RIBBON CONCORDANCE AND A PARTIAL ORDER ON S-EQUIVALENCE CLASSES

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Received 9 July 1984

A partial order is defined on S-equivalence classes which relates well to ribbon concordance of classical knots.

AMS Subj. Class.: 57M25

Ribbon concordance  S-equivalence

Introduction

In [2], Cameron Gordon introduced the notion of ribbon concordance between classical knots. One says $K_1$ is ribbon cobordant to $K_0$ (and writes $K_1 \geq K_0$) if there is a smooth concordance $C$ in $S^3 \times I$ from $K_1$ to $K_0$ such that the restriction to $C$ of the projection $S^3 \times I \to I$ is a Morse function with no local maxima. Gordon conjectured that this is a partial order on the set of classical knots as the notation suggests.

In this paper we introduce an analogous partial order on the set of $S$-equivalence classes of knots. We summarize its properties in the following theorem. Here $X$ and $Y$ will denote $S$-equivalence classes, $[K_i]$ knots, $[K_i]$ the $S$-equivalence class of $K_i$, $\Delta$ the Alexander polynomial of an $S$-equivalence class.

Theorem (0.1). There is a partial order on $S$-equivalence classes such that

1) If $X \geq Y$, then $\Delta_X \sim \Delta_Y f(t) f(t^{-1})$ where $f(t) \in \mathbb{Z}[t]$.
2) If $X \geq Y$ and $\Delta_X \sim \Delta_Y$, then $X = Y$.
3) If $K_1 \geq K_0$, then $[K_1] \geq [K_0]$.
4) If $X \geq [K_0]$, then there exists a knot $K_1$ such that $K_1 \geq K_0$ and $[K_1] = X$.
5) $X$ is minimal with respect $\geq$ if and only if it is the $S$-equivalence class of a rationally anisotropic knot (in the sense of Gordon).

In Section 1, we will define this partial order in terms of Seifert matrices and show that it is a partial order satisfying 1), 2), 3), and 4). In Section 2, we translate

* Partially supported by N.S.F. Grant MCS-8102118.
this partial order so that it is given in terms of the Seifert form of Trotter [2], which is essentially the Blanchfield pairing (as we explain in Section 3). This is necessary in order to understand the minimal elements and prove 5). It also leads to a pretty definition of the partial ordering, which goes as follows.

Recall that the Blanchfield pairing of a knot is a non-singular Hermitian pairing $\beta$ on $A$, a finitely generated $\mathbb{Z}$-torsion free, $A_0 = \mathbb{Z}[t, t^{-1}]$ torsion module for which $1-t$ is an automorphism. $\beta$ takes values in $R/A_0$ where $R$ stands for the field of rational functions. Two knots are $S$-equivalent if and only if their Blanchfield pairings are isometric [5], [12]. The partial order is given by $A_1 \succeq A_0$ if there is a $A_0$-submodule $H \subset A_1$ such that $H \subset H^+$ and $A_0$ is isometric to $H^+/H$. The original pairing on $A_1$ is well-defined on $H$ cosets in $H^+$. In the last section, we discuss the relation of the partial order to cobordism of Seifert matrices. We also discuss some further questions raised by this work. The most interesting is

**Question (0.2).** If $K_0$ and $K_1$ are $Q$-anisotropic cobordant knots, are $K_0$ and $K_1$ necessarily $S$-equivalent?

It turns out that a 'yes' answer to two questions of Gordon implies a 'yes' answer to this question (see Section 4).

The initial impetus for this work came from the following observation.

**Corollary (0.3).** If $K_1 \succeq K_0$ then $\Delta_{K_1}(t) - f(t)f(t^{-1})\Delta_{K_0}(t)$ where $f(t) \in \mathbb{Z}[t]$.

This fact was essentially already known to Fox and Milnor when they defined cobordism of knots (1957). This can be learned from Fox's review [10] p. 113 of a paper of Terasaka [4] who proves this via the free calculus at least in the case where $C$ has one local minima. I thank Daniel Silver for pointing this reference out to me. I also wish to thank Pierre Conner and particularly Neal Stoltzfus for guidance in this work.

We work in the smooth category. For us $A_0 = \mathbb{Z}[t, t^{-1}]$, $A = \mathbb{Z}[t, t^{-1}, (1-t)^{-1}]$, $R$ is the field of rational functions, $Q$ is the rationals, and $QB$ denotes $B \otimes Q$.

