Generalized $L$-KKM type theorems in topological spaces with an application∗

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Received 4 April 2006; received in revised form 19 July 2006; accepted 21 July 2006

Abstract

In this paper, we first introduce a new concept of generalized $L$-KKM mapping and establish some new generalized $L$-KKM type theorems without any convexity structure in topological spaces. As an application, an existence theorem of equilibrium points for an abstract generalized vector equilibrium problem is proved in topological spaces. The results presented in this paper unify and generalize some known results in recent literature.

Keywords: Generalized $L$-KKM mapping; Generalized $L$-diagonally quasi-subspace; $L$-subspace; Compactly open; Abstract generalized vector equilibrium problem

1. Introduction

In 1929, Knaster, Kuratowski and Mazurkiewicz [1] first established the famous KKM theorem in finite dimensional spaces. In 1961, Fan [2] extended the KKM theorem to infinite dimensional topological vector spaces and gave some applications in several directions. Since then, the study of nonlinear analysis related to the KKM principle has become a rapidly developing area in mathematics and applied science. Many authors have made important contributions to developing the KKM principle with applications (see, for example, [3–18] and the references therein). In most known KKM theorems and applications, the convexity assumptions play a crucial role which strictly restricts the applicable area of the KKM principle. In 1983, Horvath [15], replacing convex hulls by contract subsets, gave a purely topological version of the KKM theorem. Motivated by the work of Horvath mentioned above, Park and Kim [16,17] introduced the concept of generalized convex ($G$-convex) spaces and proved some KKM theorems in the generalized convex ($G$-convex) spaces. Recently, Ding [9–11] proved some new generalized $G$-KKM type theorems in $G$-convex spaces and gave some applications. Very recently, Deng and Xia [5] and Ding [12] proved some generalized...
R-KKM type theorems in general topological spaces without any convexity structure, which unified and generalized some known results.

Inspired and motivated by recent works in this research field, in this paper, we introduce a new class of generalized L-KKM mappings and establish some new generalized L-KKM type theorems without any convexity structures in topological spaces. As an application, an existence theorem of equilibrium points for an abstract generalized vector equilibrium problem is proved in topological spaces. The results presented in this paper unify and generalize some known results of Ansari, Oettli and Schläger [19], Deng and Xia [5], Ding [12], and Ding and Park [18].

2. Preliminaries

Let $X$ and $Y$ be two nonempty sets. We denote by $2^Y$ and $\langle X \rangle$ the families of all subsets of $Y$ and the family of all nonempty finite subsets of $X$, respectively. For each $A \in \langle X \rangle$, $|A|$ denotes the cardinality of $A$. Let $\Delta_n$ denote the standard $n$-dimensional simplex with vertices $\{e_0, e_1, \ldots, e_n\}$. If $J$ is a nonempty subset of $\{0, 1, \ldots, n\}$, we shall denote by $\Delta_J$ the convex hull of vertices $\{e_j : j \in J\}$. Let $X$ be a topological space. A subset $A$ of $X$ is said to be compactly open (resp. compactly closed) if for each nonempty compact subset $K$ of $X$, $A \cap K$ is open (resp. closed) in $K$. Let $X$ and $Y$ be two topological spaces. A set-valued mapping $T : X \to 2^Y$ is said to be lower (resp. upper) semicontinuous on $X$ if, for each open set $U \subseteq Y$, the set $\{x \in X : T(x) \cap U \neq \emptyset\}$ (resp. $\{x \in X : T(x) \subseteq U\}$) is open in $X$.

Let $Y$ be a topological space, $X$ and $Z$ be two nonempty sets, $F : X \times Y \to 2^Z$ and $C : Y \to 2^Z$ be two set-valued mappings. An abstract generalized vector equilibrium problem (for short, AGVEP) is to find $\hat{y} \in Y$ such that

$$F(x, \hat{y}) \not\subseteq C(\hat{y}), \quad \forall x \in X.$$ 

We would like to point out that the abstract generalized vector equilibrium problem was first considered by Ansari, Oettli and Schläger [19] in 1997.

