Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 336 (2007) 758–767

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Univalence and starlikeness of certain transforms defined by convolution of analytic functions ☆

M. Obradović [a], S. Ponnusamy [b,*]

[a] Department of Mathematics, Faculty of Civil Engineering, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia

[b] Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India

Received 28 January 2005

Available online 13 March 2007

Submitted by Richard M. Aron

Abstract

Let $\mathcal{U}(\lambda)$ denote the class of all analytic functions f in the unit disk Δ of the form $f(z) = z + a_2z^2 + \cdots$ satisfying the condition

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \leqslant \lambda, \quad z \in \Delta.$$

In this paper we find conditions on λ and on $c \in \mathbb{C}$ with $\operatorname{Re} c \geqslant 0 \neq c$ such that for each $f \in \mathcal{U}(\lambda)$ satisfying $(z/f(z)) * F(1, c; c+1; z) \neq 0$ for all $z \in \Delta$ the transform

$$G(z) = G_f^c(z) = \frac{z}{(z/f(z)) * F(1, c; c+1; z)}, \quad z \in \Delta,$$

is univalent or starlike. Here $F(a, b; c; z)$ denotes the Gauss hypergeometric function and $*$ denotes the convolution (or Hadamard product) of analytic functions on Δ.

Keywords: Univalent, starlike and convex functions

☆ The work was initiated during the visit of the first author to the University of Turku, Finland. The visit was supported by the Commission on Development and Exchanges of the International Mathematical Union and this author thanks CDE for its support.

* Corresponding author.

E-mail addresses: obrad@grf.bg.ac.yu (M. Obradović), samy@iitm.ac.in (S. Ponnusamy).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.03.020

1. Introduction

Let $\Delta := \{z \in \mathbb{C}\colon |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{A}$ be the set of all functions analytic in Δ with the usual normalization $f(0) = 0 = f'(0) - 1$. Also, we let $\mathcal{S} = \{f \in \mathcal{A}\colon f \text{ is univalent in } \Delta\}$. If $f \in \mathcal{S}$ maps Δ onto a starlike domain (with respect to the origin), i.e. if $tw \in f(\Delta)$ whenever $t \in [0,1]$ and $w \in f(\Delta)$, then we say that f is a starlike function. The class of all starlike functions is denoted by $\mathcal{S}^*$. A necessary and sufficient condition for $f \in \mathcal{A}$ to be starlike is the inequality [3,5]

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \Delta. \tag{1}$$

Let $\mathcal{U}(\lambda)$ denote the class of all functions $f \in \mathcal{A}$ satisfying the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \leqslant \lambda, \quad z \in \Delta.$$

We set $\mathcal{U} = \mathcal{U}(1)$. We remark that from $f \in \mathcal{U}(\lambda)$ it follows that $f(z)/z \neq 0$ for $z \in \Delta$. It is well known that $\mathcal{U} \subsetneq \mathcal{S}$ (see [1,10]) and so, for $0 \leqslant \lambda \leqslant 1$, one has $\mathcal{U}(\lambda) \subsetneq \mathcal{S}$. In a recent paper [9, Corollary 1.1] the authors have obtained the largest $r \in (0,1]$ such that for each $f \in \mathcal{S}$ the function $z \mapsto r^{-1} f(rz)$ is included in $\mathcal{U}$. More precisely, the authors have proved that

$$\max\left\{r \in (0,1]\colon r^{-1} f(rz) \in \mathcal{U} \text{ for every } f \in \mathcal{S}\right\} = 1/\sqrt{2}. \tag{2}$$

For the proof of our results, we need the following lemmas.

Lemma 1. *(See [8].) If $f \in \mathcal{U}(\lambda)$, $a := |f''(0)|/2 \leqslant 1$ and $0 \leqslant \lambda \leqslant \frac{\sqrt{2-a^2}-a}{2}$, then $f \in \mathcal{S}^*$.*

Recently, Fournier and Ponnusamy [4] have indicated a proof for the sharpness part of Lemma 1 by stating that there exists a nonstarlike function $f \in \mathcal{U}$ such that with $a = |f''(0)|/2$ it holds that

$$0 < \frac{\sqrt{2-a^2}-a}{2} < \sup_{z \in \Delta}\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \leqslant 1 - a.$$

A careful analysis of results in [4] implies that Lemma 1 is actually sharp (see also [15] for a detailed proof). For a general result, we refer to [13,14].

