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Oscillation of Two-Dimensional Linear Second-Order Differential Systems*

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This article is concerned with the oscillatory behavior at infinity of the solution $y: [a, \infty) \to \mathbb{R}^2$ of a system of two second-order differential equations,

$$y''(t) + Q(t)\, y(t) = 0, \qquad t \in [a, \infty);$$

Q is a continuous matrix-valued function on $[a, \infty)$ whose values are real symmetric matrices of order 2. It is shown that the solution is oscillatory at infinity if the largest eigenvalue of the matrix $\int_a^t Q(s)\, ds$ tends to infinity as $t \to \infty$. This proves a conjecture of D. Hinton and R. T. Lewis (*Rocky Mountain J. Math.* **10** (1980), 751–766) for the two-dimensional case. Furthermore, it is shown that considerably weaker forms of the condition still suffice for oscillatory behavior at infinity.
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1. INTRODUCTION

We are concerned with the differential equation

$$y''(t) + Q(t)\, y(t) = 0, \qquad t \in [a, \infty), \tag{1.1}$$

for a vector-valued function $y: [a, \infty) \to \mathbb{R}^n$. Here Q is a continuous matrix-valued function on $[a, \infty)$ whose values are real symmetric matrices of order n.

Two points $\alpha, \beta \in [a, \infty)$ are said to be conjugate relative to (1.1) if there exists a nontrivial function y which satisfies (1.1) and vanishes at α and β. Equation (1.1) is said to be oscillatory at infinity if, for any point $\alpha \in [a, \infty)$, there exists a point $\beta \in (\alpha, \infty)$ such that α and β are conjugate

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relative to (1.1). We use the notation $Q_1(t)$ for the matrix of the integrals over $[a, t]$ of the corresponding elements of $Q(t)$,

$$Q_1(t) = \int_a^t Q(s)\, ds. \tag{1.2}$$

The oscillation theory for (1.1) has received considerable attention; see, for example, the recent monograph by Reid [7, Chap. V]. It has been conjectured, see Hinton and Lewis [2], that (1.1) is oscillatory at infinity whenever

$$\lim_{t \to \infty} \lambda_1\{Q_1(t)\} = \infty, \tag{1.3}$$

where $\lambda_1\{\cdot\}$ is the largest eigenvalue of the matrix inside the braces. This conjecture is interesting because, if true, it would imply that eigenvalues other than the largest one have no impact on oscillation at infinity. At this point the conjecture is still open, although it has been established under additional growth conditions on the trace of $Q_1(t)$ by Mingarelli [5, 6] and, more recently, by Kwong *et al.* [3]. In this article we show that the conjecture is indeed true for $n = 2$. In fact, we show that, in this case, the condition (1.3) can be relaxed considerably. Unfortunately, the method we use to prove these results is rather technical and does not seem to extend in an obvious way to higher dimensions.

The oscillatory properties of the solution of (1.1) are usually studied by means of the prepared or conjoined solutions of the associated matrix differential equation, i.e., those solutions Y ($n \times n$ matrix-valued functions of t) of the equation

$$Y''(t) + Q(t)\, Y(t) = 0, \qquad t \in [a, \infty), \tag{1.4}$$

for which $Y'(t)\, Y^{-1}(t)$ is selfadjoint (Hermitian). An alternative approach, which we shall use in this article, is based on the solution of a nonlinear equation for the matrix-valued function R,

$$R(t) = R(a) + \int_a^t Q(s)\, ds + \int_a^t R^2(s)\, ds, \qquad t \in [a, \infty). \tag{1.5}$$

The solutions of Eqs. (1.4) and (1.5) are related by the change of variables,

$$R(t) = -Y'(t)\, Y^{-1}(t). \tag{1.6}$$

If Y is a conjoined solution of (1.4), then the corresponding R is a selfadjoint matrix-valued function which satisfies (1.5), and vice versa. It can be shown that, if $R(a)$ is selfadjoint (real), then the matrix $R(t)$, which is uniquely determined by (1.5), is selfadjoint (real) for all t for which it is defined. Of course, the matrix $R(t)$ may blow up at a finite value of t. This happens when the corresponding matrix $Y(t)$ becomes singular.