1. Seifert matrices and ribbon concordance

Let $K$ be a knot in $S^3$. A Seifert surface $F \subset S^3$ is an oriented surface whose boundary is $K$. There exist many Seifert surfaces for a given $K$. To each Seifert surface we can associate a Seifert pairing $\theta : H_1(F) \times H_1(F) \rightarrow \mathbb{Z}$ by the rule $\theta(x, y)$ is the linking number of a cycle representing $x$ with a translate in the positive normal direction of $F$ of a cycle representing $y$. A matrix which gives $\theta$ with respect to some basis for $H_1(F)$ is called a Seifert matrix. Any integral matrix $V$ such that $\det(V - V^*) = \pm 1$ is a Seifert matrix for some knot.
A Seifert matrix $W$ row reduces to $V$ if

$$W = \begin{bmatrix} 0 & 0 & 0 \\ 1 & x & y \\ 0 & z & V \end{bmatrix},$$

where $x$ is a number, $y$ a row vector and $z$ a column vector. $W$ column reduces to $V$ if $W'$ row reduces to $V'$. We say $V$ reduces to $W$ if there is a sequence of integral congruences and row or column reductions leading from $V$ to $W$.

Two Seifert matrices $V$ and $W$ are $S$-equivalent if there is a sequence of integral congruences and row or column reductions or expansions (the reverse of a reduction) leading from $V$ to $W$. We let $[V]$ denote the $S$-equivalence class of $V$. Any two Seifert matrices for a given knot are $S$-equivalent. See Cameron Gordon's survey article [3] section 9 for a proof. This is also a good and convenient reference for other background to this paper. Two knots are $S$-equivalent if they possess $S$-equivalent Seifert matrices.

The Alexander polynomial of a Seifert matrix $V$ is $\Delta_V(t) = \det(tV - V')$. Two Alexander polynomials $f$ and $g$ are considered equivalent $f \sim g$, if $f(t) = \pm t^ng(t)$. $S$-equivalent matrices have equivalent polynomials.

**Definition (1.1).** If a Seifert matrix $V_i$ is congruent to a matrix of the form

$$\begin{bmatrix} V_0 & 0 & U \\ 0 & 0 & N \\ U' & N' - I & J \end{bmatrix},$$

where $J = J'$, then we will write $V_i \succeq V_0$. Note $V_0$ will also be a Seifert matrix.

This definition is motivated by the following theorem.

**Theorem (1.2).** If $K_i \succeq K_0$ and $V_0$ is a Seifert matrix for $K_0$, then $K_i$ has a Seifert matrix $V_i$ such that $V_i \succeq V_0$.

**Proof.** Suppose the ribbon concordance from $K_1$ to $K_0$ has $\mu$ minima and let $L$ denote the link consisting of $K_0$ and $\mu$ unknotted unlinked circles off to the side. Using the techniques of Tristram [11], one easily sees that $K_1$ may be obtained by doing a sequence of $\mu - 1$ band modifications to $L$.

Let $F_0$ be a Seifert surface for $K_0$ with $V_0$ as an associated Seifert matrix. Let $F'_0$ denote $F_0$ less an open collar of the boundary and $F$ denote $F_0$ union some disjoint spanning disks for the other components of $L$. We may isotope the bands so that they are transverse to int $F$ and themselves and intersect int $F$ and themselves only in ribbon intersections. Moreover we can isotope the bands so that none of the bands intersect $F'_0$.
Let $G$ denote the union of $F$ and these isotoped bands. $G$ is an immersed surface with only ribbon intersections and with boundary $K_1$. Resolve each of the ribbon intersections as indicated in the figure. Let $F_1$ denote the resulting Seifert surface for $K_1$. Note $F_0' = F_1$ and $F_1 - \text{int } F_0'$ is a twice punctured surface of genus $r$ if there were $r$ ribbon intersections. This is because $F_1 - \text{int } F_0'$ is obtained by resolving the immersed annulus $G - \text{int } F_0'$ and each modification consists abstractly of cutting a hole (creating a homology class $x_i$) and adding a band (creating a dual class $y_i$). With a little care we can arrange that the $y_i$ do not intersect each other.

