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Partial Hamming graphs and expansion procedures

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Abstract

Structural properties of isometric subgraphs of Hamming graphs are presented, generalizing certain results on quasi-median graphs. Consequently, a relation on the edge set of a graph which is closely related to Winkler–Djoković's relation Θ is introduced and used for a characterization of isometric subgraphs of Hamming graphs. Moreover, some results considering semi-median graphs and expansions on isometric subgraphs of hypercubes are extended to general non-bipartite case. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Isometric subgraphs of Hamming graphs (*partial Hamming graphs*) and related classes of graphs have been considered by several authors over the last years. In particular, isometric subgraphs of hypercubes (*partial cubes*) which are precisely bipartite partial Hamming graphs have been investigated in the 1970s by Graham and Pollak [4] where they were used as a model for a communication network. Later, partial cubes have drawn attention of several other authors who proved their characterizations such as Djoković [3] and Winkler [12]. Recently, Imrich and Klavžar proposed a hierarchy of classes of partial cubes, including semi-median graphs [6]. Semi-median graphs were introduced as partial cubes for which certain sets U_{ab} are connected for every edge ab of a graph, and several properties of these graphs were established (see also a more recent paper [8]).

Usually from a result on partial Hamming graphs we quickly obtain a corollary on partial cubes. On the other hand, non-bipartite generalizations of results on partial cubes are often rather difficult. Partial Hamming graphs have been studied by Chepoi [2] and Wilkeit [11], and they proved several characterizations of these graphs (see also [5] where isometric embeddings of graphs are presented in a more general setting, and

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[7] where partial Hamming graphs and related classes of graphs are studied from the algorithmic point of view). In this paper, we will use several structural properties of partial Hamming graphs given in [2,11] to prove new characterizations of these graphs and to shed more light on the expansion procedure in partial Hamming graphs.

We will use the procedure of expansion on the partial Hamming graphs which was introduced by Chepoi [2]. More precisely, he proved that the partial Hamming graphs are precisely the graphs that can be obtained by a sequence of expansions from a one-vertex graph. We also refer to Mulder [10] who introduced quasi-median graphs along with the procedure of expansion on these and some related classes of graphs. Characterizations of graphs via an expansion procedure have been studied extensively in [9], see also references there.

In the sequel of this section, we fix the notation and state some simple preliminary results. In Section 2, we use known properties of partial Hamming graphs [2,11] to prove some new properties of these graphs. Some of the claims in this section may be viewed as generalizations of results on quasi-median graphs [1,10]. Next, because of these properties we introduce a certain relation (denoted by Δ) on the edge-set of a connected graph. The transitivity of this relation is a basic condition for new characterizations of partial Hamming graphs. These characterizations can be viewed as extensions of Winkler's result [12, Theorem 4] in which transitivity of the well-known relation Θ is used. In Section 3 a contraction of edges of a partial Hamming graph is introduced using the relation Δ. This contraction is basically the opposite operation to expansion defined by Chepoi [2]. Then, a *connected* and an *isometric* expansion are defined as special cases of Chepoi's expansion in the same way as in [6]. We introduce semi-quasi-median graphs analogous to the semi-median graphs introduced in [6]. It is proved that they can be obtained by the connected expansion procedure from a one-vertex graph. On the other hand, we show that the isometric expansion procedure always produces semi-quasi-median graphs. These two theorems are generalizations of results on semi-median graphs [6].

Let $G=(V(G),E(G))$ be a graph. The *distance* in G between vertices u,v is denoted by $d_G(u,v)$ (or $d(u,v)$) and is defined as the number of edges on a shortest u,v-path. The distance between a vertex u and a subgraph H of a graph G is denoted by $d_G(u,H)$ and is defined as $\min_{v\in V(H)}\{d_G(u,v)\}$. A subgraph H of a graph G is called *isometric* if $d_H(u,v)=d_G(u,v)$ for all $u,v\in V(H)$. For a subset U of $V(G)$ we will denote by $\langle U\rangle$ a subgraph induced by vertices of U.

A graph G is a *partial Hamming graph* if its vertices can be labeled by words (or labels) of a fixed length k over some finite alphabet Σ, so that for any two vertices in G, say u,v, the distance $d(u,v)$ between u and v equals the *Hamming distance* $H(f(u),f(v))$ between their labels $f(u)$ and $f(v)$, which is defined as the number of positions on which the two labels differ. The function $f: V(G)\rightarrow\Sigma^k$ is called a *Hamming labeling* of G. If in addition, Σ can be chosen to have only two symbols, then G is called a *partial cube*.

The Cartesian product $G=G_1\Box G_2\Box\cdots\Box G_k$ of graphs $G_1,G_2,\ldots,G_k$ has the vertex-set $V(G)=V(G_1)\times V(G_2)\times\cdots\times V(G_k)$ and two vertices $u=(u_1,u_2,\ldots,u_k)$,

$v=(v_1,v_2,\dots,v_k)$ are adjacent in G if there exists an index j $(1\leqslant j\leqslant k)$ such that

$$u_jv_j\in E(G_j)$$

and

$$u_i=v_i \quad \text{for all } i\in\{1,2,\dots,k\}\backslash\{j\}.$$

If all the factors in a Cartesian product are complete graphs then G is called a *Hamming graph* and if all factors are graphs K_2 then G is called a *hypercube* or simply a *k-cube*. It is obvious that partial Hamming graphs are precisely isometric subgraphs of Hamming graphs.