As Etgen and Pawlowski [1] have shown, Eq. (1.1) is nonoscillatory at infinity if and only if (1.4) has a nonsingular conjoined solution on $[a, \infty)$. (More precisely, Eq. (1.1) is nonoscillatory at infinity if and only if (1.4) has a nonsingular conjoined solution on $[a_0, \infty)$ for some $a_0 \geqslant a$. However, as a in (1.1) is arbitrary, there is no loss in generality if we take a_0 and a to be the same point.) Hence, a necessary and sufficient condition for (1.1) to be nonoscillatory at infinity is that (1.6) has a continuous selfadjoint solution on $[a, \infty)$. It is the latter criterion that we shall use to study the oscillatory behavior at infinity of the system (1.1) with $n = 2$.

In the next section we establish several ordering relations for the quadratic term in the two-dimensional matrix Riccati equation. We use these relations in Section 3 to prove the conjecture mentioned earlier, viz., that the solution of (1.1) is oscillatory at infinity if the larger of the two eigenvalues of $Q_1(t)$ tends to infinity as $t \to \infty$. Finally, in Section 4 we present several weaker conditions on the asymptotic behavior of this eigenvalue, under which one can prove oscillatory behavior at infinity. Our main results are formulated in Theorem 3 and Theorem 8.

2. Ordering Relations

In this section we shall establish an ordering relation for the quadratic term in the matrix Riccati equation

$$R(t) = F(t) + \int_0^t R^2(s)\, ds, \qquad t \geqslant 0. \tag{2.1}$$

Here F is a given continuous function on $[0, \infty)$, whose values are selfadjoint matrices of order 2. We shall use the standard partial ordering in the space of selfadjoint matrices, viz., $A \geqslant B$ if $A - B$ is nonnegative. The symbol I stands for the identity matrix of order 2.

Lemma 1. *Suppose R is a continuous selfadjoint matrix-valued global solution of* (2.1). *If $\lambda_1\{F(t)\} \geqslant 1$ for all $t \geqslant 0$, then*

$$\int_0^{25} R^2(s)\, ds \geqslant \frac{1}{4} I. \tag{2.2}$$

Proof. Without loss of generality we may assume that the matrix $\int_0^{25} R^2(s)\, ds$ is diagonal, $\left(\begin{smallmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{smallmatrix}\right)$, say; otherwise, an appropriate similarity transformation can be applied to both sides of the identity (2.1) to make this matrix diagonal. We use the following notation:

$$R(t) = \mathrm{mat}[r_{ij}(t)\colon i, j = 1, 2],$$

$$P(t) = \int_0^t R^2(s)\, ds = \mathrm{mat}[\rho_{ij}(t)\colon i, j = 1, 2],$$

where r_{11} and r_{22} are real-valued functions; r_{12} and r_{21} are complex-valued functions, with $r_{21}(t) = \overline{r_{12}(t)}$.

The proof is by contradiction. Assume that (2.2) does not hold. Then at least one of the numbers σ_1, σ_2 is less than c; for example,

$$\sigma_2 = \int_0^{25} ((r_{22}(s))^2 + |r_{12}(s)|^2)\, ds < c.$$

The interval $[0, 25]$ is the union of two disjoint sets S_c and S'_c, where

$$S_c = \{t \in [0, 25]\colon |r_{22}(t)| < c \text{ and } |r_{12}(t)| < c\}$$

and $S'_c = [0, 25] \backslash S_c$. Clearly, mes $S'_c < c^{-1} = 4$.

If R satisfies (2.1) and $\lambda_1\{F(t)\} \geqslant 1$, then $\lambda_1\{R(t) - P(t)\} \geqslant 1$ or, explicitly,

$$\begin{aligned}&[(r_{11} - \rho_{11} + r_{22} - \rho_{22})^2 - 4(r_{11} - \rho_{11})(r_{22} - \rho_{22}) + 4\, |r_{12} - \rho_{12}|^2]^{1/2}\\ &\quad \geqslant 2 - (r_{11} - \rho_{11} + r_{22} - \rho_{22}).\end{aligned}$$