![Diagram of before and after resolution]

**Fig. 1.**

If $V_0$ is the Seifert matrix for $F_0$ with respect to a basis $f_1, \ldots, f_m$ for $H_1(F_0) = H_1(F_0')$ then

$$(f_0, \ldots, f_m, x_1, \ldots, x_r, y_1, \ldots, y_r)$$

is a basis for $H_1(F_1)$. Since $\theta(x_i, x_j) = \theta(x_i, y_j) = 0$ and the intersection pairing is the Seifert pairing skew-symmetrized, the Seifert matrix for $F_1$ with respect to this basis has the stated form □

**Theorem (1.3).** If $V_i \geq V_0$ and $K_0$ is a knot which has $V_0$ as a Seifert matrix, then there exists a knot $K_i$ with $V_i$ as a Seifert matrix such that $K_i \cong K_0$.

**Proof.** Let $F_0$ be a Seifert surface for $K_0$ with $V_0$ as a Seifert matrix. Suppose $V_i$ is the matrix pictured in Definition (1.1), $2r = \dim V_i - \dim V_0$, and $L$ the link consisting of $K_0$ together with $r$ unlinked and unknotted other components. If we remove $r$ open disjoint disks from $F_0$ we obtain a Seifert surface $F$ for $L$ which has
the upper ‘2 × 2’ left hand corner of $V_1$ as a Seifert matrix. By attaching $r$ bands to $F$ appropriately we can arrange that the resulting Seifert surface $F_1$ has $V_1$ as a Seifert matrix. The bands will be twisted and linked. Let $K_1$ be the boundary of $F_1$. Since $K_1$ is obtained by banding together $L$, we have $K_1 \cong K_0$. □

Proposition (1.4). If $V_1 \cong V_0$ then $\Delta_{V_1}(t) \sim f(t)f(t^{-1})\Delta_{V_0}(t)$ where $f \in \mathbb{Z}[t]$.

Proof. $\Delta_{V_1}(t)$ is the determinant of

$$
\begin{bmatrix}
  tV_0 - V_0 & (t-1)U \\
  0 & (t-1)N + I
\end{bmatrix},
$$

Let $f(t) = \det((t-1)N+I)$, then

$$
\det((t-1)N'-tI) = (-1)^r \det((t^{-1}-1)N+I) = t^r f(t^{-1}).
$$

View the above matrix over the field of rational functions $\mathbb{R}$. Since $f(1) = 1$, $(t-1)N+I$ is invertible over $\mathbb{R}$. This means that we can clear out $(t-1)U$ doing row operations that leave the rest of the matrix alone and of course not change the determinant. Then since $(t-1)N'-tI$ is also nonsingular, we can clear out $(t-1)U'$ as well. Finally $\Delta_{V_1}(t) = \det(V_0 - V_0')$. □

Proposition (1.5). If $V_1 \cong V_0$ and $\Delta_{V_1}(t) \sim \Delta_{V_0}(t)$, then $V_1$ reduces to $V_0$.

Proof. We further develop the proof of the previous proposition. Since the Alexander polynomials are equal, $f(t) = \det((t-1)N+I) = \pm t^r$. Let $z = (1-t)^{-1}$, so $t = z^{-1}(z-1)$. Then $\det(N - zI) = (-z)^{-r}f(t) = \pm z^p(z-1)^q$. By Lemma 1 below, there is an $S \in GL_n(\mathbb{Z})$ such that $Q = SNS^{-1}$ is upper triangular with one’s and zero’s down the diagonal. Let $R$ be

$$
\begin{bmatrix}
  I & 0 & 0 \\
  0 & S & 0 \\
  0 & 0 & (S^{-1})^r
\end{bmatrix},
$$

then $RV_1R'$ has the form

$$
\begin{bmatrix}
  V_0 & 0 & * \\
  0 & 0 & Q \\
  * & Q' - I & *
\end{bmatrix}.
$$

If $V_0$ is $m \times m$ and $Q$ is $r \times r$, the above matrix defines a bilinear form on $\mathbb{Z}^{m+2r}$ which we will denote by $(\, , )$. If $Q_n = 0$, we have $(e_{i+m}, e_i) = 0$ for all $i$, $(e_i, e_{i+m}) = 0$ for $i \neq m + 2r$ and $(e_{i+2r}, e_i) = -1$. Thus the matrix row reduces to this matrix with the $m + r$ and $m + 2r$ rows and columns deleted. In a similar way if $Q_n = 1$, the matrix column reduces. The new matrix has the same form but the size of $Q$ has reduced by 1. Thus we can continue in this way till we reduce to $V_0$. □
Lemma 1. A square matrix over \( \mathbb{Z} \) is triangular if and only if all the roots of the characteristic polynomial are integers. Moreover the diagonal entries are just these roots (counted with multiplicities).

