Univalence and starlikeness of certain transforms defined by convolution of analytic functions

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Abstract

Let \( U(\lambda) \) denote the class of all analytic functions \( f \) in the unit disk \( \Delta \) of the form \( f(z) = z + a_2 z^2 + \cdots \) satisfying the condition

\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, \quad z \in \Delta.
\]

In this paper we find conditions on \( \lambda \) and on \( c \in \mathbb{C} \) with \( \text{Re} c \geq 0 \neq c \) such that for each \( f \in U(\lambda) \) satisfying \( (z/f(z)) \ast F(1, c; c + 1; z) \neq 0 \) for all \( z \in \Delta \) the transform

\[
G(z) = G^c_f(z) = \frac{z}{(z/f(z)) \ast F(1, c; c + 1; z)}, \quad z \in \Delta,
\]

is univalent or starlike. Here \( F(a, b; c; z) \) denotes the Gauss hypergeometric function and \( \ast \) denotes the convolution (or Hadamard product) of analytic functions on \( \Delta \).

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1. Introduction

Let \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \) be the open unit disk in the complex plane \( \mathbb{C} \) and \( \mathcal{A} \) be the set of all functions analytic in \( \Delta \) with the usual normalization \( f(0) = 0 = f'(0) - 1 \). Also, we let \( \mathcal{S} = \{ f \in \mathcal{A} : f \) is univalent in \( \Delta \} \). If \( f \in \mathcal{S} \) maps \( \Delta \) onto a starlike domain (with respect to the origin), i.e. if \( tw \in f(\Delta) \) whenever \( t \in [0, 1] \) and \( w \in f(\Delta) \), then we say that \( f \) is a starlike function. The class of all starlike functions is denoted by \( \mathcal{S}^* \). A necessary and sufficient condition for \( f \in \mathcal{A} \) to be starlike is the inequality \[ \Re \left( \frac{zf'(z)}{f(z)} \right) > 0, \quad z \in \Delta. \] (1)

Let \( \mathcal{U}(\lambda) \) denote the class of all functions \( f \in \mathcal{A} \) satisfying the condition
\[
\left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq \lambda, \quad z \in \Delta.
\]
We set \( \mathcal{U} = \mathcal{U}(1) \). We remark that from \( f \in \mathcal{U}(\lambda) \) it follows that \( f(z) / z \neq 0 \) for \( z \in \Delta \). It is well known that \( \mathcal{U} \subset \mathcal{S} \) (see \cite{1,10}) and so, for \( 0 \leq \lambda \leq 1 \), one has \( \mathcal{U}(\lambda) \subset \mathcal{S} \). In a recent paper \cite[Corollary 1.1]{9} the authors have obtained the largest \( r \in (0, 1] \) such that for each \( f \in \mathcal{S} \) the function \( z \mapsto r^{-1} f(rz) \) is included in \( \mathcal{U} \). More precisely, the authors have proved that
\[
\max \{ r \in (0, 1] : r^{-1} f(rz) \in \mathcal{U} \text{ for every } f \in \mathcal{S} \} = 1/\sqrt{2}.
\] (2)

For the proof of our results, we need the following lemmas.

**Lemma 1.** (See \cite{8}.) If \( f \in \mathcal{U}(\lambda) \), \( a := |f''(0)|/2 \leq 1 \) and \( 0 \leq \lambda \leq \sqrt{2-a^2-a} \), then \( f \in \mathcal{S}^* \).

Recently, Fournier and Ponnusamy \cite{4} have indicated a proof for the sharpness part of Lemma 1 by stating that there exists a nonstarlike function \( f \in \mathcal{U} \) such that with \( a = \frac{|f''(0)|}{2} \) it holds that
\[
0 < \frac{\sqrt{2-a^2-a}}{2} < \sup_{z \in \Delta} \left| f'(z) \left( \frac{z}{f(z)} \right)^2 - 1 \right| \leq 1 - a.
\]
A careful analysis of results in \cite{4} implies that Lemma 1 is actually sharp (see also \cite{15} for a detailed proof). For a general result, we refer to \cite{13,14}.