Two possibilities arise: either the expression in the right member is negative or, if it is positive, the expression inside the square brackets is greater than or equal to the square of the expression in the right member. In the former case, $r_{11} \geqslant 2 + \rho_{11} - r_{22} + \rho_{22}$. But ρ_{22} is nonnegative and $|r_{22}| < c$ on S_c, so in this case we certainly have the inequality

$$r_{11} \geqslant 1 + \rho_{11} \qquad \text{on } S_c. \tag{2.3}$$

In the latter case we find, after some simplification,

$$r_{11}(1 + \rho_{22} - r_{22}) \geqslant (1 + \rho_{11})(1 + \rho_{22} - r_{22}) - |r_{12} - \rho_{12}|^2. \tag{2.4}$$

The last term can be estimated in the obvious way,

$$\begin{aligned}|r_{12} - \rho_{12}|^2 &\leqslant |r_{12}|^2 + 2\, |\rho_{12}|\, |r_{12}| + |\rho_{12}|^2\\ &\leqslant |r_{12}|^2 + (1 + |\rho_{12}|^2)\, |r_{12}| + |\rho_{12}|^2.\end{aligned}$$

Because $P(t)$ is nonnegative, we necessarily have $|\rho_{12}|^2 \leqslant \rho_{11}\rho_{22}$ on $[0, 25]$.

Also, ρ_{22}—being the integral of a nonnegative function—is nondecreasing, so $\rho_{22} \leqslant \sigma_2 < c$ on $[0, 25]$. Hence, $|\rho_{12}|^2 < c\rho_{11}$ on $[0, 25]$ and therefore

$$|r_{12} - \rho_{12}|^2 < (1 + |r_{12}|)(|r_{12}| + c\rho_{11}) \qquad \text{on } [0, 25].$$

In particular,

$$|r_{12} - \rho_{12}|^2 < c(1 + c)(1 + \rho_{11}) \qquad \text{on } S_c.$$

Furthermore,

$$1 - c < 1 + \rho_{22} - r_{22} < 1 + 2c \qquad \text{on } S_c.$$

Thus we obtain the following inequality from (2.4):

$$r_{11} > \frac{1 - 2c - c^2}{1 + 2c}(1 + \rho_{11}) \qquad \text{on } S_c. \tag{2.5}$$

Notice that the constant factor in the right member of this inequality is less than one, but certainly greater than $\frac{1}{4}$ if $c = \frac{1}{4}$. The estimates (2.3) and (2.5) can thus be combined into one single estimate,

$$r_{11} > \tfrac{1}{4}(1 + \rho_{11}) \qquad \text{on } S_c. \tag{2.6}$$

Because $\rho'_{11} = r_{11}^2 + |r_{12}|^2 \geqslant r_{11}^2$ on $[0, 25]$, the function ρ_{11} therefore satisfies the following differential inequalities:

$$\rho'_{11}(t) > \tfrac{1}{16}(1 + \rho_{11}(t))^2, \qquad t \in S_c, \tag{2.7-1}$$

$$\rho'_{11}(t) \geqslant 0, \qquad t \in S'_c. \tag{2.7-2}$$

However, here we have arrived at a contradiction, as there is no differentiable function ρ_{11} on $[0, 25]$ which satisfies (2.7). To prove this last statement, one may construct a comparison problem in the following way.

The set $\{t \in [0, 25]: \rho'_{11}(t) > \frac{1}{16}(1 + \rho_{11}(t))^2\}$ is open and contains S_c. Hence, it is a countable union of open intervals and has total measure at least 21. Let U be the union of a finite number of these intervals, such that mes $U > 16$. We define ρ as the solution of the differential equation

$$\rho'(t) = \tfrac{1}{16}(1 + \rho(t))^2, \qquad t \in U,$$

$$\rho'(t) = 0, \qquad t \notin U,$$

on $[0, 25]$ which satisfies the initial condition $\rho(0) = \rho_{11}(0) = 0$. Then $0 \leqslant \rho(t) \leqslant \rho_{11}(t)$, and, if ρ_{11} is defined on $[0, 25]$, the same must be true for ρ. But a direct computation shows that ρ blows up before t reaches the right endpoint of U; hence, a contradiction. ∎

A closer examination of the proof of the lemma shows that, instead of the constant $\frac{1}{4}$, we could have taken any positive constant c less than $-1+\sqrt{2}$ in the inequality (2.2). But, as the following corollary shows, the precise value of the constant is immaterial.