Proof. The standard proof for this fact over fields (see [1] page 543) goes through. If \( \lambda \) is an eigenvalue, we can find an associated eigenvector which is primitive. A primitive vector can always be extended to a basis. 

Let \( X_1 \) and \( X_0 \) be two \( S \)-equivalence classes. We will say \( X_1 \cong X_0 \) if there exist representative Seifert matrices \( V_1 \) and \( V_0 \) such that \( V_1 \succeq V_0 \).

Theorem (1.6). This gives a partial order on the set of \( S \)-equivalence classes.

Proof. (1) Clearly \( X \cong X \). (2) If \( X \cong Y \) and \( Y \cong X \) by Proposition (1.4), they have the same polynomial. So by Proposition (1.5), they have \( S \)-equivalent representatives. For transivity we need the following lemmas whose proofs are left to the reader.

Lemma 2. If \( V_1 \) and \( V_2 \) are \( S \)-equivalent, there is a \( V_3 \) such that \( V_1 \) expands to \( V_3 \) and \( V_3 \) reduces to \( V_2 \).

Lemma 3. If \( V_1 \) reduces to \( V_2 \), then \( V_1 \cong V_2 \).

Lemma 4. If \( V_1 \cong V_2 \) and \( V_2 \cong V_3 \), then \( V_1 \cong V_3 \).

Lemma 5. If \( V_1 \cong V_2 \) and \( V_2 \) expands to \( V_3 \), then there exists \( V_4 \) which reduces to \( V_1 \) and \( V_4 \cong V_3 \).

(3) Suppose \( V_1 \cong V_2 \), \( V_2 \) is \( S \)-equivalent to \( V_3 \) and \( V_3 \cong V_4 \). By Lemma 2, there is a \( V_5 \) such that \( V_5 \) expands to \( V_1 \) and \( V_5 \) reduces to \( V_3 \). By Lemmas 3 and 4, \( V_5 \cong V_4 \). By Lemma 5, there is a \( V_6 \) which reduces to \( V_1 \) and \( V_6 \cong V_5 \). By Lemma 4, \( V_6 \cong V_4 \). Thus the class of \( V_1 \) is greater than the class of \( V_4 \).

2. Seifert forms

We recall some work of Trotter [12]. If \( V \) is a Seifert matrix, define \( A_V \) to be the \( \Lambda \)-module presented by the matrix \( M_v = (t-1)^{-1}(tV - V') \). \( A_V \) is a \( \Lambda \)-torsion ([12] Lemma (1.3)) and \( Z \) torsion-free ([12] Lemma (2.1)) module. If \( a, b \in \Lambda \), define \( a \cdot b \in R/\Lambda \) to be \( bM_v^{-1}b \), where \( b \in \Lambda \). \( a \cdot b \) defines a hermitian form on \( A_V \) called the Seifert form associated to \( V \). We will generally refer to a Seifert form \( A \) without mentioning \( \cdot \). We let \( q: \Lambda \to A \) denote the quotient map.
Trotter shows that two Seifert matrices are $S$-equivalent if and only if they define isometric Seifert forms. Thus it is natural to wonder what is the equivalent partial order is in this context. The main motivation for this is that it permits us to characterize the minimal elements in the partial order. We are also able to simplify the definition of $\geq$ in terms of Seifert matrices ((iv) of the theorem below).

Let $A$ be a Seifert form and suppose $H \subseteq A$ is an $\Lambda$-submodule, self-annihilating with respect to $\cdot$ and pure as a $\mathbb{Z}$ submodule. Let $H^\perp = \{a \in A | a \cdot h = 0 \text{ for all } h \in H\}$. Then $H \subseteq H^\perp$ and $H^\perp/H$ is a torsion $\Lambda$-module with no $\mathbb{Z}$-torsion. Moreover the pairing $\cdot$ is well defined on cosets of $H$ in $H^\perp$. Thus it defines a hermitian form on $H^\perp/H$. According to (iii) of the theorem below $H^\perp/H$ is a Seifert form. If $A_0$ is isometric to $H^\perp/H$ for some such $H$, we write $A \simeq A_0$. A Seifert form is called anisotropic if it possesses no nontrivial, self-annihilating $\Lambda$ submodules.

The proof of the if part of (i) and (ii) of the following theorem are relatively easy and should be comprehensible without reading Trotter’s paper. The proof of (iii) and the only if part of (ii) require familiarity with Trotter’s paper through Lemma (2.14), but do not rely on the truth of Trotter’s main theorem. The only if part of (i) and (iv) both require this theorem but not any understanding of its proof.