For an edge ab of a graph G we introduce the following sets:

$$W_{ab}=\{x\in V(G)\colon d(x,a)<d(x,b)\},$$
$$U_{ab}=\{x\in W_{ab}\colon x \text{ has a neighbor } y \text{ in } W_{ba}\},$$
$$F_{ab}=\{xy\in E(G)\colon x\in U_{ab},\ y\in U_{ba}\}.$$

Djoković's relation $\sim$ was originally defined as follows [3]: two edges $xy,ab\in E(G)$ are in relation $\sim$, if

$$x\in W_{ab} \quad\text{and}\quad y\in W_{ba}.$$

Note that the set of edges in F_{ab} is precisely the set of edges in relation $\sim$ with the edge ab. Obviously, $a\in U_{ab}$, $b\in U_{ba}$ so the relation $\sim$ is reflexive. If for edges $ab,uv\in E(G)$ we have $u\in U_{ab}$ and $v\in U_{ba}$ then $a\in U_{uv}$ and $b\in U_{vu}$ and the relation is also symmetric, such that

$$d(u,a)=d(v,b)=d(v,a)-1=d(u,b)-1. \tag{1}$$

However, the relation is not transitive in general (consider for instance $K_{2,3}$). We also note that in non-bipartite graphs the sets W_{ab} and W_{ba} do not necessarily cover $V(G)$. We mention that in bipartite graphs Djoković's relation is equivalent to a relation Θ introduced by Winkler [12]: $xy\Theta ab$, if

$$d(x,a)+d(y,b)\neq d(x,b)+d(y,a).$$

The relation Θ plays an important role in the theory of isometric embeddings in Hamming graphs. In the non-bipartite case, $\sim$ is contained in Θ, but $ab\Theta xy$ does not imply $ab\sim xy$ in general (K_3 is an example).

The set $I(u,v)$ of all vertices in G which lie on shortest paths between vertices $u,v\in V(G)$ is called an *interval.* A set A in $V(G)$ is called *convex* if $I(u,v)\subseteq A$ for all $u,v\in A$ and a subgraph H in G is *convex* if its vertex set is convex. A subgraph H of a graph G is called *gated* in G if for every $x\in V(G)$ there exists a vertex u in H such that $u\in I(x,v)$ for all $v\in V(H)$. If for some x such a vertex u in $V(H)$ exists, it must be unique. We denote this unique vertex by $\alpha_H(x)$ and we call it the *gate* of x in the subgraph H. As in [11] we denote for a subgraph H in G,

$$W_a(H)=\{x\in V(G)\colon a \text{ is the gate of } x \text{ in } H\} \quad (a\in V(H)),$$
$$W(H)=\{x\in V(G)\colon \text{ for each } a\in H,\ d(a,x)=d(H,x)\}.$$

If in a subgraph H of G for every $x\in V(G)$ there exists either a gate $\alpha_H(x)$ or $x\in W(H)$ then H is called *pseudo-gated* in G. We note that if H is pseudo-gated then $V(G)=\bigcup_{a\in H} W_a(H)\cup W(H)$. Obviously, every edge of a graph is pseudo-gated and every gated subgraph of G is pseudo-gated. A maximal complete subgraph of G is called a *clique* in G. A complete graph on four vertices with an edge deleted is denoted by K_4-e. We observe that the cliques in K_4-e are two triangles and they are not gated, moreover they are not even pseudo-gated.

2. The structure of partial Hamming graphs

In this section, some new results considering the structure of partial Hamming graphs are presented in a way similar to the way known for the quasi-median graphs [10]. Then, a new relation Δ on the edge set of a graph which is in a close relationship with relation Θ is introduced. We use this relation to prove new characterizations of partial Hamming graphs.

First, let us recall some properties of these graphs. They have been established by Chepoi [2] (assertions (i), (iv) and implicitly some of the rest assertions) and by Wilkeit [11].

Theorem 1 (Chepoi [2], Wilkeit [11]). *If G is a partial Hamming graph then*

(i) *if $ab\in E(G)$ then the sets W_{ab} and $W_{ab}\cup W_{ba}$ are convex*,
(ii) *for edges $xy,ab\in(G)$: $ab\sim xy\Rightarrow W_{ab}=W_{xy}$*,
(iii) *$K_{2,3},K_4-e$ and C_{2n+1} $(n\geqslant 2)$ are not isometric subgraphs in G*,
(iv) *every clique in G is pseudo-gated*,
(v) *if xy is an edge of G and if K is a clique of G, maximum with respect to containing an edge ab, such that $ab\sim xy$, then K is gated*,
(vi) *if a vertex $w\in V(G)$ has the same distance to adjacent vertices x and y of G, then any two neighbors $u\in W_{xy}$ and $v\in W_{yx}$ of w are adjacent.*

From the property (ii) of the above theorem we deduce transitivity of the relation $\sim$ in partial Hamming graphs.

Proposition 2. *Let ab be an edge of a partial Hamming graph G. The edges in F_{ab} induce a matching between U_{ab} and U_{ba}. Furthermore, a mapping $\varphi:U_{ab}\to U_{ba}$, $\varphi(x)=y$, where xy is an edge in F_{ab}, induces an isomorphism between $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$.*

Proof. Suppose that $u\in U_{ab}$ is adjacent to both v and w in U_{ba}. From the convexity of W_{ba} it follows that v and w are adjacent. Using (1) we have

$$d(b,v)=d(b,w)=d(b,u)-1$$

and a clique containing u,v and w would not be pseudo-gated, a contradiction.