Lemma 2. *(See [12, Corollary 3.2].) If $f(z) = z + a_{n+1}z^{n+1} + \cdots$ $(n \geqslant 2)$ belongs to $\mathcal{U}(\lambda)$ and*

$$0 \leqslant \lambda \leqslant \frac{n-1}{\sqrt{(n-1)^2+1}},$$

then $f \in \mathcal{S}^*$.

We observe that for $n = 2$ (i.e. $f \in \mathcal{U}(\lambda)$ with $f''(0) = 0$), Lemma 2 gives Lemma 1.

Lemma 3. *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a nonvanishing analytic function on Δ and let f be of the form*

$$f(z) = \frac{z}{\phi(z)} = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}. \tag{3}$$

Then, we have the following:

(1) *If* $\sum_{n=2}^{\infty}(n-1)|b_n| \leqslant \lambda$, *then* $f \in \mathcal{U}(\lambda)$.
(2) *If* $\sum_{n=2}^{\infty}(n-1)|b_n| \leqslant 1-|b_1|$, *then* $f \in \mathcal{S}^*$.

The first part of Lemma 3 is from [7,8] whereas the second part is obtained from [16, Theorem 1]. At this place it is important to present the following example: Consider the function

$$f(z) = \frac{z}{1+ibz+(e^{2i\beta}/2)z^3}.$$

Then, for $|b| \leqslant 1/2$ and β a real number, we have (with $b_1 = ib$, $b_2 = 0$, $b_3 = e^{2i\beta}/2$ and $b_n = 0$ for $n \geqslant 4$)

$$\mathrm{Re}\left(\frac{z}{f(z)}\right) > 1-|b|-\frac{1}{2} \geqslant 0 \quad \text{and} \quad \sum_{n=2}^{\infty}(n-1)|b_n| = 1$$

and so, by Lemma 3(1), $f \in \mathcal{U} \subseteq \mathcal{S}$. On the other hand f is not in $\mathcal{S}^*$ when $0 < b \leqslant 1/2$ and $0 < \beta < \arctan(2b)$, because

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\bigg|_{z=1} = \frac{[\sin\beta - 2b\cos\beta]\sin\beta}{|1+ib+(e^{2i\beta}/2)|^2} < 0.$$

This example shows the sharpness of the condition in part (2) of Lemma 3.

2. Results

If f and g are analytic functions on Δ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then the convolution (Hadamard product) of f and g, denoted by $f * g$, is an analytic function on Δ given by

$$(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \Delta.$$

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathcal{A}$, we have a natural convolution operator defined by

$$zF(a,b;c;z) * f(z) := \sum_{n=1}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n, \quad c \notin -\mathbb{N},\ z \in \Delta, \tag{4}$$

where $(a)_n$ denotes the Pochhammer symbol $(a)_0 = 1$, $(a)_n := a(a+1)\cdots(a+n-1)$ for $n \in \mathbb{N}$. Here $F(a,b;c;z)$ denotes the Gauss hypergeometric function which is analytic in Δ. As a special case of the Euler integral representation for the hypergeometric function, one has

$$F(1,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{1}{1-tz} t^{b-1}(1-t)^{c-b-1}\,dt, \quad z \in \Delta,\ \mathrm{Re}\,c > \mathrm{Re}\,b > 0.$$

Using this representation we have, for $f \in \mathcal{A}$,

$$zF(1,c;c+1;z) * f(z) = z\left(F(1,c;c+1;z) * \frac{f(z)}{z}\right)$$

and therefore, we obtain the following form:

$$zF(1,c;c+1;z) * f(z) = zc\int_0^1 \frac{f(tz)}{tz} t^{c-1}\,dt, \quad z \in \Delta,\ \mathrm{Re}\,c > 0. \tag{5}$$

Now, we state and prove our results.

