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Note

On a conjecture concerning optimal orientations of the cartesian product of a triangle and an odd cycle

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Abstract

Let $G \times H$ denote the cartesian product of the graphs G and H, and C_n the cycle of order n. We prove the conjecture of Konig et al. that for $n \geqslant 2$, the minimum diameter of any orientation of the graph $C_3 \times C_{2n+1}$ is $n+3$.

Keywords: Minimum diameter; Orientation; Triangle; Odd cycle; Gossip problem

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the *eccentricity* $e(v)$ of v is defined as $e(v) = \max\{d(v,x) \mid x \in V(G)\}$, where $d(v,x)$ denotes the distance from v to x. The *diameter* of G, denoted by $d(G)$, is defined as $d(G) = \max\{e(v) \mid v \in V(G)\}$. Let D be a digraph with vertex set $V(D)$ and edge set $E(D)$. For $v \in V(D)$, the notions $e(v)$ and $d(D)$ are similarly defined. An *orientation* of a graph G is a digraph obtained from G by assigning to each edge in G a direction. An orientation D of G is *strong* if every two vertices in D are mutually reachable in D. Let $\mathscr{D}(G)$ be the family of strong orientations of G. Define $\vec{d}\,(G) = \min\{d(D) \mid D \in \mathscr{D}(G)\}$.

By evaluating $\vec{d}\,(G)$, we more than refine the one-way problem of Robbins [10]. Indeed, the parameter $\vec{d}\,(G)$ also provides an upper bound for the half-duplex version of the gossip problem (see for e.g., [1–3]).

Let $G \times H$ denote the cartesian product of two graphs G and H, and P_n, C_n and K_n the path, cycle and complete graph, respectively, of order n. Roberts and Xu [11–14],

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and independently Koh and Tan [5], evaluated the quantity $\vec{d}\,(P_m \times P_n)$. Recently, Koh and Lee [4] evaluated $\vec{d}\,(P_m \times C_{2n+1})$, Koh and Tay [6–8] determined the quantities $\vec{d}\,(P_m \times C_{2n})$, $\vec{d}\,(C_{2m} \times C_{2n})$ and $\vec{d}\,(K_m \times C_{2n+1})$, where $m \geqslant 4$, and Konig et al. [9] independently enumerated $\vec{d}\,(C_m \times C_n)$ for almost all m, n but not including the case $m = 3$ and $n \geqslant 5$, where n is odd. While Koh and Tay [7] remarked that the value of $\vec{d}\,(C_3 \times C_{2n+1})$, where $n \geqslant 2$, was difficult to ascertain, Konig et al. [9] proposed the following.

Conjecture. $\vec{d}\,(C_3 \times C_{2n+1}) = n + 3$ for $n \geqslant 2$.

In this note, we shall prove that this conjecture is true.

2. Notation and terminology

Given two graphs G_1 and G_2, their *cartesian product* $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

We write $V(C_n) = \{i \mid 1 \leqslant i \leqslant n\}$ and $V(C_3 \times C_{2n+1}) = \{(i,j) \mid 1 \leqslant i \leqslant 3,\ 1 \leqslant j \leqslant 2n+1\}$. Thus, two distinct vertices (i,j) and (i', j') are adjacent in $C_3 \times C_{2n+1}$ iff either $j = j'$ or '$j - j' \equiv \pm 1 (\bmod 2n+1)$ and $i = i'$'.

Let G be a graph, $F \in \mathscr{D}(G)$ and A a subdigraph of F. The eccentricity and outdegree of a vertex v in A are denoted, respectively, by $e_A(v)$ and $s_A(v)$. The subscript A is omitted if $A = F$.

Let D be a digraph. For $X \subseteq V(D)$ or $X \subseteq E(D)$, the subdigraph of D induced by X is denoted by $D[X]$. Given $F \in \mathscr{D}(C_3 \times C_{2n+1})$, where $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 2n+1$, let $F^i = F[\{i\} \times V(C_{2n+1})]$ and $F_j = F[V(C_3) \times \{j\}]$.

For $x, y \in V(D)$, we write '$x \to y$' or '$y \leftarrow x$' if $xy \in E(D)$. Also, for $A, B \subseteq V(D)$, we write '$A \to B$' or '$B \leftarrow A$' if $x \to y$ in D for all $x \in A$ and for all $y \in B$. When $A = \{x\}$, we shall write '$x \to B$' or '$B \leftarrow x$' for $A \to B$.

