proved for $r = r' = 0$ and $r = 0$, respectively. Our proof closely follows [3].

Theorem A.7. *Let B be a ring of dimension d and $A = B[Y_1, \dots, Y_{r'}, X_1^{\pm 1}, \dots, X_r^{\pm 1}]$. Let $(P, \langle\,,\rangle)$ be a symplectic A-module of* rank $2n > 0$. *If $2n \geqslant d$, then $ESp(A^2 \perp P, \langle\,,\rangle)$ acts transitively on* $\mathrm{Um}(A^2 \oplus P)$.

Proof. Let $(g_1, g_2, p) \in \mathrm{Um}(A^2 \oplus P)$. We want to show that there exists $\Gamma \in ESp(A^2 \perp P, \langle\,,\rangle)$ such that $\Gamma(g_1, g_2, p) = (1, 0, 0)$. We prove the result by induction on r.

If $r = 0$, then the result follows from [3, Theorem 4.8]. Hence, we assume that the result is proved for $r - 1$ and $r \geqslant 1$. For the sake of simplicity, we write $R = B[Y_1, \dots, Y_{r'}, X_1^{\pm 1}, \dots, X_{r-1}^{\pm 1}]$ and $X_r = X$.

Without loss of generality, we can assume that B is reduced. Let S be the set of non-zero divisors of B. Then B_S is a finite direct product of fields and therefore, by [19,20], every projective A_S-module is free. Hence, we can find a basis $\widetilde{p_1}, \dots, \widetilde{p_n}, \widetilde{q_1}, \dots, \widetilde{q_n}$ of P_S such that $\langle \widetilde{p_i}, \widetilde{p_j}\rangle = 0 = \langle \widetilde{q_i}, \widetilde{q_j}\rangle$ for $1 \leqslant i, j \leqslant n$, $\langle \widetilde{p_i}, \widetilde{q_i}\rangle = 1$ for $1 \leqslant i \leqslant n$ and $\langle \widetilde{p_i}, \widetilde{q_j}\rangle = 0$ for $1 \leqslant i, j \leqslant n$, $i \neq j$.

We can choose some $t \in S$ such that $\widetilde{p_i} = e_i/t$, $\widetilde{q_i} = f_i/t$ for some $e_i, f_i \in P$ for $1 \leqslant i \leqslant n$. Let $s = t^2$ and $F = \sum_{i=1}^n Ae_i + \sum_{i=1}^n Af_i$. Then, by [3, Lemma 4.2], F is a free A-submodule of P of rank $2n$ and $sP \subset F$.

Let $F_1 = \sum_{i=1}^n R[X]e_i + \sum_{i=1}^n R[X]f_i$. Let P be generated by $\mu_1, \dots, \mu_l$ as an A-module such that (1) the set $\mu_1, \dots, \mu_l$ contains $e_1, \dots, e_n, f_1, \dots, f_n$, (2) $s\mu_i \in F_1$ for $1 \leqslant i \leqslant l$, and (3) $\langle \mu_i, \mu_j\rangle \in R[X]$ for $1 \leqslant i, j \leqslant l$. Let $M = \sum_{i=1}^l R[X]\mu_i$. Then $MA = P$ and $sM \subset F_1$.

Since $s \in B$ is a non-zero-divisor, $B_1 = B[X^{\pm 1}]/(s(X-1))$ is a ring of dimension d and $\overline{A} = A/(s(X-1)) = B_1[Y_1, \dots, Y_{r'}, X_1^{\pm 1}, \dots, X_{r-1}^{\pm 1}]$. Moreover, since $\operatorname{rank} P \geqslant d$, by [11, Theorem 1.19], the map $\mathrm{Um}(A^2 \oplus P) \to \mathrm{Um}(\overline{A^2} \oplus (P/s(X-1)P))$ is surjective. Therefore, by [3, Lemma 4.1] and induction hypothesis, there exists $\Psi \in ESp(A^2 \perp P, \langle\,,\rangle)$ such that $\Psi(g_1, g_2, p) = (1, 0, 0)$ modulo $s(X-1)A$.

Replacing (g_1, g_2, p) with $\Psi(g_1, g_2, p)$, we may assume that $(g_1, g_2, p) = (1, 0, 0)$ modulo $s(X-1)A$. By 2.5, there exist $h \in A$ and $p_1 \in P$ such that $\mathrm{ht}(Ag_3 + I) \geqslant d + 1$, where $g_3 = g_1 + hg_2$, $p_2 = p + g_2 p_1$ and $I = p_2(P^*) = \langle P, p_2\rangle$. Put $\alpha = g_3 + \langle p_1, p\rangle \in A$. Then

$$\sigma_{(h,p_1)}(g_1, g_2, p) = \big(g_1 + g_2 h + \langle p_1, p\rangle, g_2, p + g_2 p_1\big) = (\alpha, g_2, p_2).$$

Since $(g_3, g_2, p) = (1, 0, 0)$ modulo $s(X-1)A$, we have $\alpha = 1$ modulo $s(X-1)A$. Moreover, since $\langle p_1, p_2\rangle = \langle p_1, p\rangle \in I$, hence $(g_3, I)A = (\alpha, I)A = (\alpha, s(X-1)I)A$. Now, since $(\alpha, s(X-1)I)A$ is an ideal of A of height $> d = \dim B$, by Mandal's theorem [13], $(\alpha, s(X-1)I)A$ contains a special monic polynomial, say γ, in the variable X. We write $\gamma = \gamma(X) \in R[X]$.