Theorem 1. *Let* $f \in \mathcal{U}(\lambda)$ *and* $c \in \mathbb{C}$ *with* $\operatorname{Re} c \geqslant 0 \neq c$ *such that*

$$\big(z/f(z)\big) * F(1, c; c+1; z) \neq 0 \quad \text{in } \Delta,$$

and $G = G_f^c$ *be the transform defined by*

$$G(z) = \frac{z}{(z/f(z)) * F(1, c; c+1; z)}, \quad z \in \Delta. \tag{6}$$

Further, let A *be a nonnegative real number such that* $A = |\frac{c}{c+1}\frac{f''(0)}{2}| \leqslant 1$. *Then we have the following*:

(1) $G \in \mathcal{U}(\lambda|c|/|c+2|)$. *The result is sharp especially when* $|f''(0)/2| \leqslant 1 - \lambda$. *In particular,* $G \in \mathcal{U}$ *whenever* $0 < \lambda \leqslant |(c+2)/c|$.

(2) $G \in \mathcal{S}^*$ *whenever* $0 < \lambda \leqslant \frac{|c+2|}{2|c|}(\sqrt{2 - A^2} - A)$.
In particular, if $\lambda = 1$, $f''(0) = 0$ *and* $|c - 2| \leqslant 2\sqrt{2}$ *with* $\operatorname{Re} c \geqslant 0$, *then* $G \in \mathcal{S}^*$.

Proof. We consider the function

$$\frac{z}{G(z)} = \frac{z}{f(z)} * F(1, c; c+1; z), \quad z \in \Delta. \tag{7}$$

Differentiating $z/G(z)$ shows that

$$(c+1)\frac{z}{G(z)} - \left(\frac{z}{G(z)}\right)^2 G'(z) = c\frac{z}{G(z)} + z\left(\frac{z}{G(z)}\right)', \quad z \in \Delta. \tag{8}$$

Further, using the series expansion of $F(1, c; c+1; z)$ from (4), we have

$$F(1, c; c+1; z) = 1 + \sum_{n=1}^{\infty} \frac{(c)_n}{(c+1)_n} z^n = 1 + \sum_{n=1}^{\infty} \frac{c}{c+n} z^n, \quad z \in \Delta, \tag{9}$$

which yields

$$cF(1, c; c+1; z) + zF'(1, c; c+1; z) = \frac{c}{1-z}, \quad z \in \Delta,$$

from which in combination with (7) and (8), one obtains

$$(c+1)\frac{z}{G(z)} - \left(\frac{z}{G(z)}\right)^2 G'(z) = c\frac{z}{f(z)}, \quad z \in \Delta. \tag{10}$$

Now, we set

$$p(z) = \left(\frac{z}{G(z)}\right)^2 G'(z).$$

Then $p(z)$ is analytic on Δ (with $p(0) = 1$ and $p'(0) = 0$); for one has the relations (7) and, by (10),

$$p(z) = (c+1)\frac{z}{G(z)} - c\frac{z}{f(z)}, \quad z \in \Delta, \tag{11}$$

and $z \mapsto z/f(z)$ is analytic on Δ, as by assumption $f \in \mathcal{U}(\lambda)$ and so $f(z)/z \neq 0$ on Δ. From (8), (10) and (11) one then obtains that

$$\begin{aligned} cp(z)+zp'(z) &= (c+1)c\frac{z}{G(z)}+(c+1)z\left(\frac{z}{G(z)}\right)'-c^2\frac{z}{f(z)}-cz\left(\frac{z}{f(z)}\right)' \\ &= c\left[(c+1)\frac{z}{f(z)}-c\frac{z}{f(z)}-z\left(\frac{z}{f(z)}\right)'\right] \\ &= c\left[\frac{z}{f(z)}-z\left(\frac{z}{f(z)}\right)'\right] \\ &= c\left(\frac{z}{f(z)}\right)^2 f'(z). \end{aligned} \tag{12}$$