Definition 2.1. Let $X$ be a nonempty set and $Y$ be a topological space. A set-valued mapping $G : X \to 2^Y$ is said to be a generalized L-KKM mapping if, for any $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ (where some elements in $N$ may be same), there exists a lower semicontinuous mapping $\varphi_N : \Delta_n \to 2^Y$ such that for each $\{e_{i_0}, \ldots, e_{i_k}\} \subset \{e_0, \ldots, e_n\}$,

$$\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^k G(x_{i_j}),$$

where $\Delta_k = \text{co} \{e_{i_0}, \ldots, e_{i_k}\}$.

Example 2.1. Let $X$ be a nonempty set and $Y$ be a topological space. Suppose $G : X \to 2^Y$ is the generalized relatively KKM ($R$-KKM) mapping of Deng and Xia [5], i.e., for any $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ (where some elements in $N$ may be same), there exists a continuous mapping $\varphi_N : \Delta_n \to Y$ such that for each $\{e_{i_0}, \ldots, e_{i_k}\} \subset \{e_0, \ldots, e_n\}$,

$$\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^k G(x_{i_j}),$$

where $\Delta_k = \text{co} \{e_{i_0}, \ldots, e_{i_k}\}$. Then it is easy to see that the generalized $R$-KKM mapping $G$ is a generalized $L$-KKM mapping defined by Definition 2.1.

Remark 2.1. It follows from Example 2.1 that the generalized $L$-KKM mapping extends the generalized $R$-KKM mapping of Deng and Xia [5]. We also know that the generalized $L$-KKM mapping defined by Definition 2.1 unifies the generalized $R$-KKM mapping of Verma [7], the generalized $G$-KKM mapping of Ding [9], the generalized $L$-KKM mapping of Ding [10], and the generalized $H$-KKM mapping of Ding [13].

Definition 2.2. Let $Y$ be a topological space and $X$ be a nonempty set. A subset $D$ of $Y$ is said to be an L-subspace if for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ (where some elements in $N$ may be same), there exists a lower semicontinuous mapping $\tilde{\varphi}_N : \Delta_n \to 2^Y$ such that $\tilde{\varphi}_N(\Delta_n) \subseteq D$. 
Let $Y$ be a topological space, $X$ and $Z$ be two nonempty sets, and $C$ be set-valued mappings. We say that $F(x, y)$ is generalized $L$-diagonally quasi-convex in $x$ with respect to $C$ if, for any $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists a lower semicontinuous mapping $\varphi_N : \Delta_n \to 2^Y$ such that for any $\{e_{i_0}, \ldots, e_{i_k}\}$ and for each $y_0 \in \varphi_N(\Delta_k)$, $F(x_{i_j}, y_0) \not\subseteq C(y_0)$.

**Example 2.2.** Let $X$ be a nonempty set and $Y$ be a topological space. Suppose $F : X \times Y \to R \cup \{\pm \infty\}$ is the $\lambda$-generalized $R$-diagonally quasi-convex mapping of Deng and Xia [5], i.e., for any $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$ such that for any $\{e_{i_0}, \ldots, e_{i_k}\}$ and for each $y_0 \in \varphi_N(\Delta_k)$,

$$\max_{0 \leq j \leq k} F(x_{i_j}, y_0) \geq \lambda.$$ 

Let $Z = R \cup \{\pm \infty\}$ and $C(y) = [\infty, \lambda)$ for all $x \in X$. Then it is easy to see that the $\lambda$-generalized $R$-diagonally quasi-convex mapping of Deng and Xia [5] is generalized $L$-diagonally quasi-convex in $x$ with respect to $C$ in the sense of Definition 2.3.