Let $\beta(X) = g_2 + \gamma(X)\gamma_1$ for some suitable $\gamma_1 \in A$ such that $\beta(X) \in R[X]$ is a special monic polynomial. Let $\gamma(X)\gamma_1 = \mu\alpha + \nu$ for some $\mu \in A$ and $\nu \in s(X-1)I$. Since $I = \langle P, p_2\rangle$, there exists $p_3 \in s(X-1)P$ such that $\nu = \langle -p_3, p_2\rangle = \langle p_2, p_3\rangle$. Put $p_4 = p_2 + \alpha p_3$. Then

$$\theta_{(\mu,p_3)}(\alpha, g_2, p_2) = \big(\alpha, g_2 + \mu\alpha + \langle p_2, p_3\rangle, p_2 + \alpha p_3\big) = \big(\alpha, \beta(X), p_4\big).$$

Note that $(\alpha, p_4) = (1, 0)$ modulo $s(X-1)A$ and $\beta(X)$ is special monic polynomial.

Since $sP \subset F$, let $p_4 = (X-1)(\sum_{i=1}^n h_i e_i + \sum_{j=1}^n k_j f_j)$ for some $h_i, k_j \in A$. Let $h_1 = -\lambda X^{-r_0} + \widetilde{h}_1$, where $\widetilde{h}_1 \in A$ has X^{-1} degree $\leqslant r_0 - 1$ and $\lambda \in R$. Let $a_0 = (X-1)X^{-r_0}\lambda$. Then

$$\sigma_{(0,a_0 e_1)}\big(\alpha, \beta(X), p_4\big) = \big(\alpha + a_0\langle e_1, p_4\rangle, \beta(X), p_4 + \beta(X)a_0 e_1\big).$$

Note that if $p_4 + a_0\beta(X)e_1 = (X-1)(e_1 h_{11} + \sum_{i=2}^n h_i e_i + \sum_{j=1}^n k_j f_j)$, then degree of X^{-1} in $h_{11} \in A$ is $\leqslant r_0 - 1$. Also note that $\alpha + a_0\langle e_1, p_4\rangle = 1$ modulo $s(X-1)A$. Hence, by induction on the X^{-1} degree, applying such symplectic transvections, say $\Psi_1 \in ESp(A^2 \perp P, \langle , \rangle)$, we can assume that if $\Psi_1(\alpha, \beta(X), p_4) = (\alpha_1, \beta(X), p_5)$, then $p_5 \in (X-1)F_1$. Now, we write p_5 as $p_5(X)$. We still have $\alpha_1 = 1$ mod $s(X-1)A$. Write $\Gamma_1 = \Psi_1\theta_{(\mu,p_3)}\sigma_{(h,p_1)}$. Then $\Gamma_1(g_1, g_2, p) = (\alpha_1, \beta(X), p_5(X))$.

Since $\sigma_{(d,0)}(\alpha_1, \beta(X), p_5(X)) = (\alpha_1 + \beta(X)d, \beta(X), p_5(X))$ for $d \in A$, applying symplectic transvections of the type $\sigma_{(d,0)}$, say Ψ_2, we may assume that if $\Psi_2(\alpha_1, \beta(X), p_5(X)) = (\alpha_2, \beta(X), p_5(X))$, then $\alpha_2 \in R[X]$ and $\alpha_2 = 1$ modulo $s(X-1)R[X]$. Now, we write α_2 as $\alpha_2(X)$. Since $\beta(0) = 1$, $(\alpha_2(X), \beta(X), p_5(X)) \in \mathrm{Um}(R[X]^2 \perp F_1, \langle , \rangle)$.

Let $\beta(X) = 1 - Xw$ and $\alpha_2(X) = 1 + s(X-1)w'$ for some $w, w' \in R[X]$. Then $s = sXw + s\beta(X)$ and $\alpha_2(X) = 1 + sXw' - (sXw + s\beta(X))w'$. Let $\alpha_3(X) = 1 + sXw'(1-w)$. Then $\sigma_{(sw',0)}(\alpha_2(X), \beta(X), p_5(X)) = (\alpha_3(X), \beta(X), p_5(X))$ with $\alpha_3(X) = 1$ modulo $sXR[X]$.