**Theorem (2.1).** (i) $A_{V_i} \simeq A_{V_0}$ if and only if $[V_i] \simeq [V_0]$

(ii) $[V]$ is minimal with respect to $\geq$ if and only if $A_V$ is anisotropic.

(iii) If $A$ is a Seifert form and $H$ a $\mathbb{Z}$-pure self-annihilating $\Lambda$-submodule of $A$ then $H^\perp/H$ is a Seifert form.

(iv) $[V_i] \simeq [V_0]$ if and only if $V_i \simeq W$ with $[W] = [V_0]$.

**Proof.** (the if parts of (i) and (ii)). Let $M_i$ denote $M_{V_i}$ and $A_i$ denote $A_{V_i}$ etc. Let $V_i$ be as in Definition (1.1). So $\cdot$ is given by $M_{i}^{-1}$. $M_i$ is

\[
\begin{bmatrix}
M_0 & 0 & U
\\
0 & 0 & P
\\
U' & \bar{P}' & J
\end{bmatrix},
\]

where $P = N - zI$ and $z = (1-t)^{-1}$. Therefore $M_i^{-1}$ is

\[
\begin{bmatrix}
M_0^{-1} & -M_0^{-1}UP^{-1} & 0
\\
-(\bar{P}')^{-1}U'M_0^{-1} & (\bar{P}')^{-1}(U'M_0^{-1}UP^{-1} - JP^{-1}) & (\bar{P}')^{-1}
\\
0 & P^{-1} & 0
\end{bmatrix}.
\]

Let us assume that $M_0$ is $m \times m$ and $P$ is $r \times r$ and write $\Lambda^{m+2r}$ as $X_1 \oplus X_2 \oplus X_3$ where $X_1$ is generated by first $m$ basis vectors and $X_2$ is generated by the next $r$ basis vectors etc. Then let $H = q(X_3)$. $H$ is certainly a self annihilating $\Lambda$ submodule. We have det $P = -zf(t)$ where $f(t) \in \mathbb{Z}[t]$ and $f(1) = \pm 1$ (see proofs of (1.4) and (1.5)). Since $M_0$ and $P$ have nonzero determinants, $H$ is presented by $\bar{P}'$. Note that if $[V_i] \neq [V_0]$ then $f(t) \neq 1$ by Proposition (1.5) and thus $H$ is nonzero. (In fact $QH$ has order $f(t^{-1})$ as a $Q(t, t^{-1})$ module). We have proved the contrapositive of the if part of (ii).
Clearly \( q(X_1 \oplus X_3) \subseteq H^+ \). Suppose \( q(x_1 + x_2 + x_3) \in H^+ \) where \( x_i \in X_i \) then we have for all \( x \in X_3 \) that \( 0 = q(x_1 + x_2 + x_3) : q(x) = x'P^{-1}x_2 \). Thus \( P^{-1}X_2 \subseteq \Lambda ' \) or \( x_2 \in P\Lambda ' \). Then using the third 'column' of \( M_i \), we see \( q(x_2) \in q(X_1 \oplus X_3) \) and thus \( q(x_1 + x_2 + x_3) \in q(X_1 \oplus X_3) \), so \( q(X_1 \oplus X_3) = H^+ \).

The kernel of \( q_{X_1 \oplus X_3} \) is given by the columns of
\[
\begin{bmatrix}
M_0 & 0 \\
0 & \bar{P}'
\end{bmatrix}
\]
because \( P \) is nonsingular. Thus \( X_1 \) maps onto \( q_1(X_1 \oplus X_3)/q(X_3) = H^+/H \) and the kernel is generated by the columns of \( M_0 \). Thus \( A_0 = H^+/H \). Since \( A_0 \) is \( Z \)-torsion free, \( H \) is pure in \( H^+ \) and thus in \( A \) as well. By definition the induced form on \( H^+/H \) is given by \( M_0^{-1} \) (the upper corner of \( M_1^{-1} \)). Thus \( A_1 \cong A_0 \).