The *converse* of D, denoted by $\tilde{D}$, is the digraph obtained from D by reversing each arc in D.

3. The main result

First we state the following bounds obtained by Konig et al. [9].

Lemma 1. $n + 2 \leqslant \vec{d}\,(C_3 \times C_{2n+1}) \leqslant n + 3$.

For ease of presentation, we shall consider the case when $n = 2$ separately from when $n \geqslant 3$.

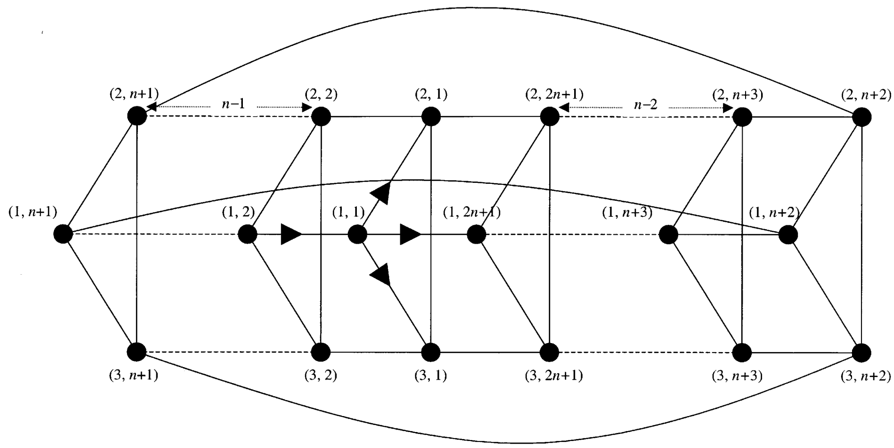

Fig. 1.

We shall first consider the general case when $n \geqslant 3$. By assuming that $\vec{d}\,(C_3 \times C_{2n+1}) = n+2$, we try to get some information about the outdegree of each vertex in $C_3 \times C_{2n+1}$.

Lemma 2. *Let* $F \in \mathscr{D}(C_3 \times C_{2n+1})$, *where* $n \geqslant 3$, *be such that* $d(F) = n+2$. *Then* $s(v) = 2$ *for all* $v \in V(F)$.

Proof. Since F is strong, $1 \leqslant s(v) \leqslant 3$ for all $v \in V(F)$. Suppose that the statement is false. As $d(\tilde{F}) = d(F)$, we assume that $s(v) = 3$ for some $v \in V(F)$. We shall split our consideration into two cases according to where the 'in' edge is.

Case 1: There exists $(i,j) \in V(F)$ such that $s((i,j)) = 3$ and either $(i,j+1) \to (i,j)$ or $(i,j-1) \to (i,j)$, where $j+1$ and $j-1$ are taken modulo $2n+1$.

We may assume that $(1,2) \to (1,1) \to \{(1,2n+1),(2,1),(3,1)\}$. (As an illustration, see Fig. 1.) We now have:

$d((2,n{+}3),(1,1)) \leqslant n{+}2$ implies that $(2,n{+}3) \to (2,n{+}4) \to \cdots \to (2,2) \to (1,2)$;
$d((3,n{+}3),(1,1)) \leqslant n{+}2$ implies that $(3,n{+}3) \to (3,n{+}4) \to \cdots \to (3,2) \to (1,2)$;
$d((1,n+3),(1,1)) \leqslant n+2$ implies that $(1,n+3) \to (1,n+) \to \cdots \to (1,2)$;
$d((1,2),(2,n+1)) \leqslant n+2$ implies that $(2,2) \to (2,3) \to \cdots \to (2,n+1)$;
$d((1,2),(3,n+1)) \leqslant n+2$ implies that $(3,2) \to (3,3) \to \cdots \to (3,n+1)$;
$d((2,1),(1,n+3)) \leqslant n+2$ implies that $(1,2n+1) \to (1,2n) \to \cdots \to (1,n+3)$;
$d((2,1),(1,n+2)) \leqslant n+2$ implies that $(2,n+1) \to (2,n+2) \to (1,n+2)$;
$d((3,1),(1,n+2)) \leqslant n+2$ implies that $(3,n+1) \to (3,n+2) \to (1,n+2)$;
$d((1,2),(2,n+2)) \leqslant n+2$ implies that $(1,n+3) \to (2,n+3) \to (2,n+2)$;
$d((1,2),(3,n+2)) \leqslant n+2$ implies that $(1,n+3) \to (3,n+3) \to (3,n+2)$.