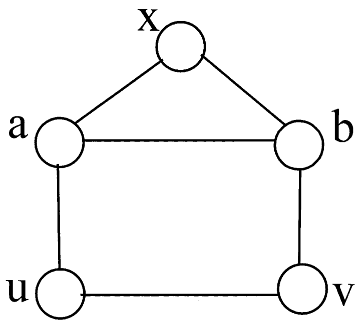

Fig. 1. In this graph $U_{ab} \neq U_{ax}$.

For the second part, we have to prove that if $uv \in E(G)$ for the vertices $u, v \in U_{ab}$ then $xy \in E(G)$ where $x = \varphi(u), y = \varphi(v) \in U_{ba}$. Suppose on the contrary that x and y are not adjacent. Since W_{ba} is convex, $d(x, y)$ must be less than 3, i.e. $d(x, y) = 2$, and let w be a mutual neighbor of x and y in W_{ba}. However, applying Theorem 1(iii) and using the fact that vertices u, v, x, w, y form a C_5, there must be another edge connecting two of these vertices. Vertices u and v cannot be endvertices of such an edge because they are already matched to vertices from U_{ba}. The remaining possibility is that x and y are adjacent and the claim is proved. □

From Theorem 1(iv) we get

Proposition 3. *For each vertex a in a partial Hamming graph G we have* $W_{ab} = W_{ax} = W_a(C)$, *where* a, b *and* x *are vertices in a common clique* C.

Proposition 3 does not hold if we replace W_{ab} with U_{ab}. However, it has been proved implicitly in [10] (see also [1]) that for the quasi-median graphs we have $U_{ab} = U_{ax}$, where b and x are any vertices forming a triangle with a. Consider, for instance, the partial Hamming graph from Fig. 1 where $u \in U_{ab}$ though $u \notin U_{ax}$. On the other hand, we deduce from Theorem 1(ii) that if $ab \sim xy$ then $U_{ab} = U_{xy}$.

In view of Proposition 3, we denote the set W_{ab} as W_a (which is also W_{ax}, and also W_{uv} for $uv \sim ab$ or by the new notation W_u) when the vertex b in consideration, which is adjacent to a, is clear from the context. We denote

$$U_a = \{x \in W_a\colon x \text{ is adjacent to a vertex in } V(G) \backslash W_a\}.$$

Obviously, $U_{ab} \subseteq U_a$; moreover:

Proposition 4. *Let ab be an edge in G and let* $K = \{x_1, x_2, \ldots, x_k\}$ *be a clique, maximum in G with respect to containing an edge* x_1x_2 *such that* $ab \sim x_1x_2$. *Then,*

(i) $U_a = \bigcup_{2<i\leqslant k} U_{x_1x_i}, U_b = \bigcup_{1\leqslant i\leqslant k; i\neq 2} U_{x_2x_i}$,
(ii) *if* $uv \sim x_ix_j$ *and* $u \in U_{x_ix_j} \cap U_{x_ix_k}$ *then* $v \in U_{x_jx_i} \cap U_{x_jx_k}$ *and there exists a vertex z in* W_{x_k} *such that* u, v *and* z *form a triangle,*
(iii) *if* $c \in \bigcup_{i=3}^{k} W_{x_i}$ *then* $d(c, a) = d(c, b)$.

Proof. The observation preceding this proposition gives us one side of the inclusion, i.e. $\bigcup_{1<i\leqslant k} U_{x_1x_i} \subseteq U_a$. By the definition of $\sim$ we infer: if uv is an edge connecting vertices in different (convex) subgraphs $\langle W_{x_i}\rangle$ and $\langle W_{x_j}\rangle$ then $uv \sim x_ix_j$. In particular, if $u \in U_a$ so that u is adjacent to $v \in W_{x_j}$, then $uv \sim x_1x_j$. In other words we have $u \in U_{x_1x_j}$, thus (ii) is proved.

For the proof of (ii) we use assertion (i) of Theorem 1 that $\langle W_{x_i} \cup W_{x_j}\rangle$ is convex in G for any i, j. Now, if v would not be adjacent to a neighbor z of x in W_{x_k}, then the path of length two from y to z would be one of the shortest y,z-paths which would not lie entirely in $\langle W_{x_j} \cup W_{x_k}\rangle$.

Claim (iii) follows from the fact that $c \notin W_{ab} \cup W_{ba}$. □

We are ready to introduce the relation $\triangle$ on the edge set of a connected graph G. We say that edges $uv, ab \in E(G)$ are in relation $\triangle$ if either $uv \sim ab$ or there exists a clique with edges $e, f \in E(G)$, such that $uv \sim e$ and $ab \sim f$. Obviously, the relation $\triangle$ is reflexive and symmetric and the relation $\sim$ is included in $\triangle$. However, $\triangle$ is not transitive in general, as an example of $K_4 - e$ shows.

We have already mentioned that in partial Hamming graphs the relation $\sim$ is an equivalence relation. The corresponding equivalence classes are the sets F_{uv}, where uv is an arbitrary edge of this class.