Since $(\alpha_3(X), s)R[X] = R[X]$ and $\beta(X)$ is monic, there exists $c \in R$ such that $1 - cs \in R \cap (\alpha_3(X), \beta(X))$. Recall that $sM \subset F_1$. Therefore, writing $b = 1$, $b' = 1 - sc$ and applying [3, Lemma 4.7], there exists $\Psi_3 \in SL_2(R[X], (sX))\, E(R[X]^2 \perp M, \langle , \rangle)$ such that

$$\Psi_3\big(\alpha_3(X), \beta(X), p_5(X)\big) = \big(\alpha_3(b'X), \beta(b'X), p_5(b'X)\big).$$

Since $\alpha_3(X) = 1$ modulo $(sX)R[X]$, we have $\alpha_3(b'X) = 1$ modulo $(sb'X)R[X]$. Moreover, $b' = 1 - cs \in R \cap (\alpha_3(b'X), \beta(b'X))$. Therefore, $[\alpha_3(b'X), \beta(b'X)]$ is a unimodular row.

Let $\Psi_3 = \Delta^{-1}\Phi$, where $\Delta \in \mathrm{SL}_2(R[X], (sX))$ and $\Phi \in E(R[X]^2 \perp M, \langle , \rangle)$. Let $\Delta(\alpha_3(b'X), \beta(b'X)) = (\alpha_4(X), \beta_1(X))$. Then

$$\Phi\big(\alpha_3(X), \beta(X), p_5(X)\big) = \big(\alpha_4(X), \beta_1(X), p_5(b'X)\big).$$

Since $\Delta \in \mathrm{SL}_2(R[X], (sX))$, hence $\alpha_4(X) = 1$ modulo $(sX)R[X]$ and $[\alpha_4(X), \beta_1(X)]$ is a unimodular row.

Write $\Gamma_2 = (\Phi \otimes A)(\sigma_{(sw',0)} \otimes A)\Psi_2\Gamma_1$. Then $\Gamma_2 \in Esp(A^2 \perp P, \langle\,,\rangle)$ and $\Gamma_2(g_1, g_2, p) = (\alpha_4(X), \beta_1(X), p_5(b'X))$ with $[\alpha_4(X), \beta_1(X)]$ a unimodular row. Therefore, by [3, Lemma 4.1], there exists $\Phi_1 \in ESp(A^2 \perp P, \langle\,,\rangle)$ such that

$$\Phi_1\big(\alpha_4(X), \beta_1(X), p_5(b'X)\big) = \big(\alpha_4(X), \beta_1(X), e_1\big).$$

Since $\langle e_1, f_1\rangle = s$, $(\alpha_4(X), e_1)$ is an element of $\mathrm{Um}(A \oplus P)$. Therefore, by [3, Lemma 4.4], there exists $\Phi_2 \in ESp(A^2 \perp P, \langle\,,\rangle)$ such that $\Phi_2(\alpha_4(X), \beta_1(X), e_1) = (1, 0, 0)$.

Let $\Gamma = \Phi_2\Phi_1\Gamma_2$. Then $\Gamma(g_1, g_2, p) = (1, 0, 0)$. Hence, the theorem is proved. □

The proof of the following result follows from [3, Lemmas 5.2 and 5.4] and A.7.

Theorem A.8. *Let R be a ring of dimension 2 and $A = R[X_1, \dots, X_r, Y_1^{\pm 1}, \dots, Y_{r'}^{\pm 1}]$. Let P be a projective A-modules of* rank 2 *with trivial determinant. If A^2 is cancellative, then P is cancellative.*

Proposition A.9. *Let R be a smooth affine domain of dimension 2 over an algebraically closed field k of characteristic 0. Let $A = R[X_1, \dots, X_n, Y^{\pm 1}]$. Then A^2 is cancellative and hence every projective A-module of* rank 2 *with trivial determinant is cancellative, by* A.8.

Proof. Let P be a stably free A-module of rank 2. By 2.3, we may assume that $P \oplus A \xrightarrow{\sim} A^3$. Since $A_{1+Yk[Y]} = \widetilde{R}[X_1, \dots, X_n]$, where $\widetilde{R}$ is a smooth affine domain over a C_1 field $k(Y)$. Hence, by [3, Theorem 5.5], $P \otimes A_{1+Yk[Y]}$ is free. There exists $h \in 1 + Yk[Y]$ such that P_h is free. Patching P and A_h^2, we get a projective $R[X_1, \dots, X_n, Y] = B$-module Q of rank 2 such that $Q_h \xrightarrow{\sim} P$. Since $(Q \oplus B)_Y$ is free, $Q \oplus B$ is free. Applying [3, Theorem 5.5], Q is free and hence P is free. This proves that A^2 is cancellative. □

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