It follows from the above sequence of arguments that $d((2,4),(1,n+3)) \geqslant n+3$, a contradiction.

Remark 1. The argument above works for $n=2$ as well. Thus if $F \in \mathscr{D}(C_3 \times C_5)$ be such that $d(F)=4$, then there does not exist $(i,j) \in V(F)$ such that $s((i,j))=1$ or 3 and either $(i,j+1) \to (i,j)$ or $(i,j-1) \to (i,j)$, where $j+1$ and $j-1$ are taken modulo 5.

Case 2: There exists $(i,j) \in V(F)$ such that $s((i,j))=3$ and either $(i+1,j) \to (i,j)$ or $(i-1,j) \to (i,j)$, where $i+1$ and $i-1$ are taken modulo 3.

We may assume that $(2,1) \to (1,1) \to \{(1,2n+1),(1,2),(3,1)\}$. The fact that $d((2,n+1),(1,1)) \leqslant n+2$ implies that $(2,n+1) \to (2,n) \to \cdots \to (2,1)$ or $(2,n+2) \to (2,n+3) \to \cdots \to (2,1)$. By symmetry, we may assume the former. The fact that $d((1,n+2),(1,1)) \leqslant n+2$ implies that $(2,2n+1) \to (2,1)$.

Suppose $(2,1) \to (3,1)$. To avoid Case 1, we must have $(3,2) \leftarrow (3,1) \to (3,2n+1)$. The fact that $d((2,2),(2,n+2)) \leqslant n+2$ implies that $(2,2) \to (1,2) \to (1,3) \to \cdots \to (1,n+1)$ or $(2,2) \to (3,2) \to (3,3) \to \cdots \to (3,n+1)$. If $(2,2) \to (1,2) \to (1,3) \to \cdots \to (1,n+1)$, then $d((1,2),(2,n+4)) \geqslant n+3$, a contradiction. If $(2,2) \to (3,2) \to (3,3) \to \cdots \to (3,n+1)$, then $d((3,2),(2,n+4)) \geqslant n+3$, a contradiction.

Thus, $(3,1) \to (2,1)$. The fact that $d((2,1),(2,n+1)) \leqslant n+2$ implies that $(1,2) \to (1,3) \to \cdots \to (1,n+1) \to (2,n+1)$.

(∗) The fact that $d((2,2),(2,n+2)) \leqslant n+2$ implies that $(2,2) \to (1,2)$ or $(2,2) \to (3,2) \to (3,3) \to \cdots \to (3,n+1)$.
Suppose $(2,2) \to (1,2)$. We have:

$d((1,2),(2,n+4)) \leqslant n+2$ implies that $(1,2) \to (3,2) \to (3,1) \to (3,2n+1)$;
$d((3,2n+1),(3,n)) \leqslant n+2$ implies that $(3,2n+1) \to (3,2n) \to \cdots \to (3,n)$;
$d((3,n+2),(1,1)) \leqslant n+2$ implies that $(3,n+2) \to (2,n+2) \to (2,n+3) \to \cdots \to (2,2n+1)$;
$d((3,2n+1),(2,n-1)) \leqslant n+2$ implies that $(3,2n+1) \to (2,2n+1)$.

To avoid Case 1, we have $(2,2n+1) \to (1,2n+1)$ and hence $d((1,2n+1),(2,n-1)) \geqslant n+3$, a contradiction. Thus, we have $(1,2) \to (2,2)$ and $(2,2) \to (3,2) \to (3,3) \to \cdots \to (3,n+1)$.

As $d((3,n+1),(1,1)) \leqslant n+2, (3,n+1) \to (2,n+1)$.
To avoid Case 1, we have $(2,n+1) \to (2,n+2)$.
As $d((2,n+2),(1,1)) \leqslant n+2$, $(2,n+2) \to (2,n+3) \to \cdots \to (2,2n+1)$.
As $d((2,1),(2,n+2)) \leqslant n+2$, $(1,2n+1) \to (1,2n) \to \cdots \to (1,n+2) \to (2,n+2)$.