Proposition 5. *In a partial Hamming graph the relation $\triangle$ is transitive and therefore an equivalence. More precisely, each $\triangle$-class is a union of some $\sim$-classes, so that for edges $ab, cd \in E(G)$ the classes F_{ab} and F_{cd} are in the same $\triangle$-class if and only if there is a clique containing edges $a'b'$, $c'd'$ such that $a'b' \in F_{ab}$, $c'd' \in F_{cd}$.*

Proof. By Theorem 1(v) there exists a gated clique K in G containing an edge from F_{cd}. We have to prove that if there exists a clique containing the edges $a'b' \in F_{ab}$ and $c'd' \in F_{cd}$, then there is an edge $a''b'' \in F_{ab}$ which is contained in K. Obviously, at least one of the vertices a', b' is not the same as one of the vertices c' or d', say $b' \neq c', d'$. Then, if c' should lie on a shortest path from b' to K, we would have $d(b',K) = d(b', \alpha_K(c')) = d(b', \alpha_K(d'))$, which is impossible since K is gated. Also, b' cannot lie on some shortest path from c' to K because c' and d' lie in different convex subsets induced by gated clique K. Therefore, $d(b',K)=d(c',K)$ and by considering two different possibilities for a', we deduce the same for a', i.e. $d(a',K)=d(b',K)=d(c',K)$. By the transitivity of $\sim$ there is an edge in K which belongs to the class F_{ab} and the proof is complete. □

The analogoue of the following proposition holds for the relation Θ in every graph. We shall use it below as a condition in the characterization of partial Hamming graphs:

Proposition 6. *Let G be a partial Hamming graph, and P a path connecting the endpoints of an edge xy. Then P contains an edge f with $xy \Delta f$.*

Proof. Every edge with one end-vertex in U_x and the other in $V(G)-W_x$ is in relation Δ with xy. Obviously, there is such an edge on P. □

Now, our aim is to find some sort of converse of Proposition 5 which would extend the following Winkler's result to general partial Hamming graphs.

Theorem 7 (Winkler [12]). *A connected graph G is an isometric subgraph of a Cartesian power of K_3 if and only if the relation Θ is transitive. G is an isometric subgraph of a hypercube if and only if, in addition, G is bipartite.*

A straightforward extension of this result is not possible since the relation $\triangle$ is transitive in odd cycles C_{2n+1} which are not partial Hamming graphs for $n \geqslant 2$. In order to avoid such cases we will use either the property (vi) of Theorem 1, or the condition of Proposition 6, or we shall forbid such isometric cycles. Also, we must require the transitivity of relation $\sim$ which is presumed in the definition of $\triangle$. In fact, we will use a stronger condition from Theorem 1(ii) to prove the following characterizations of partial Hamming graphs.

Theorem 8. *Let G be a connected graph. Then the following assertions are equivalent*:

(A) *G is a partial Hamming graph*,

(B)
  (i) *The relation $\triangle$ is transitive*,
  (ii) *for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$, and*
  (iii) *if P is a path connecting the endpoints of an edge xy, then P contains an edge f with $xy \Delta f$*,

(C)
  (i) *The relation $\triangle$ is transitive*,
  (ii) *for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$, and*
  (iii′) *if a vertex $w \in V(G)$ has the same distance to adjacent vertices x and y of G, then any two neighbors $u \in W_{xy}$* and *$v \in W_{yx}$ of w are adjacent*,

(D)
  (i) *The relation $\triangle$ is transitive*,
  (ii) *for edges $ab, xy \in E(G)$: if $ab \sim xy$ then $W_{ab} = W_{xy}$, and*
  (iii″) *there are no isometric cycles C_{2n+1} for $n \geqslant 2$.*

Proof. We already know that (A) implies all other conditions (recall Proposition 5, properties (ii), (iii) and (vi) of Theorem 1, and Proposition 6).

(D) ⇒ (B): For the proof of (iii) let xy be an edge of G and P a path between x and y. Note that if P has length 2 or 3 then (iii) is trivial. Now let $P: x \to z_1 \to z_2 \to \cdots \to z_k \to y$ be a path with the smallest number of edges in G which contradicts the claim. If the cycle $C: x \to P \to y \to x$ is isometric, then by (iii″) it is even, hence we have an edge on it which is in relation $\sim$ with xy.

Suppose that C is not isometric and let $z_i = t_1 \to \cdots \to t_m = z_j$ be a shortest path between z_i and z_j which is shorter than a path in C between these two vertices. Let us call $C': x \to z_1 \to \cdots \to z_i = t_1 \to t_2 \to \cdots \to t_{m-1} \to t_m = z_j \to \cdots \to z_k \to y \to x$

and $C'' : z_i \to z_{i+1} \to \cdots \to z_j = t_m \to t_{m-1} \to \cdots \to t_2 \to t_1 = z_i$, the two cycles into which C is splitted. Note that C', C'' both have less vertices than C. Thus xy is in relation $\triangle$ with an edge e on C'. We are done if e is also on C, so let us suppose that $e = t_l t_{l+1}$ for $1 \leqslant l \leqslant m-1$. Since C'' is smaller than C, we infer that $t_l t_{l+1}$ is in relation $\triangle$ with an edge f on C''. If f is also on C, the proof is complete, so we may suppose that $f = t_s t_{s+1}$ where $1 \leqslant t < s \leqslant m-1$. Clearly $e \nsim f$ since the two edges lie on a shortest path between t_1 and t_m. Suppose that there is clique with edges e' and f' such that $e \sim e'$ and $f \sim f'$. Obviously, $t_l, t_{l+1} \in W_{t_s t_{s+1}}$ but at least one of t_l, t_{l+1} is at equal distance to both endpoints of f'. That is in contradiction with (ii).