Now, as $f\in\mathcal{U}(\lambda)$, it follows that

$$\left|p(z)+\frac{1}{c}zp'(z)-1\right|<\lambda, \quad z\in\Delta, \tag{13}$$

and so (because $p'(0)=0$), from the work of Hallenbeck and Ruscheweyh [6] (see also [11]), we deduce that

$$\left|p(z)-1\right|\leqslant\frac{\lambda|c|}{|c+2|}|z|^2, \quad z\in\Delta.$$

The conclusion (1) follows and the bound $\lambda|c|/|c+2|$ is sharp. To prove the sharpness, we consider functions f in $\mathcal{U}(\lambda)$ of the form

$$f(z)=\frac{z}{1-a_2z+\lambda z^2}, \quad z\in\Delta,$$

where $a_2=f''(0)/2$ and $|a_2|\leqslant 1-\lambda$, so that $1-a_2z+\lambda z^2\neq 0$ for all $z\in\Delta$. Moreover, since $\operatorname{Re} c\geqslant 0$, it follows that $|c+2|>|c+1|>|c|$ and, therefore,

$$\left|1-a_2\frac{c}{c+1}z+\lambda\frac{c}{c+2}z^2\right|\neq 0$$

for all $z\in\Delta$, provided $|a_2|\leqslant 1-\lambda$. Then, by (6) and (9), a computation gives

$$G(z)=\frac{z}{1-a_2(c/(c+1))z+(\lambda c/(c+2))z^2}$$

which is analytic on Δ, $z/G(z)\neq 0$ on Δ and

$$\left(\frac{z}{G(z)}\right)^2 G'(z)-1=-\frac{\lambda c}{c+2}z^2.$$

We have that $G\in\mathcal{U}(\lambda|c|/|c+2|)$.

The second part is a consequence of Lemma 1. In fact, it suffices to observe from the definition of $G(z)$ that

$$A:=\left|\frac{G''(0)}{2}\right|=\left|\frac{c}{c+1}\frac{f''(0)}{2}\right|.$$

Then, by Lemma 1, G is starlike whenever $A\leqslant 1$ and

$$0\leqslant\frac{\lambda|c|}{|c+2|}\leqslant\frac{\sqrt{2-A^2}-A}{2}$$

and the result follows from the last inequality. $\square$

Remark. We recall first that if $|a_2| \leqslant 1-\lambda$, then it is known that [8]

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{1+|a_2|+\lambda} \geqslant \frac{1}{2} \quad \text{for } z \in \Delta. \tag{14}$$

Further, from the work of Ruscheweyh [17, Lemma 2], it follows that

$$\operatorname{Re} F(1,c;c+1;z) > \frac{1}{2}, \quad z \in \Delta,\ \operatorname{Re} c \geqslant 0. \tag{15}$$

From (14), it follows that $\operatorname{Re}(f(z)/z) > 0$, $z \in \Delta$. From this observation and (15), we obtain (using either the Herglotz representation formula for functions with positive real part or [18]) that

$$\operatorname{Re}\left(\frac{f(z)}{z} * F(1,c;c+1;z)\right) > 0, \quad z \in \Delta,\ \operatorname{Re} c \geqslant 0,$$

and so, in particular, that $(z/f(z)) * F(1,c;c+1;z) \neq 0$ for all $z \in \Delta$, $\operatorname{Re} c \geqslant 0$.

Remark. In case $\operatorname{Re} c > 0$, the formula (5) shows that the transform $G(z) = G_f^c(z)$ defined by (6) has a second representation in the form

$$G(z) = z\left(c\int_0^1 \frac{tz}{f(tz)} t^{c-1}\,dt\right)^{-1}, \quad z \in \Delta.$$

Using Lemma 2, Theorem 1 can be generalized as follows:

Theorem 2. *For a fixed $n \geqslant 2$, let $f(z) = z + a_{n+1}z^{n+1} + \cdots$ belong to $\mathcal{U}(\lambda)$ and let $c \in \mathbb{C}$ with $\operatorname{Re} c \geqslant 0 \neq c$ such that $(z/f(z)) * F(1,c;c+1;z) \neq 0$ in Δ, and $G = G_f^c$ be the transform defined by* (6). *Then we have the following*:

(1) $G \in \mathcal{U}(\lambda|c|/|c+n|)$. *In particular, $G \in \mathcal{U}$ whenever $0 < \lambda \leqslant |(c+n)/c|$.*
(2) $G \in \mathcal{S}^*$ *whenever* $0 < \lambda \leqslant \frac{|c+n|(n-1)}{|c|\sqrt{(n-1)^2+1}}$.