**Lemma 2.** (See \cite[Corollary 3.2]{12}.) If \( f(z) = z + a_{n+1}z^{n+1} + \cdots (n \geq 2) \) belongs to \( \mathcal{U}(\lambda) \) and
\[
0 \leq \lambda \leq \frac{n-1}{\sqrt{(n-1)^2+1}},
\]
then \( f \in \mathcal{S}^* \).

We observe that for \( n = 2 \) (i.e. \( f \in \mathcal{U}(\lambda) \) with \( f''(0) = 0 \)), Lemma 2 gives Lemma 1.

**Lemma 3.** Let \( \phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \) be a nonvanishing analytic function on \( \Delta \) and let \( f \) be of the form
\[
f(z) = \frac{z}{\phi(z)} = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}.
\] (3)
Then, we have the following:
(1) If \( \sum_{n=2}^{\infty} (n-1)|b_n| \leq \lambda \), then \( f \in \mathcal{U}(\lambda) \).

(2) If \( \sum_{n=2}^{\infty} (n-1)|b_n| \leq 1 - |b_1| \), then \( f \in \mathcal{S}^* \).

The first part of Lemma 3 is from [7,8] whereas the second part is obtained from [16, Theorem 1]. At this place it is important to present the following example: Consider the function

\[ f(z) = z + ibz + (e^{2i\beta/2})z^3. \]

Then, for \( |b| \leq 1/2 \) and \( \beta \) a real number, we have (with \( b_1 = ib \), \( b_2 = 0 \), \( b_3 = e^{2i\beta/2} \) and \( b_n = 0 \) for \( n \geq 4 \))

\[ \Re \left( \frac{zf'(z)}{f(z)} \right) \bigg|_{z=1} = \frac{[\sin \beta - 2b \cos \beta] \sin \beta}{|1 + ib + (e^{2i\beta/2})|^2} < 0. \]

This example shows the sharpness of the condition in part (2) of Lemma 3.

2. Results

If \( f \) and \( g \) are analytic functions on \( \Delta \) with \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=0}^{\infty} b_n z^n \), then the convolution (Hadamard product) of \( f \) and \( g \), denoted by \( f \ast g \), is an analytic function on \( \Delta \) given by

\[ (f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n, \quad z \in \Delta. \]

For \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in \( \mathcal{A} \), we have a natural convolution operator defined by

\[ zF(a, b; c; z) := \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} a_n z^n, \quad c \neq -\mathbb{N}, \quad z \in \Delta, \quad (4) \]

where \((a)_n\) denotes the Pochhammer symbol \((a)_0 = 1, (a)_n := a(a+1)\ldots(a+n-1)\) for \( n \in \mathbb{N} \).

Here \( F(a, b; c; z) \) denotes the Gauss hypergeometric function which is analytic in \( \Delta \). As a special case of the Euler integral representation for the hypergeometric function, one has

\[ F(1, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} \frac{1}{1-tz} t^{b-1}(1-t)^{c-b-1} dt, \quad z \in \Delta, \quad \Re c > \Re b > 0. \]

Using this representation we have, for \( f \in \mathcal{A} \),

\[ zF(1, c; c+1; z) \ast f(z) = z \left( F(1, c; c+1; z) \ast \frac{f(z)}{z} \right) \]

and therefore, we obtain the following form:

\[ zF(1, c; c+1; z) \ast f(z) = z c \int_{0}^{1} \frac{f(tz)}{tz} t^{c-1} dt, \quad z \in \Delta, \quad \Re c > 0. \quad (5) \]
Now, we state and prove our results.

**Theorem 1.** Let \( f \in U(\lambda) \) and \( c \in \mathbb{C} \) with \( \Re c \geq 0 \neq c \) such that
\[
\left( \frac{z}{f(z)} \right) \ast F(1, c; c + 1; z) \neq 0 \quad \text{in } \Delta,
\]
and \( G = G_c \) be the transform defined by
\[
G(z) = \frac{z}{(z/f(z)) \ast F(1, c; c + 1; z)}, \quad z \in \Delta.
\]
Further, let \( A \) be a nonnegative real number such that
\[
A = \left| \frac{c}{c+1} \frac{f''(0)}{2} \right| \leq 1.
\]
Then we have the following:

1. \( G \in U(\lambda |c|/|c + 2|). \) The result is sharp especially when \( |f''(0)/2| \leq 1 - \lambda. \) In particular, \( G \in U \) whenever \( 0 < \lambda \leq \frac{|c+2|}{2|c|} \).
2. \( G \in S^* \) whenever \( 0 < \lambda \leq \frac{|c+2|}{2|c|} \left( \sqrt{2} - A^2 - A \right) \).