(iii) Think of \( A \subset QA \) and let \( L \subset A \) be an admissible lattice, in the sense of Trotter, which generates \( A \) as a \( \Lambda \)-module. Let \( \tilde{H} = L \cap H \), then \( \tilde{H} \) is pure in \( L \) and so is a direct summand for \( L \). Consider the skew symmetric, unimodular scalar form \( [,] \) defined on \( A \) by Trotter. Since \( [x, y] = \Xi(x \cdot y) \) where \( \Xi : R/\Lambda \to Q \) is the \( Q \) linear map defined by Trotter, \( \tilde{H} \) is self annihilating for \( [,] \). So we can find a basis \( \{b_i\}_{i=1}^m \) for \( L \) so that the matrix \( S \) for \( [,] \) is
\[
\begin{bmatrix}
S_2 & 0 & 0 \\
0 & 0 & -I \\
0 & I & 0
\end{bmatrix}
\]
where \( I \) is \( r \times r \) and \( \{b_{m-r+1}, \ldots, b_m\} \) generate \( \tilde{H} \). Note \( S_2 \) is also skew-symmetric and unimodular.

Let \( \Gamma \) be the matrix for multiplication by \( z \) on \( QA \) with respect to this same basis now viewed as a basis for the rational vector space \( QA \). Since \( zL \subset L \) and \( \tilde{H} \) is a \( \Lambda \) submodule, \( \Gamma \) is integral and has the form
\[
\begin{bmatrix}
* & * & 0 \\
* & * & 0 \\
* & * & *
\end{bmatrix}
\]
By [12] (2.9 c), we must have \( \Gamma' S = S(1 - \Gamma) \). A simple matrix computation shows that \( \Gamma \) must have the form
\[
\begin{bmatrix}
\Gamma' & U \\
0 & N \\
U' S_2 & J - N'
\end{bmatrix},
\]
where \( J = J' \) and \( \Gamma' S_2 = S_2 (1 - \Gamma_2) \). If we let \( V_2 = \Gamma_2 S_2^{-1} \) and \( V = \Gamma S^{-1} \) then \( V \) is
\[
\begin{bmatrix}
V_2 & 0 & U \\
0 & 0 & N \\
U' N' - I & J
\end{bmatrix}
\]
V and V₂ are nonsingular Seifert matrices and \( V \triangleright V₂ \). Moreover (see [12] bottom of page 182) \( A = A_v \) with \( b_i = q(e_i) \) where \( \{e_i\} \) is the standard basis for \( A^n \) (as a \( A \) module). Given a matrix \( V \) of the above form, the proof of (i, the if part) above produces an \( H \subseteq A_v \) (which we will call \( H' \) to avoid confusion with the previously named \( H \) in this proof) such that \( (H')^{-1} / H' = A_v' \).

We will show \( H = H' \), completing the proof. We have that \( \{b_{m-r+1}, \ldots, b_m\} \) generate \( H' \) over \( A \), and generate \( \hat{H} = H \cap L \) over \( Z \). Thus \( A \hat{H} = H' \) and \( A \hat{H} \subset H \). Since \( \hat{H} = L \cap H, QH = Q\hat{H} \). Since \( H \) is a \( A \)-submodule, \( Q\hat{H} = QA \hat{H} \). Thus if \( h \in H \), then there is a \( q \in Z \) so that \(qh \in A \hat{H} = H' \). But \( H' \) is \( Z \)-pure in \( A \) and \( h \in A \), so \( h \in H' \).

(i, the only if part) If \( A_v \ni A_v' \) we may proceed through the proof of (iii) and find Seifert matrices \( V \) and \( V_\circ \) such that \( V \triangleright V_\circ, A_v = A_v' \) and \( A_v' = A_v'' \). By the main theorem of [12], \( [V_\circ] = [V] \) and \( [V_\circ] = [V_\circ] \). Thus \( [V_\circ] \ni [V_\circ] \).

(ii, the only if part) Again we prove the contrapositive. We suppose there is a nontrivial self-annihilating \( A \)-submodule \( \hat{H} \). Then \( H = Q\hat{H} \cap A_v \) is pure as well. By (iii) \( H^+ / H \) is a Seifert form say \( A_w \). Thus \( A_v \ni A_w \) and \( \dim_\circ QA_v > \dim_\circ QA_w \). So \( [V] \ni [W] \) and \( A_v \neq A_w \). Therefore \([V]\) is not minimal.

(iv) The if direction is trivial. Suppose \( [V_\circ] \ni [V_\circ] \), then \( A_v \ni A_v' \) and there exists an \( H \subseteq A_v \) such that \( H^+ / H = A_v' \). Let \( V_\circ \) be a nonsingular Seifert matrix obtained by reducing \( V \). Then \( A_v = A_v' \) and the presentation by \( M_v \) leads to a specific admissible lattice for \( A_v' \). Now run through the proof of (iii) only choose \( L \) to be this particular lattice at the appropriate time. Then we obtain Seifert matrices \( V \) and \( V_\circ \) (call it \( W \)) with \( V \ni W \) and \( A_v = A_v' \) and \( A_w = H^+ / H = A_w' \). Since \( V \) and \( V_\circ \) are associated to the same admissible lattice, they are congruent and \( V_\circ \ni W \). Using Lemmas 3 and 4, \( V_\circ \ni W \). By the main theorem of [12], \( [V_\circ] = [W] \).