(B) $\Rightarrow$ (C): For the proof of (iii′) let w be a vertex at the same distance from adjacent vertices x and y of G, and let $u \in W_{xy}$, $v \in W_{yx}$ be the neighbors of w. Let Q_1 be a shortest path from u to x, and Q_2 a shortest path from y to v. We note that $Q_1 \subset W_{xy}$ because $u \in W_{xy}$, and likewise $Q_2 \subset W_{yx}$. Therefore, we have a path from y to x of the form: $Q_2 \to vw \to wu \to Q_1$. Using (iii) it follows that xy is in relation Δ with an edge e of this path. If this edge would be in Q_1 we deduce using (ii) that one of its endvertices would lie in W_{yx} or $V(G) - W_{xy} \cup W_{xy}$ which is impossible. For the same reason e cannot be in Q_2, hence one of the endpoints of e must be w, and assume that $e = wu$. Obviously, $wu \nsim xy$, therefore there exists a clique with edges e', e'' such that $wu \sim e'$, $xy \sim e''$. Using condition (ii) for edges e'' and xy we deduce that e' and e'' must have a vertex in common ($u \in W_{xy}$, so it must be closer to one endpoint of e''). Let us denote the vertices of the clique by a, b and c, so that $ab \sim xy$ and $ac \sim uw$. Since $v \in W_{yx}$, we deduce from (ii) that $v \in W_{ba}$. Now, there are two possibilities: $v \in W_{bc}$ or $d(v,b) = d(v,c)$. If $v \in W_{bc}$ then v is indeed adjacent to u (because using (ii) we deduce $W_{ca} = W_{wu}$ hence we must have $d(v,u) = d(v,w)$). It remains to prove that the second possibility leads to a contradiction.

Assume that $d(v,b) = d(v,c)$, and let v' be the nearest vertex to b on a shortest path from v to b such that $d(v',b) = d(v',c)$. Thus, the rest of the path from v' to b is in W_{bc} and the rest of the path from v' to c is in W_{cb}. Observe that vertices v', b and c are in the same situation as vertices w, x and y. We deduce in a way same as above that there is a triangle a', b' and c' such that $c'\,b' \sim cb$, $v'v'' \sim a'b'$ (where v'' is a neighbor of v' in $W_{b'c'}$) and $d(a',c) = d(a',b) = d(b',b) + 1$. Since Δ is transitive, it follows that $ab \Delta a'b'$. Obviously, ab and $a'b'$ are not in relation $\sim$ (because that would imply that $v' \in W_{ab}$), so there must exist a clique with edges f' and f'' such that $ab \sim f'$, $a'b' \sim f''$. Since $b \in W_{b'a'}$ it follows by property (ii) that f' and f'' must have a common vertex, so that there is a triangle with vertices a'', b'', c'' such that $ab \sim a''b''$ and $a'b' \sim c''b''$. We infer that $d(a,b'') = d(a,c'')$ thus by (ii) also $d(a,b') = d(a,a')$. Hence, $d(a,v') = d(a,v'')$ which implies $d(v',a) < d(v',b)$, a contradiction to $v \in W_{yx} = W_{ba}$.

(C) $\Rightarrow$ (A): Let G be a graph having properties (i)–(iii′), xy be an arbitrary edge in G and let K be one of the largest cliques containing an edge from F_{xy}. We denote this edge by x_1x_2 and other eventual vertices by $x_3, x_4, \ldots, x_k$. Our first step is to prove that K is gated.

Suppose that there exists a vertex z in G and indices i,j $(1\leqslant i<j\leqslant k)$ such that $d(z,x_i)=d(z,x_j)$ and let z be one of the nearest vertices to K among all such vertices. Let $z\to a_1\to a_2\to\cdots\to a_t=x_i$ be a shortest path between z and x_i, and $z\to b_1\to b_2\to\cdots\to b_t=x_j$ be a shortest path between z and x_j. Using property (iii′), we deduce that $a_1b_1\in E(G)$. We also infer that K must have at least three vertices, and let x_s be the third vertex. Now, we note that $a_1b_1\sim x_ix_j$, so $a_1z\,\triangle\,x_ix_j$. Obviously, $x_ix_s\,\triangle\,x_ix_j$, and by the transitivity of $\triangle$ it follows $x_ix_s\,\triangle a_1z$. If $x_ix_s\sim a_1z$ then K is gated and the claim is proved. Otherwise, there exists a clique with edges $e'\in F_{x_ix_s}$ and $e''\in F_{a_1z}$. Since $x_i\in W_{a_1z}$, edges e', e'' have a common vertex a', so that $e'=a'b'$, $e''=a'c'$ and $a'b'\sim x_ix_s$, $a'c'\sim a_1z$. Now $d(x_s,a')=d(x_s,c')$ (because if $d(x_s,c')<d(x_s,a')$ we would again have $x_ix_s\sim a_1z$) therefore by (ii) we have $d(x_s,a_1)=d(x_s,z)$. Then since $x_i\in I(a_1,x_s)$ we derive $d(z,x_s)=d(z,x_i)=d(z,x_j)$ (K is pseudo-gated). Let $z\to c_1\to\cdots\to c_t\to x_s$ be a shortest path between z and x_s. By (iii′), vertices z,a_1,b_1 and c_1 form a clique. By repeating this argument for all vertices of K we obtain a clique of size $k+1$ which is in relation $\sim$ with xy, a contradiction to maximality of K.

By $W_{x_i}(K)$ we denote the set of vertices whose gate in K is x_i. Since $xy\sim x_1x_2$, using (ii) we have $W_{xy}=W_{x_1x_2}$. It is obvious that $W_{x_1x_2}=W_{x_1}(K)$ thus $W_{xy}=W_{x_1}(K)$. So far we have proved that by choosing an arbitrary edge xy in G we obtain a partition $V(G)=W_{x_1}(K)\cup W_{x_2}(K)\cup\cdots\cup W_{x_k}(K)$ where K is a largest clique containing an edge from F_{xy}. It is also clear that the edges connecting vertices in different sets $W_{x_i}(K)$ are precisely all the edges which are in relation $\triangle$ with xy.