In particular, if \( \lambda = 1, \) \( f''(0) = 0 \) and \( |c - 2| \leq 2 \sqrt{2} \) with \( \Re c \geq 0, \) then \( G \in S^*. \)

**Proof.** We consider the function
\[
\frac{z}{G(z)} = \frac{z}{f(z)} \ast F(1, c; c + 1; z), \quad z \in \Delta.
\]
Differentiating \( z/G(z) \) shows that
\[
(c + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = c \frac{z}{G(z)} + z \left( \frac{z}{G(z)} \right)', \quad z \in \Delta.
\]
Further, using the series expansion of \( F(1, c; c + 1; z) \) from (4), we have
\[
F(1, c; c + 1; z) = 1 + \sum_{n=1}^{\infty} \frac{(c)_n}{(c+1)_n} z^n = 1 + \sum_{n=1}^{\infty} \frac{c}{c+n} z^n, \quad z \in \Delta,
\]
which yields
\[
c F(1, c; c + 1; z) + z F'(1, c; c + 1; z) = \frac{c}{1-z}, \quad z \in \Delta,
\]
from which in combination with (7) and (8), one obtains
\[
(c + 1) \frac{z}{G(z)} - \left( \frac{z}{G(z)} \right)^2 G'(z) = c \frac{z}{f(z)}, \quad z \in \Delta.
\]
Now, we set
\[
p(z) = \left( \frac{z}{G(z)} \right)^2 G'(z).
\]
Then \( p(z) \) is analytic on \( \Delta \) (with \( p(0) = 1 \) and \( p'(0) = 0 \)); for one has the relations (7) and, by (10),
\[
p(z) = (c + 1) \frac{z}{G(z)} - c \frac{z}{f(z)}, \quad z \in \Delta,
\]
and \( z \mapsto z/f(z) \) is analytic on \( \Delta, \) as by assumption \( f \in U(\lambda) \) and so \( f(z)/z \neq 0 \) on \( \Delta. \) From (8), (10) and (11) one then obtains that
\[ cp(z) + zp'(z) = (c + 1)c \frac{z}{G(z)} + (c + 1)z \left( \frac{z}{G(z)} \right)' - c^2 \frac{z}{f(z)} - cz \left( \frac{z}{f(z)} \right)' = c \left[ (c + 1) \frac{z}{f(z)} - c - \frac{z}{f(z)} - z \left( \frac{z}{f(z)} \right)' \right] = c \left[ \frac{z}{f(z)} - z \left( \frac{z}{f(z)} \right)' \right] = c \left( \frac{z}{f(z)} \right)^2 f'(z). \quad (12) \]

Now, as \( f \in \mathcal{U}(\lambda) \), it follows that
\[
\left| p(z) + \frac{1}{c} z p'(z) - 1 \right| < \lambda, \quad z \in \Delta, \quad \text{(13)}
\]
and so (because \( p'(0) = 0 \)), from the work of Hallenbeck and Ruscheweyh [6] (see also [11]), we deduce that
\[
\left| p(z) - 1 \right| \leq \frac{\lambda |c|}{|c + 2|} |z|^2, \quad z \in \Delta.
\]

The conclusion (1) follows and the bound \( \lambda |c|/|c + 2| \) is sharp. To prove the sharpness, we consider functions \( f \) in \( \mathcal{U}(\lambda) \) of the form
\[
f(z) = \frac{z}{1 - a_2 z + \lambda z^2}, \quad z \in \Delta,
\]
where \( a_2 = f''(0)/2 \) and \( |a_2| \leq 1 - \lambda \), so that \( 1 - a_2 z + \lambda z^2 \neq 0 \) for all \( z \in \Delta \). Moreover, since \( \Re c \geq 0 \), it follows that \( |c + 2| > |c + 1| > |c| \) and, therefore,
\[
\left| 1 - a_2 \frac{c}{c + 1} z + \lambda \frac{c}{c + 2} z^2 \right| \neq 0
\]
for all \( z \in \Delta \), provided \( |a_2| \leq 1 - \lambda \). Then, by (6) and (9), a computation gives
\[
G(z) = \frac{z}{1 - a_2 (c/(c + 1)) z + (\lambda c/(c + 2)) z^2}
\]
which is analytic on \( \Delta \), \( z/G(z) \neq 0 \) on \( \Delta \) and
\[
\left( \frac{z}{G(z)} \right)^2 G'(z) - 1 = -\frac{\lambda c}{c + 2} z^2.
\]
We have that \( G \in \mathcal{U}(\lambda|c|/|c + 2|) \).