COROLLARY 2. *Suppose R is a continuous selfadjoint matrix-valued global solution of* (2.1). *If $\lambda_1\{F(t)\} \geqslant 1$ for all $t \geqslant 0$, then there exists a $\tau > 0$, which is independent of F, such that*

$$\int_0^t R^2(s)\,ds \geqslant I, \qquad t \geqslant \tau. \tag{2.8}$$

Proof. We can simply take $\tau = 100$. To see this, we first invoke Lemma 1,

$$\int_0^{25} R^2(s)\,ds \geqslant \frac{1}{4}\,I.$$

By shifting the origin to $t = 25$ and changing variables, we can apply the same lemma to show that

$$\int_{25}^{50} R^2(s)\,ds \geqslant \frac{1}{4}\,I.$$

We repeat this process two more times and combine the various inequalities into one single inequality,

$$\int_0^{100} R^2(s)\,ds = \left(\int_0^{25} + \int_{25}^{50} + \int_{50}^{75} + \int_{75}^{100}\right) R^2(s)\,ds \geqslant I. \quad \blacksquare$$

3. PROOF OF THE CONJECTURE

The ordering relation which we proved in the previous section enables us to compare the quadratic term in (1.5) with multiples of the identity matrix and thus to show that (1.5) does not have a global solution if (1.3) holds.

THEOREM 3. *If* (1.3) *holds, then* (1.1) *is oscillatory at infinity for $n = 2$.*

Proof. The proof is by contradiction, where we assume that the Riccati equation (1.5), which is equivalent with (1.1), has a continuous selfadjoint matrix-valued global solution R.

If (1.3) holds, then $\lambda_1\{R(a)+Q_1(t)\} \to \infty$ as $t \to \infty$, so for t sufficiently large, $t \geqslant t_0$, say, we have

$$\lambda_1\{R(a)+Q_1(t)\} \geqslant 1, \qquad t \geqslant t_0. \tag{3.1}$$

Let t_0 be kept fixed. For $t \geqslant t_0$ we have

$$R(t)=F_0(t)+\int_{t_0}^{t} R^2(s)\,ds, \qquad t \geqslant t_0, \tag{3.2}$$

where the function F_0 is defined on $[t_0, \infty)$ by the expression

$$F_0(t)=R(a)+Q_1(t)+\int_{a}^{t_0} R^2(s)\,ds, \qquad t \geqslant t_0. \tag{3.3}$$

The inequality (3.1) and the nonnegativity of the integral term imply that

$$\lambda_1\{F_0(t)\} \geqslant 1, \qquad t \geqslant t_0. \tag{3.4}$$

The change of variables $\bar{R}(t)=R(t_0+t)$ reduces (3.2) to

$$\bar{R}(t)=\bar{F}(t)+\int_{0}^{t} \bar{R}^2(s)\,ds, \qquad t \geqslant 0, \tag{3.5}$$

where $\bar{F}(t)=F_0(t_0+t)$. It follows from (3.4) that $\lambda_1\{\bar{F}(t)\} \geqslant 1$ for all $t \geqslant 0$, so according to Corollary 2 there exists a $\tau > 0$ such that $\int_0^\tau \bar{R}^2(s)\,ds \geqslant I$. Let $t_1 = t_0 + \tau$ thus determined be kept fixed. Then we have the ordering relation

$$\int_{t_0}^{t} R^2(s)\,ds \geqslant I, \qquad t \geqslant t_1. \tag{3.6}$$

We now proceed to the next step.