**Remark 2.2.** It follows from Example 2.2 that the generalized $L$-diagonally quasi-convexity in $x$ with respect to $C$ defined by Definition 2.3 extends the $\lambda$-generalized $R$-diagonally quasi-convexity of Deng and Xia [5]. We also know that Definition 2.3 generalizes Definition 2.4(2) in Ding and Park [18].

**Lemma 2.1 ([14]).** Let $X$ and $Y$ be two topological spaces, and $F : X \to 2^Y$ be a set-valued mapping. Then $F$ is lower semicontinuous if and only if for each closed set $S$ of $Y$, $F^{-1}(S) = \{x \in X : F(x) \subseteq S\}$ is a closed set of $X$.

**Lemma 2.2 ([18]).** Let $X$ be a nonempty set, $Y$ and $Z$ be two topological spaces, and $F : X \times Y \to 2^Z$ and $C : Y \to 2^Z$ be two set-valued mappings such that

1. the mapping $C(\cdot)$ has open graph in $Y \times Z$;
2. for each $x \in X$, the mapping $y \mapsto F(x, y)$ is upper semicontinuous on each compactly subset of $Y$ with nonempty compact values.

Then for each $x \in X$, the set

$$T(x) = \{y \in Y : F(x, y) \not\subseteq C(y)\}$$

is compactly closed in $Y$.

**Lemma 2.3.** Let $Y$ be a topological space, $X$ and $Z$ be two nonempty sets, and $C : Y \to 2^Z$. Then $F(x, y) : X \times Y \to 2^Z$ is generalized $L$-diagonally quasi-convex in $x$ with respect to $C$ if and only if the mapping $T : X \to 2^Y$ defined by

$$T(x) = \{y \in Y : F(x, y) \not\subseteq C(y)\}, \quad \forall x \in X$$

is a generalized $L$-KKM mapping.

**Proof.** Suppose that $F(x, y)$ is generalized $L$-diagonally quasi-convex in $x$ with respect to $C$ and $T$ is not a generalized $L$-KMM mapping. Then there exists $A = \{x_0, \ldots, x_n\} \in \langle X \rangle$ such that for any lower semicontinuous mapping $\varphi_A : \Delta_n \to 2^Y$, there exists $\{e_{i_0}, \ldots, e_{i_k}\} \subseteq \{e_0, \ldots, e_k\}$ such that

$$\varphi_A(\Delta_k) \not\subseteq \bigcup_{j=0}^k T(x_{i_j}).$$

This implies that $y_0 \not\in \varphi_A(\Delta_k)$ and $y_0 \not\in T(x_{i_j})$ for all $j \in \{0, \ldots, k\}$. It follows that

$$F(x_{i_j}, y_0) \subseteq C(y_0), \quad \forall j \in \{0, \ldots, k\},$$

which contradicts that $F(x, y)$ is generalized $L$-diagonally quasi-convex in $x$ with respect to $C$. Therefore, $T$ is a generalized $L$-KMM mapping.

Conversely, suppose $T$ is a generalized $L$-KMM mapping and $F(x, y)$ is not generalized $L$-diagonally quasi-convex in $x$ with respect to $C$. Then there exists $A = \{x_0, \ldots, x_n\} \in \langle X \rangle$ such that for any lower semicontinuous mapping $\varphi_A : \Delta_n \to 2^Y$, there exist $\{e_{i_0}, \ldots, e_{i_k}\}$ and $y_0 \in \varphi_A(\Delta_k)$ such that $F(x_{i_j}, y_0) \subseteq C(y_0)$. This implies that

$$y_0 \not\in T(x_{i_j}), \quad \forall j \in \{0, \ldots, k\}$$
and so
\[ \varphi_A(\Delta_k) \not\subseteq \bigcup_{j=0}^{k} T(x_j), \]
which contradicts that \( T \) is a generalized \( L \)-KKM mapping. Thus, \( F(x, y) \) is a generalized \( L \)-diagonally quasi-convex in \( x \) with respect to \( C \). This completes the proof. \( \square \)