The fact that $d((2,2n+1),(2,n+1)) \leqslant n+2$ implies that $(2,2n+1) \to (1,2n+1)$ or $(2,2n+1) \to (3,2n+1) \to (3,2n) \to \cdots \to (3,n+2)$. By symmetry, the argument starting from (∗) above can be analogously used to show that $(1,2n+1) \to (2,2n+1)$ and $(2,2n+1) \to (3,2n+1) \to (3,2n) \to \cdots \to (3,n+2)$. To avoid Case 1, we have $(2,n+2) \to (3,n+2)$. Then $d((3,n+2),(1,1)) \geqslant n+3$, a contradiction. □

Proposition 1. $\vec{d}\,(C_3 \times C_{2n+1}) = n+3$ *for* $n \geqslant 3$.

Proof. Suppose there exists $F \in \mathscr{D}(C_3 \times C_{2n+1})$ such that $d(F) = n+2$. By Lemma 2, $s(v) = 2$ for all $v \in V(F)$.

Suppose there exists $(i,j) \in V(F)$ such that $s_{F_j}((i,j)) = 2$. We may assume that $(i,j) = (2,1)$. Thus $(2,1) \to \{(1,1),(3,1)\}$ and we may assume that $(3,1) \to (1,1)$. By Lemma 2, $(1,1) \to \{(1,2),(1,2n+1)\}$ and $\{(2,2),(2,2n+1)\} \to (2,1)$. By symmetry and by Lemma 2, we may assume that $(3,2n+1) \to (3,1) \to (3,2)$.

The fact that $d((1,2),(2,n+4)) \leqslant n+2$ implies that $(1,2) \to (2,2)$ and the fact that $d((3,2),(2,n+4)) \leqslant n+2$ implies that $(3,2) \to (2,2)$.

By Lemma 2, $(2,2) \to (2,3)$.
As $d((2,n+1),(1,1)) \leqslant n+2$, $(2,n+1) \to (2,n+2) \to \cdots \to (2,2n+1)$.
As $d((2,1),(2,n+2)) \leqslant n+2$, $(1,2n+1) \to (1,2n) \to \cdots \to (1,n+3) \to (1,n+2) \to (2,n+2)$.
As $d((1,n+1),(1,1)) \leqslant n+2$, $(1,n+1) \to (1,n) \to \cdots \to (1,2)$.
By Lemma 2, $(1,2) \to (3,2)$.
By Lemma 2, $(3,2) \to (3,3)$.
As $d((3,n+1),(1,1)) \leqslant n+2$, $(3,n+1) \to (3,n+2) \to \cdots \to (3,2n+1)$.

Then $d((2,n+2),(2,2)) \geqslant n+3$, a contradiction.

Hence $F_j \in \mathscr{D}(C_3)$ for $1 \leqslant j \leqslant 2n+1$. For $(i,j) \in V(F)$, let i be taken modulo 3 and j be taken modulo $2n+1$. By Lemma 2, $F^i \in \mathscr{D}(C_{2n+1})$ for $1 \leqslant i \leqslant 3$. Suppose $(i,j) \to (i,j+1)$ for all $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 2n+1$. Then $d((1,1),(2,n+3)) \geqslant n+3$, a contradiction. The argument is similar if $(i,j) \to (i,j-1)$ for all $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 2n+1$. So we may assume that $(i,j) \to (i,j+1)$ for $1 \leqslant j \leqslant 2n+1$ if and only if $i = 1,3$. Call F_j *clockwise* if $(i,j) \to (i+1,j)$ for all $1 \leqslant i \leqslant 3$ and *anti-clockwise* otherwise. Without loss of generality, let F_1 be clockwise. If F_{n+3} is also clockwise, then $d((3,1),(1,n+3)) \geqslant n+3$, a contradiction. Thus F_{n+3} must be anti-clockwise. Using the argument repeatedly, we conclude that $F_{1+2p(n+2)}$ is clockwise and $F_{1+(2p-1)(n+2)}$ is anti-clockwise, where $p \geqslant 1$. Suppose $n+2$ and $2n+1$ have a common factor q. Since $2(n+2) = (2n+1)+3$, q divides 3 as well and so $q = 3$. We shall write $n = 3k+1$ and so $n+2 = 3k+3$ and $2n+1 = 6k+3$. Now, $(2n+1)/(n+2) = (6k+3)/(3k+3) = (2k+1)/(k+1)$ which is in lowest terms since the fact that $2(k+1) = (2k+1)+1$ implies that $2k+1$ and $k+1$ are coprime. Hence after $2k+1$ applications of the argument, we would return to F_1 for the first time and orient it anti-clockwise since $2k+1$ is odd. But this is a contradiction. Thus $n+2$ and $2n+1$ are coprime. Then after $2n+1$ applications of the argument, we would return to F_1 for the first time and orient it anti-clockwise since $2n+1$ is odd, a contradiction again.