The second part is a consequence of Lemma 1. In fact, it suffices to observe from the definition of \( G(z) \) that
\[
A := \left| \frac{G''(0)}{2} \right| = \left| \frac{c}{c + 1} \frac{f''(0)}{2} \right|.
\]
Then, by Lemma 1, \( G \) is starlike whenever \( A \leq 1 \) and
\[
0 \leq \frac{\lambda |c|}{|c + 2|} \leq \frac{\sqrt{2 - A^2} - A}{2}
\]
and the result follows from the last inequality. \( \square \)
Remark. We recall first that if \(|a_2| \leq 1 - \lambda\), then it is known that [8]
\[
\text{Re} \left( \frac{f(z)}{z} \right) > \frac{1}{1 + |a_2| + \lambda} \geq \frac{1}{2} \quad \text{for } z \in \Delta. \tag{14}
\]
Further, from the work of Ruscheweyh [17, Lemma 2], it follows that
\[
\text{Re} F(1, c; c + 1; z) > \frac{1}{2}, \quad z \in \Delta, \quad \text{Re} c \geq 0. \tag{15}
\]
From (14), it follows that \(\text{Re} \left( \frac{f(z)}{z} \right) > 0, \quad z \in \Delta\). From this observation and (15), we obtain
(\text{using either the Herglotz representation formula for functions with positive real part or [18]})
\[
\text{Re} \left( \frac{f(z)}{z} F(1, c; c + 1; z) \right) > 0, \quad z \in \Delta, \quad \text{Re} c \geq 0,
\]
and so, in particular, that \((z/f(z)) F(1, c; c + 1; z) \neq 0\) for all \(z \in \Delta, \text{Re} c \geq 0\).

Remark. In case \(\text{Re} c > 0\), the formula (5) shows that the transform \(G(z) = G^c_f(z)\) defined by (6) has a second representation in the form
\[
G(z) = z \left( c \int_{0}^{1} \frac{t z}{f(tz)} t^{c-1} dt \right)^{-1}, \quad z \in \Delta.
\]

Using Lemma 2, Theorem 1 can be generalized as follows:

**Theorem 2.** For a fixed \(n \geq 2\), let \(f(z) = z + a_{n+1} z^{n+1} + \cdots\) belong to \(U(\lambda)\) and let \(c \in \mathbb{C}\) with \(\text{Re} c \geq 0 \neq c\) such that \((z/f(z)) F(1, c; c + 1; z) \neq 0\) in \(\Delta\), and \(G = G^c_f\) be the transform defined by (6). Then we have the following:

1. \(G \in U(\lambda |c|/(c + n))\). In particular, \(G \in U\) whenever \(0 < \lambda \leq |(c + n)/c|\).
2. \(G \in S^*\) whenever \(0 < \lambda \leq \frac{c+n(n-1)}{|c| \sqrt{(n-1)^2+1}}\).

**Proof.** We note that
\[
\frac{z}{f(z)} = \frac{1}{1 + a_{n+1} z^n + \cdots} = 1 - a_{n+1} z^n + \cdots,
\]
so that
\[
\frac{z}{f(z)} F(1, c; c + 1; z) = 1 - a_{n+1} \left( \frac{c}{c + n} \right) z^n + \cdots.
\]
Thus, \(G\) can be written in the form
\[
G(z) = z + a_{n+1} \left( \frac{c}{c + n} \right) z^{n+1} + \cdots
\]
and therefore, as in the proof of Theorem 1, the function \(p\) defined by
\[
p(z) = \left( \frac{z}{G(z)} \right)^2 G'(z) = 1 + (n - 1) a_{n+1} \left( \frac{c}{c + n} \right) z^n + \cdots
\]
is analytic in $\Delta$ such that $p(0) = 1$, $p'(0) = \cdots = p^{(n-1)}(0) = 0$. As $f \in U(\lambda)$, $p$ satisfies (13). Consequently (see [6,11]),

$$|p(z) - 1| \leq \frac{\lambda|c||z|^n}{|c + n|}, \quad z \in \Delta,$$

and the proof of part (1) is complete. The second part is a consequence of Lemma 2. \(\square\)