To complete the proof we define a Hamming labeling f on the vertex-set of G. Let $u\in V(G)$ and let s be the number of all equivalence classes induced by the relation $\triangle$. The ith coordinate of $f(u)$ corresponds to the ith $\triangle$-class $(1\leqslant i\leqslant s)$, so that the largest clique $K=\{x_1,x_2,\ldots,x_k\}$ in the corresponding $\triangle$-class provides the following labeling:

$$f_i(u)=j-1 \quad \text{if } u\in W_{x_j}(K).$$

This labeling is well defined, since $W_{x_1}(K),W_{x_2}(K),\ldots,W_{x_k}(K)$ form a partition of the vertex set. We need to prove that this is a Hamming labeling, i.e., for $u,v\in V(G)$: $H(f(u),f(v))=d_G(u,v)$. First, we notice that all the edges lying on a shortest path between the two vertices belong to different $\triangle$-classes. The proof is similar to the part of the proof that K is gated which we described in detail above, so we shall omit it here. It follows that the labelings of u and v differ in at least $d_G(u,v)$ coordinates, so $H(f(u),f(v))\geqslant d_G(u,v)$. On the other hand, if $f_i(u)\neq f_i(v)$ then on every path between u and v there must clearly be an edge from the ith $\triangle$-class hence $d_G(u,v)\geqslant H(f(u),f(v))$. The proof is complete. □

Let us now consider Theorem 8 in comparison with Winkler's Theorem 7. As we noted in the first section, if G is bipartite then $\sim=\Theta$ and also $\sim=\triangle$. Wilkeit proved [11, Corollary 7.3] that in bipartite graphs, the condition (ii) of Theorem 8 is equivalent to the transitivity of $\sim$. Since in bipartite graphs the condition (iii) of the above theorem is fulfilled we deduce

Corollary 9 (Winkler [12]). *A connected graph G is an isometric subgraph of a hypercube if and only if the relation Δ $(=\Theta)$ is transitive and G is bipartite.*

For the first part of Winkler's theorem the situation is more complex. We can directly deduce only one direction of that part. But first we need an additional result which can be of independent interest. It easily follows from Proposition 4 and definitions of relations Δ and Θ.

Lemma 10. *Let G be a partial Hamming graph and $ab, cd \in E(G)$. If $ab\Theta cd$ then $ab\Delta cd$. The converse holds precisely when any clique which contains an edge from F_{ab} (or F_{cd}) has at most three vertices.*

The first part of the following corollary follows directly from Theorem 8 while the second part (that $\Theta = \Delta$) is derived from Lemma 10.

Corollary 11. *If G is an isometric subgraph of a power of K_3 then the relation Δ is transitive. Furthermore, $\Theta = \Delta$.*

Combining this and Theorem 8(iv) we get a sort of converse of Corollary 11:

Corollary 12. *Let G be a connected graph such that*

(i) *relation Δ is transitive,*
(ii) *for edges $ab, xy \in E(G)$: if $ab \sim x$ then $W_{ab} = W_{xy}$,*
(iii) *there are no isometric cycles C_{2n+1} for $n \geqslant 2$, and the largest clique in G has at most three vertices.*

Then G is an isometric subgraph of a power of K_3. Furthermore, $\Theta = \Delta$.

The situation is more complicated in this case since the definition of relation Δ is rather complex and relies on the definition of $\sim$. However, it seems that Theorem 8 presents a natural extension of Winkler's theorem to the class of partial Hamming graphs.

3. Expansions on partial Hamming graphs

Motivated by the structure of partial Hamming graphs we will define a contraction of edges of such a graph. This operation has been already performed by Chepoi [2] in the proof of his expansion theorem though he used a slightly different approach by taking a largest clique of a graph.

A partial Hamming graph G is transformed to a graph G' by a *contraction* with respect to the edge ab if G' has the structure as follows. For the chosen edge ab, we contract each edge which is in relation $\triangle$ with ab to a single vertex so that the edges which are in the same clique are contracted to the same vertex. Let us call W_i' the set of vertices in G' which corresponds to W_{x_i} in G. We note that the subgraph $\langle W_i' \rangle$ is isomorphic to the subgraph $\langle W_{x_i} \rangle$ for all $i = 1,\dots,k$. The sets W_i' are not pairwisely disjoint since the edges between vertices in different W_{x_i} and W_{x_j} are transformed to vertices in the intersection $W_i' \cap W_j'$. More precisely, the subgraph $\langle U_{x_ix_j} \cup U_{x_jx_i} \rangle$ in G is contracted to the subgraph $\langle W_i' \cap W_j' \rangle$ in G' which is isomorphic to $\langle U_{x_ix_j} \rangle$. Since the subgraphs $\langle W_{x_i} \rangle$ and $\langle W_{x_i} \cup W_{x_j} \rangle$ are convex in G, it is clear that the subgraphs $\langle W_i' \rangle$ and $\langle W_i' \cup W_j' \rangle$ are isometric in G'. Also it is obvious that there are no edges between $W_i' - W_j'$ and $W_j' - W_i'$ for different $i,j = 1,2,\dots,k$. We can obtain the following definition of an *expansion* by observing the contraction from the opposite side.

Definition 13 (*Chepoi [2]*). Let G' be a connected graph and $W_1',\dots,W_k'$ be subsets in $V(G')$ such that

- $W_i' \cap W_j' \neq \emptyset$ for all $i,j = 1,2,\dots,k$,
- $\bigcup_{i=1}^{k} W_i' = V(G')$,
- there are no edges between sets $W_i' - W_j'$ and $W_j' - W_i'$ for all $i,j = 1,\dots,k$,
- subgraphs $\langle W_i' \rangle, \langle W_i' \cup W_j' \rangle$ are isometric in G' for all $i,j = 1,\dots,k$.