3. Generalized \( L \)-KKM type theorems

**Theorem 3.1.** Let \( X \) be a nonempty set, \( Y \) be a topological space, and \( G : X \to 2^Y \) be a generalized \( L \)-KKM mapping such that for each \( x \in X \) and \( N = \{x_0, \ldots, x_n\} \in (X) \) (where some elements in \( N \) may be same), \( G(x) \cap \varphi_N(\Delta_n) \) is closed in \( \varphi_N(\Delta_n) \), where \( \varphi_N : \Delta_n \to 2^Y \) is the lower semicontinuous mapping in touch with \( N \) in Definition 2.1. Then
\[ \varphi_N(\Delta_n) \cap \left( \bigcap_{j=0}^{n} G(x_j) \right) \neq \emptyset. \]

**Proof.** Since \( G \) is a generalized \( L \)-KKM mapping, for each \( \{e_{i_0}, \ldots, e_{i_k}\} \subset \{e_0, \ldots, e_n\} \), we have
\[ \varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^{k} G(x_j), \]
where \( \Delta_k = \text{co}(\{e_{i_0}, \ldots, e_{i_k}\}) \). Let
\[ E_{ij} = \varphi_N^{-1}(G(x_j) \cap \varphi_N(\Delta_n)), \quad j = \{0, 1, \ldots, k\}. \]
For each \( z \in \Delta_k \), we have \( \varphi_N(z) \in \varphi_N(\Delta_k) \subset \varphi_N(\Delta_n) \). On the other hand, we know that
\[ \varphi_N(z) \in \varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^{k} G(x_j). \]
Hence there exists \( r \in \{0, \ldots, k\} \) such that \( \varphi_N(z) \in G(x_r) \cap \varphi_N(\Delta_n) \). This implies that
\[ z \in \varphi_N^{-1}(G(x_r) \cap \varphi_N(\Delta_n)) \]
and so
\[ \Delta_k = \text{co}(\{e_{i_0}, \ldots, e_{i_k}\}) \subseteq \bigcup_{j=0}^{k} \varphi_N^{-1}G(x_j) \cap \varphi_N(\Delta_n) = \bigcup_{j=0}^{k} E_{ij}. \]
Since \( G(x) \cap \varphi_N(\Delta_n) \) is closed in \( \varphi_N(\Delta_n) \) for each \( j = 0, \ldots, k \), it follows from Lemma 2.1 that \( E_{ij} \) is closed in \( \Delta_k \). By the classical KKM theorem [14], we have \( \bigcap_{j=0}^{k} E_{ij} \neq \emptyset \). Hence,
\[ \varphi_N(\Delta_n) \cap \left( \bigcap_{j=0}^{n} G(x_j) \right) \neq \emptyset. \]
This completes the proof. \( \square \)

**Remark 3.1.** Theorem 3.1 improves Theorem 3.1 of Deng and Xia [5] in the following two aspects: (1) from generalized \( R \)-KKM mapping to generalized \( L \)-KKM mapping; (2) the condition in Theorem 3.1 is weaker than the assumption that \( G \) has compactly closed values. Theorem 3.1 also generalizes (i) of Theorem 2.1 in Ding [12] from the generalized \( R \)-KKM mapping to the generalized \( L \)-KKM mapping.
Theorem 3.2. Let $X$ be a nonempty set, $Y$ be a topological space, $G : X \rightarrow 2^Y$ be a generalized $L$-KKM mapping with nonempty compactly closed values, and $K$ be a compact $L$-subspace of $Y$ such that, for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ (where some elements in $N$ may be the same),

$$\varphi_N(\Delta_n) \subseteq \tilde{\varphi}_N(\Delta_n),$$

where $\Delta_n = \text{co}(\{e_0, \ldots, e_n\})$ and $\varphi_N, \tilde{\varphi}_N : \Delta_n \rightarrow 2^Y$ are the lower semicontinuous mappings in touch with $N$ in Definition 2.1 and Definition 2.2, respectively. Then