Proof. We note that

$$\frac{z}{f(z)} = \frac{1}{1+a_{n+1}z^n+\cdots} = 1 - a_{n+1}z^n + \cdots,$$

so that

$$\frac{z}{f(z)} * F(1,c;c+1;z) = 1 - a_{n+1}\left(\frac{c}{c+n}\right)z^n + \cdots.$$

Thus, G can be written in the form

$$G(z) = z + a_{n+1}\left(\frac{c}{c+n}\right)z^{n+1} + \cdots$$

and therefore, as in the proof of Theorem 1, the function p defined by

$$p(z) = \left(\frac{z}{G(z)}\right)^2 G'(z) = 1 + (n-1)a_{n+1}\left(\frac{c}{c+n}\right)z^n + \cdots$$

is analytic in Δ such that $p(0)=1$, $p'(0)=\cdots=p^{(n-1)}(0)=0$. As $f\in\mathcal{U}(\lambda)$, p satisfies (13). Consequently (see [6,11]),

$$\bigl|p(z)-1\bigr|\leqslant\frac{\lambda|c||z|^n}{|c+n|},\quad z\in\Delta,$$

and the proof of part (1) is complete. The second part is a consequence of Lemma 2. □

3. Sufficient conditions for functions in $\mathcal{U}$ or in $\mathcal{S}^*$

We recall that $\mathcal{U}\subsetneq\mathcal{S}$. Next we consider the following question: *Given a univalent function f, is it possible to generate functions in $\mathcal{U}$ or in $\mathcal{S}^*$?* Our next result actually provides a method of obtaining functions in $\mathcal{U}$.

Theorem 3. *Let $h(z)=1+\sum_{n=1}^{\infty}c_nz^n$ be an analytic function on Δ and $a_2\in\mathbb{C}$ such that*

$$|c_1a_2|+\left(\sum_{n=2}^{\infty}\frac{|c_n|^2}{n-1}\right)^{1/2}\leqslant 1\quad\text{and}\quad\lambda:=\left(\sum_{n=2}^{\infty}(n-1)|c_n|^2\right)^{1/2}<+\infty.\tag{16}$$

Then for every function $f\in\mathcal{S}$ with $f''(0)/2=a_2$ the function H_f defined by

$$\frac{z}{H_f(z)}=\left(\frac{z}{f(z)}\right)*h(z)$$

belongs to $\mathcal{U}(\lambda)$, and thus to $\mathcal{S}$ if $\lambda\leqslant 1$, and even to $\mathcal{S}^*$ if $\lambda\leqslant 1-|a_2c_1|$.

Proof. Let $f\in\mathcal{S}$ and be of the form (3). Then $a_2=f''(0)/2=-b_1$,

$$\frac{z}{H_f(z)}=\left(\frac{z}{f(z)}\right)*h(z)=1+\sum_{n=1}^{\infty}b_nc_nz^n$$

and from the well-known Area Theorem [5, Theorem 11, p. 193, Vol. 2] we have

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2\leqslant 1.\tag{17}$$

Now, by the triangle inequality, we see for all $z\in\Delta$ that

$$\begin{aligned}\left|\frac{z}{H_f(z)}\right|&\geqslant 1-|c_1b_1||z|-\sum_{n=2}^{\infty}\bigl(\sqrt{n-1}|b_n|\bigr)\left(\frac{|c_n|}{\sqrt{n-1}}\right)|z|^n\\&\geqslant 1-|c_1a_2||z|-|z|^2\sum_{n=2}^{\infty}\bigl(\sqrt{n-1}|b_n|\bigr)\left(\frac{|c_n|}{\sqrt{n-1}}\right)\\&\geqslant 1-|c_1a_2||z|-|z|^2\left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty}\frac{|c_n|^2}{n-1}\right)^{1/2}\\&\qquad\text{(by Cauchy–Schwarz inequality)}\\&\geqslant 1-|c_1a_2|-\left(\sum_{n=2}^{\infty}\frac{|c_n|^2}{n-1}\right)^{1/2}\quad\text{by (17)}\\&\geqslant 0\quad\text{by (16).}\end{aligned}$$

Using this and the first inequality in (16), we obtain that $z/H_f(z) \neq 0$ in Δ. Next we find that

$$\sum_{n=2}^{\infty}(n-1)|c_n b_n| = \sum_{n=2}^{\infty}\big(\sqrt{n-1}|b_n|\big)\big(\sqrt{n-1}|c_n|\big) \leqslant \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty}(n-1)|c_n|^2\right)^{1/2} \leqslant \lambda \quad \text{by (17) and (16).}$$