We define the function F_1 on $[t_1, \infty)$ by adding the integral of R^2 over the interval $[t_0, t_1]$ to the function F_0:

$$F_1(t)=F_0(t)+\int_{t_0}^{t_1} R^2(s)\,ds, \qquad t \geqslant t_1. \tag{3.7}$$

The inequalities (3.4) and (3.6) together imply that

$$\lambda_1\{F_1(t)\} \geqslant 2. \tag{3.8}$$

Furthermore, if R satisfies (3.2), then

$$R(t)=F_1(t)+\int_{t_1}^{t} R^2(s)\,ds, \qquad t \geqslant t_1. \tag{3.9}$$

The change of variables $\bar{R}(t)=\frac{1}{2}R(t_1+\frac{1}{2}t)$ transforms this equation into (3.5), where $\bar{F}(t)=\frac{1}{2}F_1(t_1+\frac{1}{2}t)$. Thus, $\lambda_1\{\bar{F}(t)\}\geqslant 1$ for all $t\geqslant 0$ and Corollary 2 applies. Using the same value τ as in the first step, we conclude that $\int_0^t \bar{R}^2(s)\,ds\geqslant I$ for $t\geqslant\tau$, i.e.,

$$\int_{t_1}^{t} R^2(s)\,ds\geqslant 2I, \qquad t\geqslant t_2, \tag{3.10}$$

where $t_2=t_1+\frac{1}{2}\tau=t_0+\frac{3}{2}\tau$.

Continuing this procedure we find, after n steps,

$$\int_{t_{n-1}}^{t} R^2(s)\,ds\geqslant 2^{n-1}I, \qquad t\geqslant t_n, \tag{3.11}$$

where $t_n=t_{n-1}+2^{-(n-1)}\tau=t_0+2(1-2^{-n})\,\tau$. Thus, when we add the contributions from each of the intervals $[a,t_0]$, $[t_0,t_1]$,..., $[t_{n-1},t_n]$ we obtain the estimate

$$\int_{a}^{t} R^2(s)\,ds\geqslant (2^n-1)\,I, \qquad t\geqslant t_n. \tag{3.12}$$

But t_n tends to the finite limit $t_0+2\tau$ as $n\to\infty$, so we conclude that there exists a finite number T such that $\int_a^t R^2(s)\,ds$ blows up as $t\uparrow T$. This conclusion, however, contradicts the assumption that (1.5) has a global solution. ∎

4. Generalizations

A closer examination of the proof of Lemma 1 reveals that all that is needed for the lemma to hold is that the estimate (2.6) is satisfied on a sufficiently large set. The estimate (2.6) followed from the inequality $\lambda_1\{F(t)\}\geqslant 1$, provided $t\in S_c$. Hence, if the same inequality holds on a sufficiently large set, the lemma is stil true. This observation is made more precise in the following lemma, which we state without proof.

Lemma 4. *Suppose R is a continuous selfadjoint matrix-valued global solution of* (2.1). *Then there exists a $\tau>0$, which is independent of F, such that*

$$\int_0^t R^2(s)\,ds\geqslant I \tag{4.1}$$

whenever $\operatorname{mes}\{s\in[0,t]\colon \lambda_1\{F(s)\}\geqslant 1\}\geqslant\tau$.

With this lemma we can generalize the result of Theorem 3. We shall use the following notation:

$$S(\mu) = \{t \in [a, \infty): \lambda_1\{Q_1(t)\} \geqslant \mu\}, \qquad \mu > 0. \tag{4.2}$$

THEOREM 5. *If*

$$\lim_{\mu \to \infty} \mu \operatorname{mes} S(\mu) = \infty, \tag{4.3}$$

then (1.1) *is oscillatory at infinity for* $n = 2$.

Proof. The proof is by contradiction, where we assume that the Riccati equation (1.5) has a continuous selfadjoint matrix-valued global solution R.

If μ is sufficiently large, then

$$\lambda_1\{R(a) + Q_1(t)\} \geqslant \tfrac{1}{2}\mu, \qquad t \in S(\mu).$$

Repeating the arguments used in the proof of Theorem 3, we find that there exists a finite number T such that $\int_a^t R^2(s)\, ds$ blows up as $t \uparrow T$, unless $\operatorname{mes} S_T(\mu) \leqslant 4\tau/\mu$, where

$$S_T(\mu) = \{s \in [a, T]: \lambda_1\{R(a) + Q_1(s)\} \geqslant \tfrac{1}{2}\mu\}.$$