$$K \cap (\bigcap_{x \in X} G(x)) \neq \emptyset.$$

Proof. Define the mapping $T : X \rightarrow 2^K$ as follows

$$T(x) = G(x) \cap K, \quad \forall x \in X.$$

Since $G$ is a generalized $L$-KKM mapping, for $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ (where some elements in $N$ may be same), we have

$$\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^{k} G(x_{ij})$$

for each $\Delta_k = \text{co}(\{e_{i_0}, \ldots, e_{i_k}\})$, where $\{e_{i_0}, \ldots, e_{i_k}\} \subseteq \{e_0, \ldots, e_n\}$. By the assumption that $K$ is an $L$-subspace of $Y$, it follows from $\varphi_N(\Delta_n) \subseteq \tilde{\varphi}_N(\Delta_n)$ that

$$\varphi_N(\Delta_k) \subseteq \varphi_N(\Delta_n) \subseteq \tilde{\varphi}_N(\Delta_n) \subseteq K.$$

This implies that

$$\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^{k} (G(x_{ij}) \cap K) = \bigcup_{j=0}^{k} (T(x_{ij})).$$

Thus, $T$ is also a generalized $L$-KKM mapping with closed valued in $K$. By Theorem 3.1, we have

$$\bigcap_{j=0}^{k} T(x_{ij}) \neq \emptyset.$$

Since $K$ is compact and $T(x) = G(x) \cap K$ is closed in $K$, we know that $\{Tx : x \in X\}$ is a family of compact subsets of $K$. Hence

$$K \cap \left( \bigcap_{x \in X} G(x) \right) = \bigcap_{x \in X} T(x) \neq \emptyset.$$

This completes the proof. □

Remark 3.2. Theorem 3.2 generalizes Theorem 3.3 of Deng and Xia [5] in the following two aspects: (1) from the generalized $R$-KKM mapping to the generalized $L$-KKM mapping; (2) from the compact topological space to the noncompact topological space. Theorem 3.2 also improves Theorem 2.2 of Ding [12] from the generalized $R$-KKM mapping to the generalized $L$-KKM mapping.

Theorem 3.3. Let $X$ be a nonempty set, $Y$ be a topological space, $K$ be a compact subset of $Y$, $G : X \rightarrow 2^Y$ be a generalized $L$-KKM mapping with nonempty compactly closed values, and $S : X \rightarrow 2^Y$ be a set-valued mapping satisfying the following condition:

(i) there exists a compact $L$-subspace $L_M$ of $Y$ such that, for each $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ (where some elements in $N$ may be same), $N \subseteq S^{-1}(L_M)$,

$$L_M \cap \left( \bigcap_{x \in S^{-1}(L_M)} G(x) \right) \subset K,$$
and \( \varphi_N(\Delta_n) \subseteq \bar{\varphi}_N(\Delta_n) \) for \( \Delta_n = \text{co}(\{e_0, \ldots, e_n\}) \), where \( \varphi_N, \bar{\varphi}_N : \Delta_n \rightarrow 2^Y \) are the lower semicontinuous mappings in touch with \( N \) in Definition 2.1 and Definition 2.2, respectively.

Then \( K \cap (\bigcap_{x \in X} G(x)) \neq \emptyset \).

**Proof.** We first prove that the family \( \{G(x) \cap K : x \in X\} \) has the finite intersection property. Define the set-valued mapping \( F : S^{-1}(L_M) \rightarrow 2^{L_M} \) by

\[
F(x) = G(x) \cap L_M, \quad \forall x \in X.
\]