Thus, $H_f \in \mathcal{U}(\lambda)$ by Lemma 3(1), and, in particular, $H_f \in \mathcal{U} \subseteq \mathcal{S}$ if $\lambda \leqslant 1$. By Lemma 3(2), we obtain the last part of the conclusion. □

Example 1. Choose $h(z) = 1/(1-az)$ with $|a| = r < 1$. Then, according to (16), r has to satisfy the condition

$$|a_2|r + r\big(\log\big(1/\big(1-r^2\big)\big)\big)^{1/2} \leqslant 1 \quad \text{and} \quad \lambda = r^2/\big(1-r^2\big).$$

Then for each function $f \in \mathcal{S}$ with $f''(0)/2 = a_2$ the function $a^{-1}f(az)$ belongs to $\mathcal{U}(\lambda)$ and thus to $\mathcal{S}$ if $\lambda \leqslant 1$, and even to $\mathcal{S}^*$ if $\lambda \leqslant 1 - |a_2|r$. In particular, it is a simple exercise to see that

$$f \in \mathcal{S} \quad \text{with } f''(0) = 0 \quad \Rightarrow \quad a^{-1}f(az) \in \mathcal{U} \cap \mathcal{S}^*$$

whenever $0 < |a| = r \leqslant 1/\sqrt{2}$. At this place it is interesting to compare with (2).

Example 2. Choose $h(z) = 1/(1-az^2)$ with $|a| = r < 1$. Then, by (16), r has to satisfy the condition

$$\frac{r}{2}\log\left(\frac{1+r}{1-r}\right) \leqslant 1 \quad \text{and} \quad \lambda = \frac{r\sqrt{1+r^2}}{1-r^2}.$$

Therefore, if $f \in \mathcal{S}$ then the function $z/((z/f(z)) * h(z))$ belongs to $\mathcal{U}(\lambda)$ and thus to $\mathcal{S}^*$ if $\lambda \leqslant 1$ (since $h'(0) = 0$). In fact, it is a simple exercise to see that the second condition $\lambda \leqslant 1$ is equivalent to $r \leqslant 1/\sqrt{3}$, while the first condition is equivalent to the inequality

$$g(r) = (1-r)e^{2/r} - 1 - r \geqslant 0$$

which holds if $r \leqslant 1/\sqrt{3}$. Thus, if ω and ω' denote the two square roots of a and if $f \in \mathcal{S}$, then the function H_f defined by

$$\frac{z}{H_f(z)} = \frac{z}{f(z)} * h(z) = \frac{1}{2}\left(\frac{\omega z}{f(\omega z)} + \frac{\omega' z}{f(\omega' z)}\right)$$

belongs to $\mathcal{S}^*$ for $r \leqslant 1/\sqrt{3}$.

Corollary 1. *Let $f \in \mathcal{S}$ be of the form* (3) *with $a_2 = f''(0)/2$, and*

$$h(z) = 1 + c_1 z + a\sum_{n=2}^{\infty}\frac{1}{(n+1)\sqrt{n-1}}z^n$$

for some complex constant a, such that

$$|c_1 a_2| + |a|\sqrt{\frac{\pi^2}{12} - \frac{11}{16}} \leqslant 1 \quad \textit{and} \quad \lambda = |a|\sqrt{\frac{\pi^2}{6} - \frac{5}{4}}.$$

Then the function H_f defined by $z/H_f(z) = (z/f(z)) * h(z)$ belongs to $\mathcal{U}(\lambda)$, and thus to $\mathcal{S}$ if $\lambda \leqslant 1$, and even to $\mathcal{S}^*$ if $\lambda \leqslant 1 - |c_1 a_2|$.