3. Anisotropy and the Blanchfield Pairing

Let \( V \) be a Seifert matrix for \( K \), \( X \) the exterior of \( K \) and \( \tilde{X} \) the infinite cyclic cover of \( X \). Let \( \beta : H_1(\tilde{X}) \times H_1(\tilde{X}) \to R/A_0 \) denote the Blanchfield pairing and \( \pi : R/A_0 \to R/A \) the obvious quotient map. If \( t \) denotes the action of the covering transformation on \( H_1(\tilde{X}) \), \( 1 - t \) is invertible. In this way, \( H_1(\tilde{X}) \) becomes a \( A \)-module.

**Proposition 3.1.** [5], [8], [12]. There is a \( A \)-module isomorphism between \( A_v \) and \( H_1(\tilde{X}) \) under which \( \cdot \) corresponds to \(-\pi \circ \beta \).

The proof can be put together using the remark on p. 179 of [12], together with 14.2 and 14.3 of [8]. (Warning: We are using Trotter's convention for a Seifert matrix. Levine's convention leads to the transpose. Levine makes a slight error and should have said (p. 44) using his notation that \( d \) has a matrix representative \( tA' + (-1)^g A' \).
Proposition (3.2). If $\cdot : A \times A \to R/\Lambda$ is a Seifert form, there exists a unique lift to a $\Lambda_0$ sesquilinear form $\beta : A \times A \to R/\Lambda_0$. Moreover a submodule $H$ is self-annihilating for $\beta$ if and only if it is self annihilating for $\cdot$.

Proof. A Seifert form by definition arises from a Seifert matrix $V$ and $M^{-1}$ defines a form $\beta$ on $\Lambda_0^{2h}/(tV + V')\Lambda_0^{2h}$ with values in $R/\Lambda_0$. Using the finite generation of $A$ together with the fact that $1-t$ is an automorphism of $A$, one can easily show the uniqueness of the lift. The same argument applied to $H$ gives the final statement.

The point of these last two propositions is that the Seifert form and Blanchfield pairing of a knot are essentially the same thing. There is an analogous partial ordering on Blanchfield pairings which was given in the introduction. The Seifert form is anisotropic if and only if the Blanchfield pairing is anisotropic.

We now wish to see that anisotropy for the Blanchfield pairing is equivalent to anisotropy for the Milnor duality pairing. First note that the Blanchfield pairing is anisotropic if and only if the rational Blanchfield pairing is anisotropic. By Proposition A3 (ii) and Theorem $A-1$ of [9], this equivalent to the Milnor duality pairing being $Q$-anisotropic. Thus we have

Proposition (3.3). The $S$-equivalence class of a knot is minimal with respect to $\leq$ if and only if the knot is $Q$-anisotropic in the sense of Gordon (i.e. the rational Milnor duality pairing is anisotropic).

4. Relation to cobordism and some questions

We begin by remarking that a Seifert matrix $V$ is null-cobordant in the sense of Levine [7] if and only if $V \geq 0$. Here $0$ indicates the empty Seifert matrix. Also if $V \geq W$, it is easy to see that $V \oplus - W$ is null cobordant. Finally suppose $V$ and $W$ are cobordant Seifert matrices then $V \oplus (-W \oplus W) = (V \oplus - W) \oplus W$ is greater than both $V$ and $W$. Thus we have:

Proposition (4.1). $V$ and $W$ are cobordant if and only if there exists a $U$ with $U \geq V$ and $U \geq W$.

Recall the questions asked by Cameron Gordon [2].

Question (6.1). Let $K_0$ be minimal with respect to $\geq$. Does $K$ concordant to $K_0$ imply $K \geq K_0$? Equivalently if $K' \geq K$ and $K' \geq K_0$, is $K \geq K_0$?

Question (6.2). If $K_1 \geq K_2 \geq \ldots$ does there exist some $m$ such that $K_n = K_m$ for all $n \geq m$?
It is natural to consider the analogous questions for $S$-equivalence classes. The analogous question to (6.2) clearly has a 'yes' answer since the Alexander polynomial must stabilize. On the other hand, as we will see, the question analogous to (6.1) has a 'no' answer.