We now show that \( F \) is also a generalized \( L \)-KKM mapping. Since \( G \) is a generalized \( L \)-KKM mapping, for each \( N = \{x_0, \ldots, x_n\} \in \langle X \rangle \) (where some elements in \( N \) may be same), there exists a lower semicontinuous mapping \( \varphi_N : \Delta_n \rightarrow 2^Y \) such that, for each \( \{e_{i_0}, \ldots, e_{i_k}\} \subseteq \{e_0, \ldots, e_n\} \),

\[
\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^k G(x_{i_j}),
\]

where \( \Delta_k = \text{co}(\{e_{i_0}, \ldots, e_{i_k}\}) \). On the other hand, by condition (i), \( L_M \) is a compact \( L \)-subspace of \( Y \) such that for \( N = \{x_0, \ldots, x_n\} \in \langle X \rangle \), \( N \subseteq S^{-1}(L_M) \) and \( \varphi_N(\Delta_n) \subseteq \bar{\varphi}_N(\Delta_n) \). Thus, we have

\[
\varphi_N(\Delta_k) \subseteq \varphi_N(\Delta_n) \subseteq \bar{\varphi}_N(\Delta_n) \subseteq L_M.
\]

It follows that

\[
\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^k (G(x_{i_j}) \cap L_M) = \bigcup_{j=0}^k (F(x_{i_j})).
\]

This implies that \( F \) is also a generalized \( L \)-KKM mapping. Noting that \( L_M \) is compact and \( F(x) \) is closed in \( L_M \), it follows from Theorem 3.2 that

\[
\bigcap_{x \in S^{-1}(L_M)} F(x) = L_M \bigcap \left( \bigcap_{x \in S^{-1}(L_M)} G(x) \right) \neq \emptyset.
\]

Taking \( y \in L_M \cap (\bigcap_{x \in S^{-1}(L_M)} G(x)) \), condition (i) implies that \( y \in K \) and \( N \subseteq S^{-1}(L_M) \). Therefore,

\[
y \in K \bigcap \left( \bigcap_{x \in S^{-1}(L_M)} G(x) \right) \subseteq \bigcap_{x \in N} (G(x) \cap K).
\]

This implies that the family \( \{G(x) \cap K : x \in X\} \) has the finite intersection property. Since \( K \) is compact and each \( G(x) \) is compactly closed, we have

\[
K \bigcap \left( \bigcap_{x \in X} G(x) \right) \neq \emptyset.
\]

This completes the proof. \( \square \)

**Remark 3.3.** Theorem 3.3 improves Theorem 2.3 of Ding [12] in the following two aspects: (1) from the generalized \( R \)-KKM mapping to the generalized \( L \)-KKM mapping; (2) from the single-valued mapping \( s \) to the set-valued mapping \( S \).

### 4. An application

In this section, we will give an existence theorem of the equilibrium point of the abstract generalized vector equilibrium problem to show the application of the obtained results in Section 3.

Let \( X \) be a nonempty set, \( Y \) and \( Z \) be two topological spaces, and \( K \) be a compact subset of \( Y \). Let \( F, G : X \times Y \rightarrow 2^Z, S : X \rightarrow 2^Y \), and \( C : Y \rightarrow 2^Z \) be four set-valued mappings satisfying the following conditions:
(i) the mapping \( C(\cdot) \) has open graph in \( Y \times Z \);
(ii) for each \( x \in X \), the mapping \( y \mapsto F(x, y) \) is upper semicontinuous on each compactly subset of \( Y \) with nonempty compact values;
(iii) for each \( (x, y) \in X \times Y \), \( G(x, y) \not\subseteq C(y) \) implies that \( F(x, y) \not\subseteq C(y) \);
(iv) \( G(x, y) \) is generalized \( L \)-diagonally-quasi-convex in \( x \) with respect to \( C \).

Let \( H : X \to 2^Y \) be a set-valued mapping defined by

\[
H(x) = \{ y \in Y : G(x, y) \not\subseteq C(y) \}, \quad \forall x \in X.
\]

Then it follows from condition (iv) and Lemma 2.3 that \( H \) is a generalized \( L \)-KKM mapping.

Now we give the following result.