Proof. Set $c_n = a/((n+1)\sqrt{n-1})$ for all $n \geqslant 2$. The condition (16) takes the form

$$|c_1 a_2| + |a| \left(\sum_{n=2}^{\infty} \frac{1}{(n^2-1)^2} \right)^{1/2} \leqslant 1 \quad \text{and} \quad \lambda = |a| \left(\sum_{n=2}^{\infty} \frac{1}{(n+1)^2} \right)^{1/2}.$$

Recall that

$$\sum_{n=2}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - \frac{5}{4}.$$

Now, if we write

$$\frac{1}{(n^2-1)^2} = \frac{1}{4} \left[\frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} - \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \right],$$

then it is a simple exercise to see that

$$\sum_{n=2}^{\infty} \frac{1}{(n^2-1)^2} = \frac{1}{4} \left[2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - \frac{1}{4} - \frac{3}{2} \right] = \frac{\pi^2}{12} - \frac{11}{16}.$$

The conclusion follows from Theorem 3. □

Finally, it would be appropriate to recall the recent result of the authors in [2] in which a number of interesting applications are also derived.

Theorem 4. *(See [2, Theorem 3.9].) Let $f, g \in \mathcal{S}$ with the representations*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \qquad \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

If

$$\Phi(z) = \frac{z}{f(z)} * \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} b_n c_n z^n \neq 0$$

for every $z \in \Delta$, then $F(z) = \frac{z}{\Phi(z)} \in \mathcal{U}$, and, in particular, F is univalent in Δ.

Acknowledgment

The authors thank the referee for detailed and valuable comments.

References

[1] L.A. Aksentiev, Sufficient conditions for univalence of regular functions, Izv. Vysš. Učebn. Zaved. Mat. 3 (4) (1958) 3–7 (in Russian).

[2] R.W. Barnard, S. Naik, M. Obradović, S. Ponnusamy, Two parameter families of close-to-convex functions and convolution theorems, Analysis (Munich) 24 (2004) 71–94.

[3] P.L. Duren, Univalent Functions, Grundlehren Math. Wiss., vol. 259, Springer-Verlag, New York, 1983.

[4] R. Fournier, S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, Complex Var. Elliptic Equ. 52 (1) (2007) 1–8.
[5] A.W. Goodman, Univalent Functions, vols. 1–2, Mariner, Tampa, FL, 1983.
[6] D.J. Hallenbeck, St. Ruscheweyh, Subordination by convex functions, Proc. Amer. Math. Soc. 52 (1975) 191–195.
[7] M. Obradović, S. Ponnusamy, New criteria and distortion theorems for univalent functions, Complex Var. Theory Appl. 44 (2001) 173–191; also Reports of the Department of Mathematics, preprint 190, University of Helsinki, Finland, June 1998.
[8] M. Obradović, S. Ponnusamy, V. Singh, P. Vasundhra, Univalency, starlikesess and convexity applied to certain classes of rational functions, Analysis (Munich) 22 (3) (2002) 225–242.
[9] M. Obradović, S. Ponnusamy, Radius properties for subclasses of univalent functions, Analysis (Munich) 25 (2005) 183–188.
[10] S. Ozaki, M. Nunokawa, The Schwarzian derivative and univalent functions, Proc. Amer. Math. Soc. 33 (1972) 392–394.
[11] S. Ponnusamy, Differential subordination and Bazilevič functions, Proc. Ind. Acad. Sci. Math. Sci. 105 (1995) 169–186.
[12] S. Ponnusamy, P. Sahoo, Geometric properties of certain linear integral transforms, Bull. Belg. Math. Soc. Simon Stevin 12 (2005) 95–108.
[13] S. Ponnusamy, P. Sahoo, Special classes of univalent functions with missing coefficients and integral transforms, Bull. Malays. Math. Sci. Soc. (2) 280 (2005) 141–156.
[14] S. Ponnusamy, P. Vasundhra, Criteria for univalence, starlikeness and convexity, Ann. Polon. Math. 85 (2005) 121–133.
[15] S. Ponnusamy, P. Vasundhra, Sharpness results of certain class of analytic functions, preprint.
[16] M.O. Reade, H. Silverman, P.G. Todorov, On the starlikeness and convexity of a class of analytic functions, Rend. Circ. Mat. Palermo 33 (1984) 265–272.
[17] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975) 109–115.
[18] S. Stankiewicz, Z. Stankiewicz, Some applications of Hadamard convolutions in the theory of functions, Ann. Univ. Mariae Curie-Skłodowska 40 (1986) 251–265.