Then to each vertex $x \in V(G')$ we associate a tuple $(i_{j_1}, i_{j_2},\dots,i_{j_l})$ of all indexes i_j, where $x \in W_{i_j}$. Graph G is called an *expansion of G' relative to the sets $W_1', W_2',\dots,W_k'$* if it is obtained in the following way:

- we replace a vertex x in $V(G')$ with vertices $x_{i_1}, x_{i_2},\dots,x_{i_l}$ so that they form a clique of size l
- if an index i_t belongs to both tuples $(i_{j_1}, i_{j_2},\dots,i_{j_l})$, $(i'_{j_1}, i'_{j_2},\dots,i'_{j_l})$ corresponding to adjacent vertices x and y then in the graph G let $x_{i_t} y_{i_t} \in E(G)$.

We obtain new definitions by imposing extra conditions:

- If $W_i' \cap W_j'$ induce connected (respectively isometric) subgraphs for all $i,j = 1,\dots,k$, then this is called a *connected* (respectively, *isometric*) *expansion*.

Loosely speaking we obtain the expansion G from the graph G' by pulling all the subgraphs W_i' each at its side and the traces of previous intersections remain in the form of edges. Note that the connected and isometric expansions have been analogously defined for bipartite graphs by Imrich and Klavžar [6].

There exist several results which characterize a certain class of graphs via a certain type of expansion. The first theorem of that kind was a characterization of median graphs [10] as graphs obtainable from K_1 by a sequence of certain convex expansions, a result also known as Mulder's convex expansion theorem. However, for some classes only one direction can be proved, such as "if a graph belongs to a certain class then it can be obtained by a sequence of certain types of expansions from K_1". In the case of partial Hamming graphs both directions are true as the Chepoi's expansion theorem shows:

Theorem 14 (Chepoi [2]). *A graph G is a partial Hamming graph if and only it can be obtained from K_1 by a sequence of expansions.*

According to Definition 13 we denote by W_i the set in G that naturally corresponds to W_i' in G' for all $i=1,\ldots,k$. We shall use this notation without further notice.

Lemma 15. *Let G' be a partial Hamming graph and let G be an expansion of G' relative to the sets $W_1', W_2', \ldots, W_k'$, and let $u, v \in V(G)$ be adjacent vertices such that $u \in W_i$, $v \in W_j$. Then, in F_{uv} there are precisely all the edges connecting vertices from W_i and W_j.*

Proof. It follows from Theorem 14 that G is also a partial Hamming graph. As we know that the sets W_i and $W_i \cup W_j$ induce convex subgraphs of G it follows that the edges between different sets W_i, W_j are in the same $\sim$-class. We denote this class by F_{uv}, where $u \in W_i$ and $v \in W_j$. We must prove that there are no other edges in F_{uv} but those connecting W_i and W_j. Suppose that there is an edge $xy \in F_{uv}$ which is contradicting this claim. First, observe that xy has to lie entirely outside W_i (use convexity of W_i and convexity of $W_i \cup W_k$ for any k, $k \neq j$), and with the same argument xy has to lie entirely outside W_j. Suppose that xy lies entirely in some W_k ($k \neq i$, $k \neq j$). Then, in G' let u' be one of the nearest vertices in $W_i' \cap W_j'$ to x, thus the shortest path from u' to x lies outside W_j' and the shortest path from u' to y lies outside W_i'. Using Theorem 1(vi), the first vertices on the paths from u' to x and u' to y must be adjacent which is a contradiction since they are in $W_i' - W_j'$ and $W_j' - W_i'$. Then, the remaining possibility is that $x \in W_k$ and $y \in W_l$ where i, j, k, l are pairwise different indexes. But this is a contradiction to the convexity of $\langle W_i \cup W_l \rangle$ since the shortest y, u-path would contain x. □

In [6] *semi-median* graphs were introduced as partial cubes for which the subgraphs $\langle U_{ab} \rangle$ are connected for all edges ab in G. We consider the following results in bipartite graphs:

Theorem 16 (Imrich and Klavžar [6]). (i) *If G is a semi-median graph then it can be obtained from a one-vertex graph by a sequence of connected expansions.*

(ii) *If G is obtained from a one-vertex graph by a sequence of isometric expansions then G is a semi-median graph.*

It is not yet known whether the converse of the second part of Theorem 16 is true, i.e. is there a graph which is semi-median so that it cannot be obtained from K_1 by a sequence of isometric expansions?

We will now consider partial Hamming graphs for which the subgraphs $\langle U_{ab} \rangle$ are connected for all $ab \in E(G)$. Let us call them *semi-quasi-median* graphs, as they lie

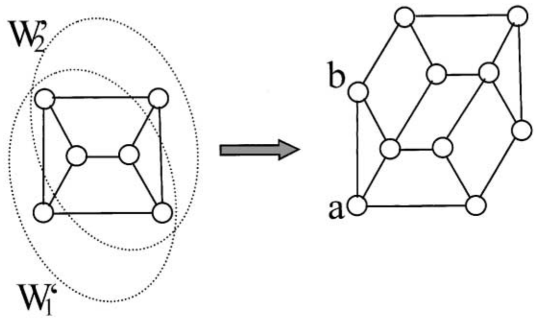

Fig. 2. Connected expansion of $K_2\Box K_3$.

between partial Hamming graphs and quasi-median graphs. We shall extend Theorem 16 to the class of semi-quasi-median graphs with the following two theorems.