**Proposition (4.2).** (Kervaire). Let $V$ and $W$ be nonsingular Seifert matrices representing anisotropic $S$-equivalence classes then $V$ and $W$ are cobordant if and only if they are rationally congruent.

**Proof.** Kervaire [6] p. 93 gave this as an exercise. We will give a proof making use of the Blanchfield pairing as that seems most natural in this paper. However a proof along the same lines can be given using Seifert matrices reformulated as isometric structures (this is probably what Kervaire intended). If $V$ and $W$ are cobordant then $V \oplus W \simeq 0$. So there is a $\Lambda$-submodule $H \subset A_V \oplus A_{-W}$ with $H = H^\perp$. $H \cap A_V$ is a self-annihilating $\Lambda$-submodule. Since $A_V$ is anisotropic, $H \cap A_V = 0$. Similarly $H \cap A_{-W} = 0$. Let $P_i$ defined on $A_V \oplus A_{-W}$ denote projection on the $i$th factor. Then $P_i|H$ is injective. Thus $\dim QH$ is less than $\dim QA_V$ and $\dim QA_{-W}$. On the other hand $2\dim QH = \dim QA_V \oplus QA_{-W}$. Thus $P_i|QH$ are isomorphisms and $f = P_{i|QH}(P_{i|QH})^{-1}: QA_V \to QA_{-W}$ is an isomorphism. Clearly $x \oplus f(x) \in QH$ for all $x$. Thus $x \cdot y = f(x) \cdot f(y)$. Thus $f$ gives an isometry between $QA_V$ and $QA_{-W}$. By Trotter [12] Proposition 2.12, $V$ and $W$ are congruent over $Q$. The 'if' direction is easy and we leave it to the reader.

Trotter has shown [12] corollary (4.7) that if $V$ is a Seifert matrix with $|\det V|$ either 1 or a prime, that every nonsingular matrix $S$-equivalent to $V$ is integrally congruent to $V$. Thus if we can find two Seifert matrices $V$ and $W$ such that (1) $|\det V|$ is one or a prime (2) $V$ is anisotropic (3) $V$ is rationally but not integrally congruent to $W$ then the $S$-equivalence classes of $V$ and $W$ would be minimal, and cobordant. But neither would be greater than the other. Note that for a $2 \times 2$ Seifert matrix anisotropic simply means not null-cobordant. We can find many such examples. In fact Trotter's examples (5.2) and (5.3) both serve. We have proved

**Proposition (4.3).** There exist $S$-equivalence classes $X$, $Y$, $Z$, such that $X \succeq Y$, $X \succeq Z$, $Y$ and $Z$ minimal but $Y \neq Z$.

If we could find cobordant knots with these Seifert matrices, we could conclude that either 6.1 or 6.2 had a no answer. Put another way

**Proposition (4.4).** If (6.1) and (6.2) have a yes answer then the following statement is true. If $K_0$ and $K_1$ are anisotropic cobordant knots then $K_0$ and $K_1$ are $S$-equivalent.

**Proof.** By (6.2), there exist a minimal knot $K$ with $K_0 \succeq K$. Since $K_0$ is anisotropic, $K$ and $K_0$ are $S$-equivalent by (3.3). By (6.1) $K_1 \succeq K$ and again since $K_1$ is anisotropic $K_1$ and $K$ are $S$-equivalent. 

\qed
I have not been able to find a counter example to this statement. This leads to

**Question (4.5).** If $V$ and $W$ are cobordant Seifert matrices can we find cobordant knots with these Seifert matrices?

Note (4.5) and (0.2) cannot both have ‘yes’ answers.

We close with a few more questions. Gordon asks Question (6.4): Does $K_1 \succeq K_0$ imply $V(K_1) \geq V(K_0)$? Here $V$ denotes the Gromov–Thurston notion of volume. This would follow if there was a degree one map from the exterior of $K_1$ to that of $K_0$. Such a degree one map would also imply that $A_0$ be a quotient of $A_1$. Thus we ask

**Question (4.6).** Does $A_1 \succeq A_0$ imply there is a surjective $A$-module homomorphism from $A_1$ to $A_0$?

Finally one may define an analogous partial order on $S$-equivalence classes of $n$-knots where $n = 3 \mod 4$. High-odd-dimensional simple knots are classified by their $S$-equivalence classes.

**Question (4.7).** Is there a geometrically defined restricted type of cobordism between high-odd-dimensional simple knots which corresponds to this partial order?

References