Hence $\vec{d}\,(C_3C_{2n+1}) \geqslant n+3$ for $n \geqslant 3$ and the result follows from Lemma 1. □

The single case when $n = 2$ is surprisingly difficult and laborious. We present the proof here for completeness.

Proposition 2. $\vec{d}\,(C_3 \times C_5) = 5$.

Proof. Suppose there exists an $F \in \mathscr{D}(C_3 \times C_5)$ such that $d(F)=4$. Let $L_j = \{i \mid (i,j-1) \to (i,j) \text{ in } F\}$ where j and $j-1$ are taken modulo 5.

Claim 1. $1 \leqslant |L_j| \leqslant 2$.

Proof. If $|L_j|=0$, then $d((1,j-1),(2,j)) \geqslant 5$. If $|L_j|=3$, then $d((2,j),(1,j-1)) \geqslant 5$.

Claim 2. If $|L_j|=1$, then $|L_{j-1}|=1$.

Proof. Suppose to the contrary that $|L_j|=1$ and $|L_{j-1}|=2$ and assume for simplicity that $j=2$. By symmetry, we may assume $L_1=\{1,2\}$ and $(2,1) \to (1,1)$. There are three cases to consider.

Case 1: $L_2=\{1\}$.

Then $d((1,1),(2,5)) \leqslant 4$ implies that $(1,1) \to (3,1)$ and $d((3,1),(2,2)) \leqslant 4$ implies that $(3,1) \to (2,1)$. Thus $d((2,1),(2,4)) \geqslant 5$, a contradiction.

Case 2: $L_2=\{2\}$.

Then $d((1,1),(1,3)) \geqslant 5$, a contradiction.

Case: 3 $L_2=\{3\}$.

Since F is strong, we must have $(1,1) \to (3,1)$. The fact that $d((2,1),(2,4)) \leqslant 4$ implies that $(2,1) \to (3,1)$.

Suppose $(1,5) \to (3,5)$. We now have:

$d((1,2),(1,5)) \leqslant 4$ implies that $(1,2) \to (1,3) \to (1,4) \to (1,5)$;
by Remark 1, $(2,5) \to (1,5)$;
$d((1,5),(1,2)) \leqslant 4$ implies that $(3,2) \to (1,2)$;
$d((3,5),(1,2)) \leqslant 4$ implies that $(3,5) \to (3,4) \to (3,3) \to (3,2)$;
$d((3,2),(2,5)) \leqslant 4$ implies that $(3,2) \to (2,2) \to (2,3) \to (2,4) \to (2,5)$;
by Remark 1, $(3,5) \to (2,5)$;
$d((1,4),(2,2)) \leqslant 4$ implies that $(1,4) \to (3,4)$;
by Remark 1, $(3,4) \to (2,4) \to (1,4)$.

Thus $d((2,4),(2,2)) \geqslant 5$, a contradiction.