Theorem 17. *If G is a semi-quasi-median graph then G can be obtained from a one-vertex graph by a sequence of connected expansions.*

Proof. Since G is a partial Hamming graph it can be obtained from K_1 by a sequence of expansions (Theorem 14). We claim that each expansion in the sequence is connected and prove the claim by induction on the number of expansions in the sequence. Let G be a graph obtained by an expansion (with respect to the sets $W_1',\dots,W_k'$) from a semi-quasi-median graph G'. Let W_i be a set in G corresponding to the set W_i' in G' for all $i=1,\dots,k$. Using Lemma 15 it follows that the subgraph $\langle W_i'\cap W_j'\rangle$ in G' is isomorphic to the subgraph $\langle U_{ab}\rangle$, where $a\in W_i$ and $b\in W_j$ are adjacent vertices. Since $\langle U_{ab}\rangle$ is connected for every edge ab, the expansion is connected. $\Box$

The converse of the theorem above is not true. Consider the graph $K_2\Box K_3$ where for the sets W_1', W_2' we take subgraphs of $K_2\Box K_3$, each of them with a vertex missing so that the missing vertices are at distance 2 (see Fig. 2). The intersection $W_1'\cap W_2'$ is obviously connected, but expanded graph has a set U_{ab} which is not connected.

Theorem 18. *If G can be obtained by a sequence of isometric expansions from K_1 then G is a semi-quasi-median graph.*

Proof. Assume that G can be obtained from K_1 by a sequence of isometric expansions. By Theorem 14 we know that G is a partial Hamming graph. We need to show that if G is obtained by an isometric expansion (relative to the sets $W_1', W_2',\dots,W_k'$) from a semi-quasi-median graph G' then the sets $\langle U_{uv}\rangle$ are connected for any edge uv in G. If uv is a new edge, i.e., is obtained by the expansion from a vertex in the intersection $W_i'\cap W_j'$, then $\langle U_{uv}\rangle$ is isomorphic to $\langle W_i'\cap W_j'\rangle$ which is connected by the definition of expansion (in fact, it is even isometric).

Let uv be an edge which lies entirely in at least one set W_i. We shall check what happens to the set of edges F_{uv} in G' after we make the expansion. The subgraph of W_i' which consists of edges in F_{uv} and their ending vertices is isomorphic to the subgraph of W_i consisting of naturally corresponding edges and their end-vertices. Therefore, $\langle U_{uv} \rangle$ in W_i is isomorphic to the subgraph of $\langle U_{uv} \rangle$ in W_i' consisting of all vertices of U_{uv} such that their neighboring vertex in U_{vu} also lies in W_i'.

We shall now consider only the sets W_i' which contain at least one edge of F_{uv}. Let $xy \in F_{uv} \cap W_i'$ be an edge which lies in a neighborhood of an edge $ab \in F_{uv}$ such that both a and b are not in W_i'. We claim that then exists an index j $(1 \leqslant j \leqslant k)$ such that both $x, y \in W_i' \cap W_j'$. Indeed, if both vertices a and b are outside W_i' then they must both lie in some W_j' and because there are no edges between $W_i' - W_j'$ and $W_j' - W_i'$, also $x, y \in W_j'$ and the claim holds. Let us now assume that $a \notin W_i$ and $b \in W_i$. With the same argument as in the previous case, we derive that a, b and x are in some W_j'. Since both x and b belong to $W_i' \cap W_j'$, which is isometric, we must have a path of length 2 inside $W_i' \cap W_j'$. If y should not belong to W_j', we would have another vertex $t \in W_i' \cap W_j'$ joining x and b. Hence, either a, b, x, y, t induce a $K_{2,3}$ of vertices, or we get an induced $K_4 - e$ by adding an edge.

From the above observation, we are now able to describe the structure of F_{uv} in G. It is obtained from F_{uv} in G' by expanding all the edges which lie in some intersection $W_j' \cap W_k'$ $(j, k = 1, 2, \ldots, n)$ to the two corresponding edges which are also in F_{uv} by the construction. It is easy to prove that there are no other edges in F_{uv}. The set U_{uv} is thus connected. □

Analogous to the subclass of partial cubes [6], we note that the partial Hamming graphs include graphs obtainable by a sequence of connected expansions from K_1. The latter class includes the class of semi-quasi-median graphs, which in turn includes the class of quasi-median graphs. The inclusions between these classes are strict. On the left-hand side of Fig. 3 a partial Hamming graph is depicted which cannot be obtained by a sequence of connected expansions from K_1. As we have

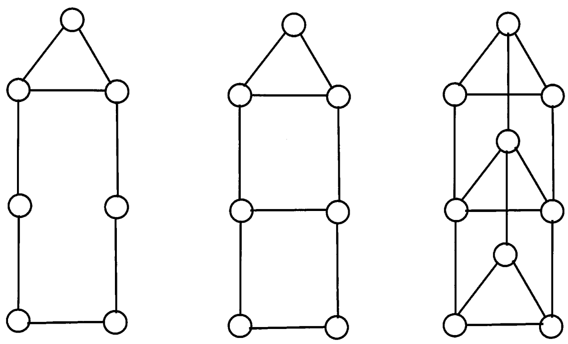

Fig. 3. Partial Hamming graphs.

already noted, the graph in Fig. 2 is obtainable by a sequence of connected expansions from K_1 but is not semi-quasi-median. The second graph in Fig. 3 is an example of semi-quasi-median graph which is not quasi-median. Finally, the third graph in the same figure is quasi-median.

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