3. Sufficient conditions for functions in $U$ or in $S^*$

We recall that $U \subset S$. Next we consider the following question: Given a univalent function $f$, is it possible to generate functions in $U$ or in $S^*$? Our next result actually provides a method of obtaining functions in $U$.

**Theorem 3.** Let $h(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$ be an analytic function on $\Delta$ and $a_2 \in \mathbb{C}$ such that

$$|c_1 a_2| + \left( \sum_{n=2}^{\infty} \frac{|c_n|^2}{n-1} \right)^{1/2} \leq 1 \quad \text{and} \quad \lambda := \left( \sum_{n=2}^{\infty} (n-1)|c_n|^2 \right)^{1/2} < +\infty. \quad (16)$$

Then for every function $f \in S$ with $f''(0)/2 = a_2$ the function $H_f$ defined by

$$\frac{z}{H_f(z)} = \left( \frac{z}{f(z)} \right) * h(z)$$

belongs to $U(\lambda)$, and thus to $S$ if $\lambda \leq 1$, and even to $S^*$ if $\lambda \leq 1 - |a_2c_1|$. \(\text{Proof.}\)

Let $f \in S$ and be of the form (3). Then $a_2 = f''(0)/2 = -b_1$,

$$\frac{z}{H_f(z)} = \left( \frac{z}{f(z)} \right) * h(z) = 1 + \sum_{n=1}^{\infty} b_n c_n z^n$$

and from the well-known Area Theorem [5, Theorem 11, p. 193, Vol. 2] we have

$$\sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1. \quad (17)$$

Now, by the triangle inequality, we see for all $z \in \Delta$ that

$$\left| \frac{z}{H_f(z)} \right| \geq 1 - |c_1b_1||z| - \sum_{n=2}^{\infty} (\sqrt{n-1}|b_n|) \left( \frac{|c_n|}{\sqrt{n-1}} \right) |z|^n$$

$$\geq 1 - |c_1 a_2||z| - |z|^2 \sum_{n=2}^{\infty} (\sqrt{n-1}|b_n|) \left( \frac{|c_n|}{\sqrt{n-1}} \right)$$

$$\geq 1 - |c_1 a_2||z| - |z|^2 \left( \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right)^{1/2} \left( \sum_{n=2}^{\infty} \frac{|c_n|^2}{n-1} \right)^{1/2}$$

(by Cauchy–Schwarz inequality)

$$\geq 1 - |c_1 a_2| - \left( \sum_{n=2}^{\infty} \frac{|c_n|^2}{n-1} \right)^{1/2} \quad \text{by (17)}$$

$$\geq 0 \quad \text{by (16)}.$$
Using this and the first inequality in (16), we obtain that $z/H_f(z) \neq 0$ in $\Delta$. Next we find that
\[
\sum_{n=2}^{\infty} (n-1)|c_n b_n| = \sum_{n=2}^{\infty} (\sqrt{n-1}|b_n|)(\sqrt{n-1}|c_n|)
\leq \left( \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right)^{1/2} \left( \sum_{n=2}^{\infty} (n-1)|c_n|^2 \right)^{1/2}
\leq \lambda \quad \text{by (17) and (16)}.
\]
Thus, $H_f \in U(\lambda)$ by Lemma 3(1), and, in particular, $H_f \in U \subseteq S$ if $\lambda \leq 1$. By Lemma 3(2), we obtain the last part of the conclusion.