Hence $(3,5) \to (1,5)$. Suppose $(2,5) \to (3,5)$. We now have:

$d((1,1),(2,1)) \leqslant 4$ implies that $(3,2) \to (2,2)$;
$d((2,2),(2,5)) \leqslant 4$ implies that $(2,2) \to (2,3) \to (2,4) \to (2,5)$;
by Remark 1, $(1,5) \to (2,5)$;
$d((3,5),(2,2)) \leqslant 4$ implies that $(3,5) \to (3,4) \to (3,3) \to (3,2)$;
$d((3,3),(3,5)) \leqslant 4$ implies that $(3,3) \to (2,3)$;
by Remark 1, $(2,3) \to (1,3) \to (3,3)$;
$d((1,1),(1,3)) \leqslant 4$ implies that $(3,2) \to (1,2) \to (1,3)$;
$d((1,1),(2,4)) \leqslant 4$ implies that $(3,4) \to (2,4)$;
by Remark 1, $(2,4) \to (1,4) \to (3,4)$;
$d((1,1),(1,4)) \leqslant 4$ implies that $(1,5) \to (1,4)$.

Thus $d((1,4),(1,5)) \geqslant 5$, a contradiction.
Hence $(3,5) \to (2,5)$. We now have:

by Remark 1, $(3,4) \to (3,5)$;
$d((1,1),(3,4)) \leqslant 4$ implies that $(3,2) \to (3,3) \to (3,4)$;
$d((1,1),(2,4)) \leqslant 4$ implies that $(2,5) \to (2,4)$;
$d((1,1),(1,4)) \leqslant 4$ implies that $(1,5) \to (1,4)$;

Suppose $(2,2) \to (3,2)$. We now have:

by Remark 1, $(3,2) \to (1,2)$;
$d((2,5),(2,2)) \leqslant 4$ implies that $(2,4) \to (2,3) \to (2,2)$;
by Remark 1, $(1,2) \to (2,2)$;
$d((1,1),(2,3)) \leqslant 4$ implies that $(3,3) \to (2,3)$;
by Remark 1, $(2,3) \to (1,3) \to (3,3)$;
$d((1,1),(1,3)) \leqslant 4$ implies that $(1,2) \to (1,3)$.

Thus $d((1,3),(3,1)) \geqslant 5$, a contradiction.
Hence $(3,2) \to (2,2)$. We now have:

by Remark 1, $(1,2) \to (3,2)$;
$d((1,1),(1,2)) \leqslant 4$ implies that $(2,2) \to (1,2)$;
$d((1,1),(1,3)) \leqslant 4$ implies that $(3,3) \to (1,3)$;
by Remark 1, $(2,3) \to (3,3)$;
$d((1,3),(1,5)) \leqslant 4$ implies that $(1,3) \to (1,4) \to (3,4)$;
by Remark 1, $(3,4) \to (2,4)$;
$d((2,4),(1,5)) \leqslant 4$ implies that $(2,4) \to (1,4)$.

Thus $d((1,4),(1,2)) \geqslant 5$, a contradiction and Claim 2 is proved.

Remark 2. If $|L_j| = 1$ for some j, $1 \leqslant j \leqslant 5$, then by induction $|L_k| = 1$ for all $1 \leqslant k \leqslant 5$. As $d(\tilde{F}) = d(F)$, we may assume $|L_j| = 1$ for $1 \leqslant j \leqslant 5$.

Claim 3. $L_j = L_{j-1}$ for $1 \leqslant j \leqslant 5$.

Proof. Assume the contrary and by symmetry let $L_2 = \{1\}$ and $L_1 = \{3\}$.

The fact that $d((2,5),(2,2)) \leqslant 4$ implies that $(2,5) \to (2,4) \to (2,3) \to (2,2)$ and $d((1,4),(1,1)) \leqslant 4$ implies that $(3,1) \to (1,1)$.
Suppose $(1,1) \to (2,1)$. We now have:

by Remark 1, $(2,1) \to (3,1)$;
$d((3,1),(2,3)) \leqslant 4$ implies that $(1,2) \to (1,3) \to (2,3)$;
by Remark 1, $(2,3) \to (3,3)$;
$d((1,3),(1,1)) \leqslant 4$ implies that $(1,3) \to (3,3) \to (3,2)$;
by Remark 1, $(3,3) \to (3,4)$ and $(1,4) \to (1,3)$;

$d((3,4),(1,1)) \leqslant 4$ implies that $(3,4) \to (3,5)$;
by Remark 2, $(1,5) \to (1,4)$;
$d((1,4),(1,1)) \leqslant 4$ implies that $(1,4) \to (3,4)$;
by Remark 1, $(3,4) \to (2,4) \to (1,4)$.

Thus $d((2,4),(1,1)) \geqslant 5$, a contradiction.