**Theorem 4.1.** Let \( F, G : X \times Y \to 2^Z \), \( S : X \to 2^Y \), and \( C : Y \to 2^Z \) be four set-valued mappings satisfying the following conditions (i)–(iv). Let \( H : X \to 2^Y \) be a set-valued mapping defined by (4.1). Moreover, suppose that

\[
\varphi_N(\Delta_n) \subseteq \varphi_N'(\Delta_n) \text{ for } \Delta_n = co\{e_0, \ldots, e_n\}, \quad \text{where } \varphi_N : \Delta_n \to 2^Y \text{ is the lower semicontinuous mapping in touch with } N \text{ in Definition 2.1 for the mapping } H \text{ and } \varphi_N' : \Delta_n \to 2^Y \text{ is the lower semicontinuous mappings in touch with } N \text{ in Definition 2.2 for the } L \text{-subspace } L_M.
\]

Then there exists \( \hat{y} \in K \) such that

\[
F(x, \hat{y}) \not\subseteq C(\hat{y}), \quad \forall x \in X.
\]

**Proof.** Define the set-valued mapping \( T : X \to 2^Y \) by

\[
T(x) = \{ y \in Y : F(x, y) \not\subseteq C(y) \}, \quad \forall x \in X.
\]

Condition (iii) implies that \( H(x) \subseteq T(x) \). It follows from conditions (i), (ii) and Lemma 2.2 that \( T \) is compactly close in \( Y \). Since \( H(x) \) is a generalized \( L \)-KKM mapping, for each \( N = \{x_0, \ldots, x_n\} \in \langle X \rangle \) (where some elements in \( N \) may be the same), there exists a lower semicontinuous mapping \( \varphi_N : \Delta_n \to 2^Y \) such that for each \( \{e_{i_0}, \ldots, e_{i_k}\} \subset \{e_0, \ldots, e_n\} \),

\[
\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^k H(x_{i_j}),
\]

where \( \Delta_k = co\{e_{i_0}, \ldots, e_{i_k}\} \). Hence,

\[
\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^k H(x_{i_j}) \subseteq \bigcup_{j=0}^k T(x_{i_j}) \subseteq \bigcup_{j=0}^k T(x_{i_j}).
\]

This implies that \( T(x) \) is also a generalized \( L \)-KKM mapping. Moreover, by condition (v), there exists a compact \( L \)-subspace \( L_M \) of \( Y \) such that, for \( N = \{x_0, \ldots, x_n\} \in \langle X \rangle \), \( N \subset S^{-1}(L_M) \),

\[
L_M \cap \bigcap_{x \in S^{-1}(L_M)} T(x) \subseteq K,
\]

and \( \varphi_N(\Delta_n) \subseteq \varphi_N'(\Delta_n) \text{ for } \Delta_n = co\{e_0, \ldots, e_n\} \). Thus, all conditions of Theorem 3.3 are satisfied and so Theorem 3.3 implies that

\[
K \cap \left( \bigcap_{x \in X} T(x) \right) \neq \emptyset.
\]

Therefore, there exists \( \hat{y} \in K \) such that \( F(x, \hat{y}) \not\subseteq C(\hat{y}) \) for all \( x \in X \). This completes the proof. \( \square \)
Remark 4.1. Theorem 4.1 generalizes Theorem 3.2 in Ding and Park [18] in the following aspects: (1) from the $G$-convex space to the general topological space without any convexity; (2) from generalized $S$-diagonally quasi-convex in $x$ with respect to $C$ to generalized $L$-diagonally quasi-convex in $x$ with respect to $C$; (3) condition (v) of Theorem 4.1 is weaker than condition (b) of Theorem 3.2 in Ding and Park [18].

Remark 4.2. Theorem 4.1 also improves Theorem 1 of Ansari, Oettli and Schläger [19] in the following ways: (1) drop the assumption that $A \subseteq X$, $B \subseteq Y$ are nonempty, convex and compact; (2) the condition that $C$ is a fixed set is replaced by the condition that $C$ is a set-valued mapping.

Acknowledgements

The authors would like to express their thanks to the referees for their valuable comments and suggestions that improved the presentation of this paper.

References