Example 1. Choose $h(z) = 1/(1 - az)$ with $|a| = r < 1$. Then, according to (16), $r$ has to satisfy the condition
\[
|a^2 r + r(\log(1/(1-r^2)))^{1/2} \leq 1 \quad \text{and} \quad \lambda = r^2/(1-r^2).
\]
Then for each function $f \in S$ with $f''(0)/2 = a_2$ the function $a^{-1} f(az)$ belongs to $U(\lambda)$ and thus to $S$ if $\lambda \leq 1$, and even to $S^*$ if $\lambda \leq 1 - |a_2|r$. In particular, it is a simple exercise to see that
\[
f \in S \quad \text{with} \quad f''(0) = 0 \quad \Rightarrow \quad a^{-1} f(az) \in U \cap S^*
\]
whenever $0 < |a| = r \leq 1/\sqrt{2}$. At this place it is interesting to compare with (2).

Example 2. Choose $h(z) = 1/(1 - az^2)$ with $|a| = r < 1$. Then, by (16), $r$ has to satisfy the condition
\[
\frac{r}{2} \log \left( \frac{1+r}{1-r} \right) \leq 1 \quad \text{and} \quad \lambda = \frac{r \sqrt{1+r^2}}{1-r^2}.
\]
Therefore, if $f \in S$ then the function $z/(z/f(z) \ast h(z))$ belongs to $U(\lambda)$ and thus to $S^*$ if $\lambda \leq 1$ (since $h'(0) = 0$). In fact, it is a simple exercise to see that the second condition $\lambda \leq 1$ is equivalent to $r \leq 1/\sqrt{3}$, while the first condition is equivalent to the inequality
\[
g(r) = (1-r)e^{2/r} - 1 - r \geq 0
\]
which holds if $r \leq 1/\sqrt{3}$. Thus, if $\omega$ and $\omega'$ denote the two square roots of $a$ and if $f \in S$, then the function $H_f$ defined by
\[
\frac{z}{H_f(z)} = \frac{z}{f(z) \ast h(z)} = \frac{1}{2} \left( \frac{\omega z}{f(\omega z)} + \frac{\omega' z}{f(\omega' z)} \right)
\]
begins to $S^*$ for $r \leq 1/\sqrt{3}$.

Corollary 1. Let $f \in S$ be of the form (3) with $a_2 = f''(0)/2$, and
\[
h(z) = 1 + c_1 z + a \sum_{n=2}^{\infty} \frac{1}{(n+1)\sqrt{n-1}} z^n
\]
for some complex constant $a$, such that
\[
|c_1 a_2| + |a| \sqrt{\frac{\pi^2}{12} - \frac{11}{16}} \leq 1 \quad \text{and} \quad \lambda = |a| \sqrt{\frac{\pi^2}{6} - \frac{5}{4}}.
\]
Then the function $H_f$ defined by $z/H_f (z) = (z/f (z)) \ast h(z)$ belongs to $U(\lambda)$, and thus to $S$ if $\lambda \leq 1$, and even to $S^*$ if $\lambda \leq 1 - |c_1a_2|$. 

**Proof.** Set $c_n = a/((n + 1)\sqrt{n - 1})$ for all $n \geq 2$. The condition (16) takes the form

$$|c_1a_2| + |a| \left( \sum_{n=2}^{\infty} \frac{1}{(n^2 - 1)^2} \right)^{1/2} \leq 1 \quad \text{and} \quad \lambda = |a| \left( \sum_{n=2}^{\infty} \frac{1}{(n+1)^2} \right)^{1/2}.$$

Recall that

$$\sum_{n=2}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - \frac{5}{4}.$$

Now, if we write

$$\frac{1}{(n^2 - 1)^2} = \frac{1}{4} \left[ \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} - \left( \frac{1}{n-1} - \frac{1}{n+1} \right) \right],$$

then it is a simple exercise to see that

$$\sum_{n=2}^{\infty} \frac{1}{(n^2 - 1)^2} = \frac{\pi^2}{12} - \frac{11}{16}.$$

The conclusion follows from Theorem 3. \qed

Finally, it would be appropriate to recall the recent result of the authors in [2] in which a number of interesting applications are also derived.

**Theorem 4.** (See [2, Theorem 3.9].) Let $f, g \in S$ with the representations

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

If

$$\Phi(z) = \frac{z}{f(z)} \ast \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} b_n c_n z^n \neq 0$$

for every $z \in \Delta$, then $F(z) = \frac{z}{\Phi(z)} \in U$, and, in particular, $F$ is univalent in $\Delta$.

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**References**