Hence $(2,1) \to (1,1)$. By Remark 1, $(3,1) \to (2,1)$. Suppose $(1,2) \to (3,2)$. We now have:

$d((3,2),(3,4)) \leqslant 4$ implies that $(3,2) \to (3,3)$;
by Remark 2, $(1,3) \to (1,2)$;
$d((2,5),(3,2)) \leqslant 4$ implies that $(2,2) \to (3,2)$;
by Remark 1, $(1,2) \to (2,2)$;
$d((3,1),(1,3)) \leqslant 4$ implies that $(1,5) \to (1,4) \to (1,3)$;
by Remark 2, $(3,3) \to (3,4) \to (3,5)$;
$d((1,4),(1,1)) \leqslant 4$ implies that $(1,4) \to (3,4)$;
by Remark 1, $(3,4) \to (2,4) \to (1,4)$;
$d((1,3),(1,5)) \leqslant 4$ implies that $(1,3) \to (3,3)$ and $(3,5) \to (1,5)$;
by Remark 1, $(3,3) \to (2,3) \to (1,3)$ and $(1,5) \to (2,5) \to (3,5)$.

Thus $d((1,3),(2,5)) \geqslant 5$, a contradiction.

Hence $(3,2) \to (1,2)$. The fact that $d((1,2),(1,5)) \leqslant 4$ implies that $(1,2) \to (1,3) \to (1,4) \to (1,5)$ or $(1,2) \to (2,2)$.

Suppose $(1,2) \to (1,3) \to (1,4) \to (1,5)$. We now have:

by Remark 2, $(3,5) \to (3,4) \to (3,3) \to (3,2)$;
$d((1,5),(1,2)) \leqslant 4$ implies that $(1,5) \to (3,5)$;
$d((1,5),(2,2)) \leqslant 4$ implies that $(1,5) \to (2,5)$;
by Remark 1, $(2,5) \to (3,5)$;
$d((3,2),(3,4)) \leqslant 4$ implies that $(1,4) \to (3,4)$;
by Remark 1, $(1,2) \to (2,2) \to (3,2)$ and $(3,4) \to (2,4) \to (1,4)$;
$d((3,3),(3,5)) \leqslant 4$ implies that $(3,3) \to (1,3)$.

Thus $d((2,2),(3,3)) \geqslant 5$, a contradiction.

Hence $(1,2) \to (2,2)$. We now have:

by Remark 1, $(2,2) \to (3,2)$;
$d((3,1),(3,3)) \leqslant 4$ implies that $(1,2) \to (1,3) \to (3,3)$;
by Remark 2, $(3,3) \to (3,2)$;
$d((1,5),(1,2)) \leqslant 4$ implies that $(1,5) \to (3,5)$;
$d((2,5),(1,2)) \leqslant 4$ implies that $(2,5) \to (3,5)$;
by Remark 1, $(1,5) \to (2,5), (1,4) \to (1,5)$ and $(3,5) \to (3,4)$;
$d((3,3),(3,5)) \leqslant 4$ implies that $(3,3) \to (3,4) \to (1,4)$;
by Remark 2, $(1,4) \to (1,3)$.

Thus $d((3,2),(1,4)) \geqslant 5$, a contradiction and Claim 3 is proved.

By Claim 3, we may assume that $L_j = \{3\}$ for $1 \leqslant j \leqslant 5$. Because 5 is an odd number, there must be a j such that either $(2, j-1) \to (1, j-1)$ and $(2, j) \to (1, j)$, or $(1, j-1) \to (2, j-1)$ and $(1, j) \to (2, j)$. We may assume $(2,1) \to (1,1)$ and $(2,2) \to (1,2)$. We now have:

$d((1,5),(2,1)) \leqslant 4$ implies that $(3,1) \to (2,1)$;
$d((1,1),(2,2)) \leqslant 4$ implies that $(1,1) \to (3,1)$ and $(3,2) \to (2,2)$;
$d((2,1),(1,2)) \leqslant 4$ implies that $(3,2) \to (1,2)$.

Thus $d((1,2),(2,3)) \geqslant 5$, a contradiction.

Hence $\vec{d}\,(C_3 \times C_5) \geqslant 5$ and the proposition follows from Lemma 1. □

The conjecture is proven true from Propositions 1 and 2.

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