Clearly, $S_T(\mu) \supset (S(\mu) \cap [a, T])$, so, if T is sufficiently large, then $\operatorname{mes} S_T(\mu) \geqslant \frac{1}{2} \operatorname{mes} S(\mu)$. Hence, $\mu \operatorname{mes} S_T(\mu) \geqslant \frac{1}{2}\mu \operatorname{mes} S(\mu)$, and (4.3) implies that $\mu \operatorname{mes} S_T(\mu)$ must grow beyond any finite bound as $\mu \to \infty$. The inequality $\operatorname{mes} S_T(\mu) \geqslant 4\tau/\mu$ is therefore not satisfied for sufficiently large μ and we have a contradiction. ∎

Some special cases of Theorem 5 are of interest. Let J be an unbounded subset of $[a, \infty)$ and f a function defined on $[a, \infty)$. We say that $\lim_{t \to \infty, t \in J} f(t) = \infty$ if, for any positive number N, there exists a $t_0 \in (a, \infty)$ such that $f(t) \geqslant N$ for all $t \in J \cap [t_0, \infty)$.

COROLLARY 6. *If there is a measurable subset* J *of* $[a, \infty)$ *of infinite measure, such that*

$$\lim_{t \to \infty, t \in J} \lambda_1\{Q_1(t)\} = \infty, \tag{4.4}$$

then (1.1) *is oscillatory at infinity for* $n = 2$.

COROLLARY 7. *If there is a measurable subset J of $[a, \infty)$ of finite measure and a $\gamma \in (0, 1)$ such that*

$$\int_J [\lambda_1\{Q_1(t)\}]_+^\gamma \, dt = \infty, \tag{4.5}$$

where $[\cdot]_+ = \max\{\cdot, 0\}$, then (1.1) is oscillatory at infinity for $n = 2$.

Proof. That (4.3) follows from (4.5) is shown in Kwong and Zettl [4, Corollary 3]. ∎

Corollary 7 shows that oscillatory behavior at infinity may result even if $\lambda_1\{Q_1(t)\}$ is negative on a set of infinite measure.

An example to which Corollary 7 applies is a two-dimensional system with $\lambda_1\{Q_1(t)\} \geqslant t^\alpha \sin t$, $\alpha > 0$.

In the scalar case, $Q_1(t) = \sin t$ results in oscillatory behavior at infinity. Whether the same result is true for systems, if $\lambda_1\{Q_1(t)\} = \sin t$, is not known. This problem seems to be nontrivial even in the two-dimensional case.

We remark that by changing variables one can sometimes extend the applicability of oscillation criteria. For instance, the techniques used in Kwong and Zettl [4, Sect. 6] apply also to systems. We state without proof the following extension of Theorem 5 to Zlamal-type oscillation conditions.

THEOREM 8. *If there exist a C^1-function f: $[a, \infty) \to [0, \infty)$ and a set $J \subset [a, \infty)$, such that*

$$\int_J (f(t))^{-1} \, dt = \infty$$

and

$$\lim_{t \to \infty, t \in J} \lambda_1 \left\{ \frac{1}{2} f'(t) I + \int_a^t \left[f(s) Q(s) - \frac{(f'(s))^2}{4f(s)} I \right] ds \right\} = \infty, \tag{4.6}$$

then (1.1) is oscillatory at infinity for $n = 2$.

This theorem generalizes a result of Reid for two-dimensional systems [7, Sect. V.15.20]. It has the following interesting corollary.

COROLLARY 9. *If there exist a set $J \subset [a, \infty)$ and a nonincreasing C^1-function f: $[a, \infty) \to [0, \infty)$ such that*

$$\int_J \frac{dt}{f(t)} = \infty \qquad \text{and} \qquad \int_J \frac{(f'(t))^2}{f(t)} \, dt < \infty, \tag{4.7}$$

then the condition

$$\lim_{t \to \infty, t \in J} \lambda_1 \left\{ \int_a^t f(s)\, Q(s)\, ds \right\} = \infty \tag{4.8}$$

implies that (1.1) *is oscillatory at infinity for* $n = 2$.

Examples of functions f satisfying (4.7) are $f(t) = t^\gamma$ for $\gamma < 1$, and $f(t) = t(\ln t)^\alpha$, $t(\ln t)^{-1} (\ln \ln t)^\alpha$, $t(\ln t \ln \ln t)^{-1} (\ln \ln \ln t)^\alpha$, etc., for $\alpha < -1$.

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